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Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$ II*Sous-groupes commutateur de puissance sous-groupes $H(\lambda_q)$ II*Recep Sahin^a, Özden Koruoğlu^b^a Balıkesir Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, 10145 Balıkesir, Turkey^b Balıkesir Üniversitesi, Necatibey Eğitim Fakültesi, İlköğretim Bölümü, 10100 Balıkesir, Turkey

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ABSTRACT

Let $q \geq 3$ be an odd number and let $H(\lambda_q)$ be the Hecke group associated to q . Let m be a positive integer and let $H^m(\lambda_q)$ be the m -th power subgroup of $H(\lambda_q)$. In this work, we study the commutator subgroups of the power subgroups $H^m(\lambda_q)$ of $H(\lambda_q)$.

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R É S U M É

Soit $q \geq 3$ un nombre impair et soit $H(\lambda_q)$ le groupe de Hecke associé à q . Soit m un entier positif et soit $H^m(\lambda_q)$ le sous-groupe des puissances m -ièmes de $H(\lambda_q)$. Dans ce travail, nous étudions les sous-groupes commutateurs des puissances sous-groupes $H^m(\lambda_q)$ de $H(\lambda_q)$.

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1. Introduction

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z + \lambda},$$

where λ is a fixed positive real number.

E. Hecke showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2 \cos \frac{\pi}{q}$, for $q \geq 3$ integer, or $\lambda \geq 2$. These groups have come to be known as the *Hecke groups*. We consider the former case $q \geq 3$ integer and we denote it by $H(\lambda_q)$. Then the Hecke group $H(\lambda_q)$ is the discrete group generated by T and S , and it has a presentation, see [3]:

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_q.$$

Also $H(\lambda_q)$ has the signature $(0; 2, q, \infty)$, that is they are infinite triangle groups. The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$.

Let m be a positive integer. Let us define $H^m(\lambda_q)$ to be the subgroup generated by the m -th powers of all elements of $H(\lambda_q)$. The subgroup $H^m(\lambda_q)$ is called the *m -th power subgroup* of $H(\lambda_q)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

The power subgroups of the Hecke groups $H(\lambda_q)$ have been studied and classified in [1,5,2]. For $q \geq 3$ odd integer and for m positive integer, they proved the following results in [2].

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i) If $(m, 2) = 1$ and $(m, q) = 1$, then the normal subgroup $H^m(\lambda_q)$ is $H(\lambda_q)$, i.e.,

$$H^m(\lambda_q) = H(\lambda_q).$$

ii) If $(m, 2) = 1$ and $(m, q) = 1$, then the normal subgroup $H^m(\lambda_q)$ is the free product of two finite cyclic groups of order q , i.e.,

$$H^m(\lambda_q) = \langle S, TST \mid S^q = (TST)^q = I \rangle \cong \mathbb{Z}_q * \mathbb{Z}_q, \quad (1)$$

and the signature of $H^m(\lambda_q)$ is $(0; q^{(2)}, \infty)$.

iii) If $(m, 2) = 1$ and $(m, q) = d$, then the normal subgroup $H^m(\lambda_q)$ is the free product of d finite cyclic groups of order two and the finite cyclic group of order q/d , i.e.,

$$H^m(\lambda_q) = \langle T \rangle * \langle STS^{q-1} \rangle * \langle S^2TS^{q-2} \rangle * \dots * \langle S^{d-1}TS^{q-d+1} \rangle * \langle S^d \rangle, \quad (2)$$

and it has the signature $(0; 2^{(d)}, q/d, \infty)$.

iv) If $(m, 2) = 2$ and $(m, q) = d > 2$, then we do not know anything about the normal subgroup $H^m(\lambda_q)$. But, if $d = q$, then

$$H^m(\lambda_q) \subset H'(\lambda_q) \quad (3)$$

and the normal subgroup $H^m(\lambda_q)$ is a free group.

In iii) and iv), if $d = q$ then we can restate these cases as the following.

iii*) If $(m, 2) = 1$ and $(m, q) = q$, then the normal subgroup $H^m(\lambda_q)$ is the free product of q finite cyclic groups of order two, i.e.,

$$H^m(\lambda_q) = \langle T \rangle * \langle STS^{q-1} \rangle * \langle S^2TS^{q-2} \rangle * \dots * \langle S^{q-1}TS \rangle.$$

iv*) If $(m, 2) = 2$ and $(m, q) = q$, then

$$H^m(\lambda_q) \subset H'(\lambda_q) \quad (4)$$

and the normal subgroup $H^m(\lambda_q)$ is a free group.

