

VRANCEANU SURFACE IN \mathbb{E}^4 WITH POINTWISE 1- TYPE GAUSS MAP

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In this article we investigate Vranceanu rotation surfaces with pointwise 1-type Gauss map in Euclidean 4-space \mathbb{E}^4 . We show that a Vranceanu rotation surface M has harmonic Gauss map if and only if M is a part of a plane. Further, we give necessary and sufficient conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map.

Key words : Rotation surface, Gauss map, finite type, pointwise 1-type.

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1. INTRODUCTION

Since the late 1970's, the study of submanifolds of Euclidean space or pseudo-Euclidean space with the notion of finite type immersion has been extensively carried out. An isometric immersion $x : M \rightarrow \mathbb{E}^m$ of a submanifold M in Euclidean m -space \mathbb{E}^m is said to be of finite type if x identified with the position vector field of M in \mathbb{E}^m can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M , that is;

$$x = x_0 + \sum_{i=1}^k x_i$$

where x_0 is a constant map, x_1, x_2, \dots, x_k non-constant maps such that $\Delta x = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. If $\lambda_1, \lambda_2, \dots, \lambda_k$ are different, then M is said to be of k -type. Similarly, a smooth map ϕ of an n -dimensional Riemannian manifold M of \mathbb{E}^m is said to be of finite type if ϕ is a finite sum of \mathbb{E}^m -valued eigenfunctions of Δ (of. [5], [6]). Granted, this notion of finite type immersion is naturally extended to the Gauss map G on M in Euclidean space [8]. Thus, if a submanifold M of Euclidean space has 1-type Gauss map G , then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C (of. [1], [2], [3], [10]). However, the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and a right cone in Euclidean 3-space \mathbb{E}^3 take a somewhat different form; namely, $\Delta G = f(G + C)$ for some non-constant function f and some constant vector C . Therefore, it is worth studying the class of solution surfaces satisfying such an equation. A submanifold M of a Euclidean space \mathbb{E}^m is said to have pointwise 1-type Gauss map if its Gauss map G satisfies

$$\Delta G = f(G + C) \tag{1}$$

for some non-zero smooth function f on M and a constant vector C . A pointwise 1-type Gauss map is called proper if the function f defined by (1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind ([7], [9], [11], [12]). In particular, if the Gauss map G is a constant vector $-C$, we regard it of pointwise 1-type of the second kind. Granted, it is of the first kind as well in this case. In [9], one of the present authors characterized the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Also, together with Chen, they proved that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with surfaces of revolution with constant mean curvature.

Moreover, they characterized the rational surfaces of revolution with pointwise 1-type Gauss map [7]. In [18] D. W. Yoon studied Vranceanu rotation surfaces in Euclidean 4-space \mathbb{E}^4 . He proved that the flat Vranceanu rotation surfaces with pointwise 1-type Gauss map is a Clifford torus, i.e. it is the product of two plane circles with the same radius. For more details see also [14, 17]. In this article investigate Vranceanu rotation surfaces with pointwise 1-type Gauss map in Euclidean 4-space \mathbb{E}^4 . We show that a Vranceanu rotation surface M has harmonic Gauss map if and only if M is a part of a plane. Further, we give the complete classification of flat Vranceanu rotation surfaces with pointwise 1-type Gauss map. Finally, we give the necessary and sufficient conditions for non-flat Vranceanu rotation surface to have pointwise 1-type Gauss map.

All functions under consideration are assumed to be smooth and the manifolds are connected unless otherwise stated.

2. PRELIMINARIES

In the present section we recall definitions and results of [4]. Let $x : M \rightarrow \mathbb{E}^m$ be an immersion from an n -dimensional connected Riemannian manifold M into an m -dimensional Euclidean space \mathbb{E}^m . We denote by g the metric tensor of \mathbb{E}^m as well as the induced metric on M . Let $\tilde{\nabla}$ be the Levi-Civita connection of \mathbb{E}^m and ∇ the induced connection on M . Then the Gaussian and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2)$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \quad (3)$$

where X, Y are vector fields tangent to M and ξ normal to M . Moreover, h is the second fundamental form, D is the linear connection induced in the normal bundle $T^\perp M$, called normal connection and A_ξ the shape operator in the direction of ξ that is related with h by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle. \quad (4)$$

If we define a covariant differentiation $\bar{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^\perp M$ of M by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for any vector fields X, Y and Z tangent to M . Then we have the Codazzi equation

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z). \quad (5)$$

We denote by R the curvature tensor associated with ∇ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (6)$$

The equation of Gauss and Ricci are given by

$$\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \quad (7)$$

$$\langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle \quad (8)$$

for vectors X, Y, Z, W tangent to M and ξ, η normal to M .

Let us now define the Gauss map G of a submanifold M into $G(n, m)$ in $\wedge^n \mathbb{E}^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n -planes through the origin of \mathbb{E}^m and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in \mathbb{E}^m . In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space \mathbb{E}^N where $N = \binom{m}{n}$. Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ be an adapted local orthonormal frame field in \mathbb{E}^m such that e_1, e_2, \dots, e_n , are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ normal to M . The map $G : M \rightarrow G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_n)(p)$ is called the Gauss map of M that is a smooth map which carries a point p in M into the oriented n -plane in \mathbb{E}^m obtained from the parallel translation of the tangent space of M at p in \mathbb{E}^m .

For any real function f on M the Laplacian of f is defined by

$$\Delta f = - \sum_i (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i} e_i} f). \quad (9)$$

The mean curvature vector field \vec{H} of M is defined by

$$\vec{H} = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i). \quad (10)$$

3. VRANCEANU SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

Rotation surfaces were studied in [16] by Vranceanu as surfaces in \mathbb{E}^4 which are defined by the following parametrization

$$M : X(s, t) = (r(s) \cos s \cos t, r(s) \cos s \sin t, r(s) \sin s \cos t, r(s) \sin s \sin t) \quad (11)$$

We choose a moving frame Let e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as given in the following (see, [13, 15]):

$$\begin{aligned} e_1 &= \frac{1}{r} \frac{\partial}{\partial t} \\ &= (-\cos s \sin t, \cos s \cos t, -\sin s \sin t, \sin s \cos t), \\ e_2 &= \frac{1}{A} \frac{\partial}{\partial s} \\ &= \frac{1}{A} (B \cos t, B \sin t, C \cos t, C \sin t), \\ e_3 &= \frac{1}{A} (-C \cos t, -C \sin t, B \cos t, B \sin t), \\ e_4 &= (-\sin s \sin t, \sin s \cos t, \cos s \sin t, -\cos s \cos t), \end{aligned} \quad (12)$$

where

$$\begin{aligned} A &= \sqrt{r^2(s) + (r'(s))^2}, \quad B = r'(s) \cos s - r(s) \sin s, \\ C &= r'(s) \sin s + r(s) \cos s. \end{aligned} \quad (13)$$

Furthermore, by covariant differentiation with respect to e_1 and e_2 , a straightforward calculation gives:

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= -\lambda(s)k(s)e_2 + \lambda(s)e_3, \\ \tilde{\nabla}_{e_2} e_2 &= \mu(s)e_3, \\ \tilde{\nabla}_{e_2} e_1 &= -\lambda(s)e_4, \\ \tilde{\nabla}_{e_1} e_2 &= \lambda(s)k(s)e_1 - \lambda(s)e_4, \\ \tilde{\nabla}_{e_1} e_3 &= -\lambda(s)e_1 - \lambda(s)k(s)e_4, \\ \tilde{\nabla}_{e_1} e_4 &= \lambda(s)e_2 + \lambda(s)k(s)e_3, \\ \tilde{\nabla}_{e_2} e_3 &= -\mu(s)e_2, \\ \tilde{\nabla}_{e_2} e_4 &= \lambda(s)e_1, \end{aligned} \quad (14)$$

where

$$\begin{aligned} k &= \frac{r'(s)}{r(s)}, \\ \lambda &= \frac{1}{A} = \frac{1}{\sqrt{r^2(s) + (r'(s))^2}}, \\ \mu &= \frac{2(r'(s))^2 - r(s)r''(s) + r^2(s)}{(r^2 + (r')^2)^{3/2}}, \end{aligned} \quad (15)$$

are differentiable functions.