In [6], according to all the cases m positive integer, Sahin and Koruoğlu obtained the group structures and the signatures of commutator subgroups of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, $q \geq 3$ a prime.

In this note, we generalize the results given in [6] for $q \geq 3$ a prime number to $q \geq 3$ an odd number. In fact, there are similar cases between the case of prime q and the case of odd q . In the above cases i), ii), iii*) and iv*), then all results for $q \geq 3$ an odd number coincide with the ones for $q \geq 3$ a prime number in [6]. In the case $q \geq 3$ an odd number, there are only two cases different from $q \geq 3$ prime number case in [6]. These are the cases iii) and iv) except for the cases iii*) and iv*). Since we don't know anything about the normal subgroup $H^m(\lambda_q)$ in the case iv) except for the case iv*), we investigate only the case iii) and obtain the commutator subgroups of the power subgroups $H^m(\lambda_q)$ of the Hecke groups $H(\lambda_q)$.

2. Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$

In this section, we study the commutator subgroups of the power subgroups $H^2(\lambda_q)$ and $H^q(\lambda_q)$ of the Hecke groups $H(\lambda_q)$, for $q \geq 3$ odd number.

Theorem 2.1. *Let $q \geq 3$ be odd number. If $(m, 2) = 2$ and $(m, q) = 1$, then*

- i) $|H^m(\lambda_q) : (H^m)'(\lambda_q)| = q^2$,
- ii) the group $(H^m)'(\lambda_q)$ is a free group of rank $(q-1)^2$ with basis $[S, TST], [S, TS^2T], \dots, [S, TS^{q-1}T], [S^2, TST], [S^2, TS^2T], \dots, [S^2, TS^{q-1}T], \dots, [S^{q-1}, TST], [S^{q-1}, TS^2T], \dots, [S^{q-1}, TS^{q-1}T]$,
- iii) the group $(H^m)'(\lambda_q)$ is of index q in $H'(\lambda_q)$,
- (iv) for $n \geq 2$, $|H^m(\lambda_q) : (H^m)^{(n)}(\lambda_q)| = \infty$.

Theorem 2.2. *Let $q \geq 3$ be odd number. If $(m, 2) = 1$ and $(m, q) = d$, then*

- i) $|H^m(\lambda_q) : (H^m)'(\lambda_q)| = 2^d \cdot \frac{q}{d}$,
- ii) the group $(H^m)'(\lambda_q)$ is a free group of rank $1 + (q-2)2^{d-1}$,
- iii) the group $(H^m)'(\lambda_q)$ is of index 2^{d-1} in $H'(\lambda_q)$,
- (iv) for $n \geq 2$, $|H^m(\lambda_q) : (H^m)^{(n)}(\lambda_q)| = \infty$.

Proof. i) From (1.1), let $k_1 = T$, $k_2 = STS^{q-1}$, $k_3 = S^2TS^{q-2}$, ..., $k_d = S^{d-1}TS^{q-d+1}$, $k_{d+1} = S^d$. The quotient group $H^m(\lambda_q)/(H^m)'(\lambda_q)$ is the group obtained by adding the relation $k_i k_j = k_j k_i$ to the relations of $H^m(\lambda_q)$, for $i \neq j$ and $i, j \in \{1, 2, \dots, d+1\}$. Then

$$H^m(\lambda_q)/(H^m)'(\lambda_q) \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{d \text{ times}} \times \mathbb{Z}_{q/d}.$$

Therefore, we obtain $|H^m(\lambda_q) : (H^m)'(\lambda_q)| = 2^d \cdot \frac{q}{d}$.

ii) Now we choose $\Sigma = \{I, k_1, k_2, \dots, k_d, k_{d+1}, k_{d+1}^2, \dots, k_{d+1}^{\frac{q}{d}-1}, k_1 k_2, k_1 k_3, \dots, k_1 k_d, k_1 k_{d+1}, k_1 k_{d+1}^2, \dots, k_1 k_{d+1}^{\frac{q}{d}-1}, k_2 k_3, \dots, k_2 k_d, k_2 k_{d+1}, k_2 k_{d+1}^2, \dots, k_2 k_{d+1}^{\frac{q}{d}-1}, \dots, k_d k_{d+1}, k_d k_{d+1}^2, \dots, k_d k_{d+1}^{\frac{q}{d}-1}, k_1 k_2 k_3, \dots, k_1 k_2 k_d, k_1 k_2 k_{d+1}, k_1 k_2 k_{d+1}^2, \dots, k_1 k_2 k_{d+1}^{\frac{q}{d}-1}, \dots, k_1 k_d k_{d+1}, k_1 k_d k_{d+1}^2, \dots, k_1 k_d k_{d+1}^{\frac{q}{d}-1}, k_2 k_3 k_4, \dots, k_2 k_d k_{d+1}, k_2 k_d k_{d+1}^2, \dots, k_1 k_d k_{d+1}^{\frac{q}{d}-1}, \dots, k_1 k_2 \dots k_d k_{d+1}, k_1 k_2 \dots k_d k_{d+1}^2, \dots, k_1 k_2 \dots k_d k_{d+1}^{\frac{q}{d}-1}\}$ as a Schreier transversal for $(H^m)'(\lambda_q)$. According to the Reidemeister–Schreier method, we get the generators of $(H^m)'(\lambda_q)$ as the followings.

There are $\binom{d}{2}$ generators of the form $k_i k_t k_i k_t$ where $i < t$ and $i, t \in \{1, 2, \dots, d\}$. There are $2 \times \binom{d}{3}$ generators of the form $k_i k_t k_l k_t k_i k_l$, or $k_i k_t k_l k_i k_l k_t$ where $i < t < l$ and $i, t, l \in \{1, 2, \dots, d\}$. Similarly, there are $(d-1) \times \binom{d}{d}$ generators of the form $k_1 k_2 \dots k_d k_1 k_d k_{d-1} \dots k_2$, or $k_1 k_2 \dots k_d k_2 k_d k_{d-1} \dots k_3 k_1$, or ..., or $k_1 k_2 \dots k_d k_{d-1} k_d k_{d-2} \dots k_2 k_1$.

Also, there are $\binom{d}{1} \binom{\frac{q}{d}-1}{1}$ generators of the form $k_i k_{d+1}^j k_i k_{d+1}^{d-j}$ where $i \in \{1, 2, \dots, d\}$ and $j \in \{1, 2, \dots, \frac{q}{d}-1\}$. There are $2 \times \binom{d}{2} \binom{\frac{q}{d}-1}{1}$ generators of the form $k_i k_t k_{d+1}^j k_t k_{d+1}^{d-j} k_i$, or $k_i k_t k_{d+1}^j k_i k_{d+1}^{d-j} k_t$ where $i, t \in \{1, 2, \dots, d\}$, $i < t$ and $j \in \{1, 2, \dots, \frac{q}{d}-1\}$. Similarly, there are $d \times \binom{d}{d} \binom{\frac{q}{d}-1}{1}$ generators of the form $k_1 k_2 \dots k_d k_{d+1}^j k_1 k_{d+1}^{d-j} k_d k_{d-1} \dots k_2$, or $k_1 k_2 \dots k_d k_{d+1}^j k_2 k_{d+1}^{d-j} k_d k_{d-1} \dots k_3 k_1$, ..., or $k_1 k_2 \dots k_d k_{d+1}^j k_{d-1} k_{d+1}^{d-j} k_d k_{d-2} \dots k_2 k_1$ where $j \in \{1, 2, \dots, \frac{q}{d}-1\}$. In fact, there are totally $1 + (q-2)2^{d-1}$ generators from the theorem of Nielsen.

iii) Since $|H(\lambda_q) : H^m(\lambda_q)| = d$, $|H(\lambda_q) : H'(\lambda_q)| = 2q$ and $|H(\lambda_q) : (H^m)'(\lambda_q)| = q \cdot 2^d$, we obtain

$$|H'(\lambda_q) : (H^m)'(\lambda_q)| = 2^{d-1}.$$

iv) Please see the proof of the Theorem 2.2 iv). □

Also, we obtain the signature of $(H^m)'(\lambda_q)$ as

$$((q-2)2^{d-2} - 2^{(d-2) \cdot q/d} + 1; \underbrace{\infty, \infty, \dots, \infty}_{2^{d-1} \text{ times}}) = ((q-2)2^{d-2} - 2^{(d-2) \cdot q/d} + 1; \infty^{(2^{d-1})}).$$

Remark 2.1. In Theorem 2.2, if $d = q$ then $\frac{q}{d} - 1 = 0$. Then there is no $\binom{d}{1} \binom{\frac{q}{d}-1}{1} + 2 \times \binom{d}{2} \binom{\frac{q}{d}-1}{1} + \dots + d \times \binom{d}{d} \binom{\frac{q}{d}-1}{1}$ generators given in the last paragraph of the proof of ii). Thus there are only $\binom{d}{2} + 2 \times \binom{d}{3} + \dots + (d-1) \times \binom{d}{d} = 1 + (d-2)2^{d-1}$ generators of $(H^m)'(\lambda_q)$. In this case, this result coincides with the Theorem 2.2 in [6].