By the use of (4) the coefficients of the second fundamental form h become

$$h_{ij}^k = \langle h(e_i, e_j), e_k \rangle = \langle A_{e_k} e_i, e_j \rangle \quad 1 \leq i, j \leq 2, \quad k = 3, 4. \quad (16)$$

Moreover, combining (2), (14) and (16) we get

$$\begin{aligned} h_{11}^3 &= \lambda, & h_{12}^3 &= h_{21}^3 = 0, \\ h_{22}^3 &= \mu, & h_{11}^4 &= h_{22}^4 = 0, \\ h_{12}^4 &= h_{21}^4 = -\lambda. \end{aligned} \quad (17)$$

By the use of (17) together with (14) we get the following result.

Lemma 1 — Let M a Vranceanu surface given with the surface patch (11). Then, we get

$$A_{e_3} = \begin{bmatrix} \lambda(s) & 0 \\ 0 & \mu(s) \end{bmatrix}, \quad A_{e_4} = \begin{bmatrix} 0 & -\lambda(s) \\ -\lambda(s) & 0 \end{bmatrix}. \quad (18)$$

The Gauss curvature K and the mean curvature vector field \vec{H} are respectively given by

$$\begin{aligned} K &= \det A_{e_3} + \det A_{e_4} \\ &= \lambda(\mu - \lambda) \\ &= \frac{(r')^2 - rr''}{(r^2 + (r')^2)^2} \end{aligned} \quad (19)$$

and

$$\begin{aligned}
\vec{H} &= \frac{1}{2} \sum_{i=1}^2 h(e_i, e_i) \\
&= \frac{1}{2}(\lambda + \mu)e_3 \\
&= \frac{2r^2 + 3(r')^2 - rr''}{2(r^2 + (r')^2)^{3/2}} e_3.
\end{aligned} \tag{20}$$

The Gauss map G of the M is defined by $G = e_1 \wedge e_2$. By using (14), (15) and straight-forward computation the Laplacian of the Gauss map can be expressed as

$$\begin{aligned}
-\Delta G &= -(3\lambda^2(s) + \mu^2(s))e_1 \wedge e_2 + (3\lambda^2(s)k(s) + \lambda'(s))e_2 \wedge e_4 \\
&\quad + (\mu'(s) + \lambda(s)\mu(s)k(s) - 2\lambda^2(s)k(s))e_1 \wedge e_3 \\
&\quad + 2(\lambda(s)\mu(s) - \lambda^2(s))e_3 \wedge e_4.
\end{aligned} \tag{21}$$

Suppose that the Gauss map of M is harmonic, i.e. $\Delta G = \vec{0}$. Firstly, if $G = -C$, M is part of a plane which is not a Vranceanu surface. Secondly, suppose the function f identically zero. From (21) we get

$$3\lambda^2(s) + \mu^2(s) = 0.$$

It implies that $\lambda = 0$ and $\mu = 0$, a contradiction. So, this case cannot occur.

Thus we have

Theorem 1 *There is no Vranceanu surface with harmonic Gauss map.*

Now, we suppose that the Vranceanu surface given by the patch (11) is of pointwise 1-type Gauss map in \mathbb{E}^4 . From (1) and (21)

$$\begin{aligned}
f + f \langle C, e_1 \wedge e_2 \rangle &= +3\lambda^2(s) + \mu^2(s), \\
f \langle C, e_1 \wedge e_3 \rangle &= 2\lambda^2(s)\mu(s) - \mu'(s) - \lambda(s)\mu(s)k(s), \\
f \langle C, e_2 \wedge e_4 \rangle &= -3\lambda^2(s)k(s) - \lambda'(s), \\
f \langle C, e_3 \wedge e_4 \rangle &= -2\lambda(s)\mu(s) + 2\lambda^2(s),
\end{aligned} \tag{22}$$

where f is a smooth non-zero function. Since λ is non-zero everywhere on M , so is the function f . Then we obtain from (21)

$$\begin{aligned}\langle C, e_1 \wedge e_4 \rangle &= 0, \\ \langle C, e_2 \wedge e_3 \rangle &= 0.\end{aligned}\tag{23}$$

Differentiating (23) and using (22) we get

$$f = 5\lambda^2(s)(k^2(s)+1) - \lambda(s)\mu(s)(2+k^2(s)) + k(s)(\lambda'(s) - \mu'(s)) + \mu^2(s).\tag{24}$$

By using the equations of Gauss (7), Codazzi (5) we get

$$\lambda'(s)k(s) + \lambda(s)k'(s) + \lambda^2(s)k^2(s) = \lambda^2(s) - \lambda(s)\mu(s),\tag{25}$$

and

$$\lambda'(s) = \lambda(s)k(s) (\mu(s) - 2\lambda(s)),\tag{26}$$

respectively.

Further, substituting (26) into (25) we obtain

$$(\lambda(s) - \mu(s))(k^2(s) + 1) - k'(s) = 0.\tag{27}$$

We now suppose that the Vranceanu surface M has the Gauss map with pointwise 1-type of the first kind, i.e., C is zero. Then, (22) implies $\lambda = \mu$ on M and thus M is flat. Combining (15) and (19), we get k is constant. By using (26) with $\lambda = \mu$, we get $\lambda'(s) = -k\lambda^2(s)$ and thus $\lambda(s) = \frac{1}{ks+a}$ for some constant a . Since the Gauss map is of pointwise 1-type of the first kind, the mean curvature $\|\vec{H}\|$ is constant and λ is constant (for details, see [12]). Therefore, $k = 0$ and r is a constant function. Hence, the Vranceanu surface M is part of Clifford torus.

We now consider the Gauss map G of M is of pointwise 1-type of the second kind, i.e., $C \neq 0$. We distinguish the following cases:

Case I: M is flat, or, equivalently, $\lambda(s) = \mu(s)$ for all s .

Case II: M is non flat, or, equivalently, $\lambda \neq \mu$.

Let us consider these in turn:

Case I: Suppose that M is flat. Thus, by using (27), we get $k'(s) = 0$. So, from the first equation of (15) we get $r(s) = \alpha \exp(\beta s)$, for some constants $\alpha \neq 0$ and

β . By applying homothetic transformation if necessary we may assume that $\alpha = 1$. Thus, using (15) we get

$$\begin{aligned} r(s) &= e^{(\beta s)}, \\ \lambda(s) &= \mu(s) = \frac{-e^{(\beta s)}}{\sqrt{1+k^2}}. \end{aligned} \quad (28)$$

Consequently, substituting (28) into (24) we obtain

$$\begin{aligned} f &= 4\lambda^2(s)(k^2 + 1) \\ &= 4e^{(-2\beta s)}. \end{aligned} \quad (29)$$

Thus, summing up the following theorem is proved.

Theorem 2 — *Let M be a flat Vranceanu surface given with the parametrization (11). If M has pointwise 1-type Gauss map of the second kind, then M is represented by the function $f = 4e^{(-2\beta s)}$, where β is a real constant.*

Case II: Suppose that M is a non flat surface. Then, the open subset $\mathbf{M}_0 = \{\mathbf{p} \in \mathbf{M} | \mathbf{K}(\mathbf{p}) \neq \mathbf{0}\}$ is not empty. Then, the functions $\lambda \neq \mu$ everywhere on \mathbf{M}_0 . Substituting (15) into (27), we get

$$(r(s)r''(s) - (r'(s))^2) \left(1 - (r^2(s) + (r'(s))^2)^{1/2}\right) = 0 \quad (30)$$

on \mathbf{M}_0 . On \mathbf{M}_0 , by (19) $r(s)r''(s) - (r'(s))^2 \neq 0$. So, from the equation (30) we get

$$r^2(s) + (r'(s))^2 = 1. \quad (31)$$

Thus, differentiating (31) with respect to s we get $r'(s)(r(s) + r''(s)) = 0$ on \mathbf{M}_0 . Suppose $(r + r'')(p) \neq 0$ at $p \in \mathbf{M}_0$. On a component \mathbf{O} containing p in \mathbf{M}_0 , $r' = 0$. Thus, $k = 0$ on \mathbf{O} . If we make use of (27), $\lambda = \mu$ on \mathbf{O} , which is a contradiction. Therefore, we have

$$r(s) + r''(s) = 0$$

on \mathbf{M}_0 , which has non-trivial solutions

$$r(s) = A \sin(s + s_0).$$

for some constants $A \neq 0$ and s_0 . By connectedness of M and the continuity of K , $M = \mathbf{M}_0$ that means M does not have a flat point.

Thus, the following theorem is proved.

Theorem 3— *Let M be a non flat Vranceanu surface given with the parametrization (11). If M has pointwise 1-type Gauss map of the second kind, then M is given up to homothety by*

$$X(s, t) = \left(\frac{1}{2} \sin s \cos t, \frac{1}{2} \sin s \sin t, \sin^2 s \cos t, \sin^2 s \sin t \right).$$

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