Example 2.1. Let $q = 9$ and $d = 3$. Then $|H^3(\lambda_9) : (H^3)'(\lambda_9)| = 24$. We choose $\Sigma = \{I, k_1, k_2, k_3, k_4, k_4^2, k_1 k_2, k_1 k_3, k_1 k_4, k_1 k_4^2, k_2 k_3, k_2 k_4, k_2 k_4^2, k_3 k_4, k_3 k_4^2, k_1 k_2 k_3, k_1 k_2 k_4, k_1 k_2 k_4^2, k_1 k_3 k_4, k_1 k_3 k_4^2, k_2 k_3 k_4, k_2 k_3 k_4^2, k_1 k_2 k_3 k_4, k_1 k_2 k_3 k_4^2\}$ as a Schreier transversal for $(H^3)'(\lambda_9)$. Using the Reidemeister–Schreier method, we get the generators of $(H^3)'(\lambda_9)$ as the following. There are 3 generators of the form,

$$k_1 k_2 k_1 k_2, \quad k_1 k_3 k_1 k_3, \quad k_2 k_3 k_2 k_3,$$

2 generators of the form,

$$k_1 k_2 k_3 k_2 k_3 k_1, \quad k_1 k_2 k_3 k_1 k_3 k_2,$$

6 generators of the form,

$$k_1 k_4 k_1 k_4^2, \quad k_2 k_4 k_2 k_4^2, \quad k_3 k_4 k_3 k_4^2, \\ k_1 k_4^2 k_1 k_4, \quad k_2 k_4^2 k_2 k_4, \quad k_3 k_4^2 k_3 k_4,$$

12 generators of the form,

$$k_1 k_2 k_4 k_1 k_4^2 k_2, \quad k_1 k_2 k_4 k_2 k_4^2 k_1, \quad k_1 k_2 k_4^2 k_1 k_4 k_2, \quad k_1 k_2 k_4^2 k_2 k_4 k_1, \\ k_1 k_3 k_4 k_1 k_4^2 k_3, \quad k_1 k_3 k_4 k_3 k_4^2 k_1, \quad k_1 k_3 k_4^2 k_1 k_4 k_3, \quad k_1 k_3 k_4^2 k_3 k_4 k_1, \\ k_2 k_3 k_4 k_2 k_4^2 k_3, \quad k_2 k_3 k_4 k_3 k_4^2 k_2, \quad k_2 k_3 k_4^2 k_2 k_4 k_3, \quad k_2 k_3 k_4^2 k_3 k_4 k_2,$$

and 6 generators of the form,

$$\begin{aligned} k_1 k_2 k_3 k_4 k_1 k_4^2 k_3 k_2, & \quad k_1 k_2 k_3 k_4^2 k_1 k_4 k_3 k_2, \\ k_1 k_2 k_3 k_4 k_2 k_4^2 k_3 k_1, & \quad k_1 k_2 k_3 k_4^2 k_2 k_4 k_3 k_1, \\ k_1 k_2 k_3 k_4 k_3 k_4^2 k_2 k_1, & \quad k_1 k_2 k_3 k_4^2 k_3 k_4 k_2 k_1. \end{aligned}$$

Therefore, the group $(H^3)'(\lambda_9)$ is a free group of rank 29.

References

- [1] I.N. Cangül, D. Singerman, Normal subgroups of Hecke groups and regular maps, *Math. Proc. Camb. Phil. Soc.* 123 (1998) 59–74.
- [2] I.N. Cangül, R. Sahin, S. İkikardes, Ö. Koruoğlu, Power subgroups of some Hecke groups. II, *Houston J. Math.* 33 (1) (2007) 33–42.
- [3] R.J. Evans, A new proof on the free product structure of Hecke groups, *Publ. Math. Debrecen* 22 (1–2) (1975) 41–42.
- [4] E. Hecke, Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichungen, *Math. Ann.* 112 (1936) 664–699.
- [5] S. İkikardes, Ö. Koruoğlu, R. Sahin, Power subgroups of some Hecke groups, *Rocky Mountain J. Math.* 36 (2) (2006) 497–508.
- [6] R. Sahin, Ö. Koruoğlu, Commutator subgroups of the power subgroups of Hecke groups $H(\lambda_q)$, *Ramanujan J.*, doi:10.1007/s11139-010-9246-1, in press.