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TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN GENERALIZED LEBESGUE SPACES WITH VARIABLE EXPONENT

ТРИГОНОМЕТРИЧНЕ НАБЛИЖЕННЯ ФУНКЦІЙ В УЗАГАЛЬНЕНИХ ПРОСТОРАХ ЛЕБЕГА ЗІ ЗМІННОЮ ЕКСПОНЕНТОЮ

We investigate the approximation properties of the trigonometric system in $L_{2\pi}^{p(\cdot)}$. We consider the fractional order moduli of smoothness and obtain direct, converse approximation theorems together with a constructive characterization of a Lipschitz-type class.

Досліджено властивості наближення тригонометричної системи в $L_{2\pi}^{p(\cdot)}$. Розглянуто модулі гладкості дробового порядку та отримано пряму і обернену теореми наближення разом із конструктивною характеризацією класу типу Ліпшиця.

1. Introduction. Generalized Lebesgue spaces $L^{p(x)}$ with variable exponent and corresponding Sobolev-type spaces have waste applications in elasticity theory, fluid mechanics, differential operators [31, 10], nonlinear Dirichlet boundary-value problems [24], nonstandard growth and variational calculus [33].

These spaces appeared first in [28] as an example of modular spaces [14, 26] and Sharapudinov [36] has been obtained topological properties of $L^{p(x)}$. Furthermore if $p^* := \operatorname{ess\,sup}_{x\in\boldsymbol{T}} p(x) < \infty$, then $L^{p(x)}$ is a particular case of Musielak – Orlicz spaces [26]. Later various mathematicians investigated the main properties of these spaces [36, 24, 32, 12]. In $L^{p(x)}$ there is a rich theory of boundedness of integral transforms of various type [22, 33, 9, 37].

For $p(x) := p$, $1 < p < \infty$, $L^{p(x)}$ is coincide with Lebesgue space L^p and basic problems of trigonometric approximation in L^p are investigated by several mathematicians (among others [39, 19, 30, 40, 6, 4], . . .). Approximation by algebraic polynomials and rational functions in Lebesgue spaces, Orlicz spaces, symmetric spaces and their weighted versions on sufficiently smooth complex domains and curves was investigated in [1 – 3, 15, 18, 16]. For a complete treatise of polynomial approximation we refer to the books [5, 8, 41, 29, 35, 23].

In harmonic and Fourier analysis some of operators (for example partial sum operator of Fourier series, conjugate operator, differentiation operator, shift operator $f \to \to f(\cdot + h)$, $h \in \mathbb{R}$) have been extensively used to prove direct and converse type approximation inequalities. Unfortunately the space $L^{p(x)}$ is not $p(\cdot)$-continuous and not translation invariant [24]. Under various assumptions (including translation invariance) on modular space Musielak [27] obtained some approximation theorems in modular spaces with respect to the usual moduli of smoothness. Since $L^{p(x)}$ is not translation invariant using Butzer – Wehrens type moduli of smoothness (see [7, 13]) Israfilov et all. [17] obtained direct and converse trigonometric approximation theorems in $L^{p(x)}$.

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In the present paper we investigate the approximation properties of the trigonometric system in $L_{2\pi}^{p(\cdot)}$. We consider the fractional order moduli of smoothness and obtain direct, converse approximation theorems together with a constructive characterization of a Lipschitz-type class.

Let $\boldsymbol{T} := [-\pi, \pi]$ and $\mathcal{P}$ be the class of 2π-periodic, Lebesgue measurable functions $p = p(x)\colon\ \boldsymbol{T} \to (1, \infty)$ such that $p^* < \infty$. We define class $L_{2\pi}^{p(\cdot)} := L_{2\pi}^{p(\cdot)}(\boldsymbol{T})$ of 2π-periodic measurable functions f defined on $\boldsymbol{T}$ satisfying

$$\int\limits_{\boldsymbol{T}} |f(x)|^{p(x)}\, dx < \infty.$$

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space [24] with norms

$$\|f(x)\|_{p,\pi} := \|f\,(x)\|_{p,\pi,\boldsymbol{T}} := \inf\left\{\alpha > 0 : \int\limits_{\boldsymbol{T}} \left|\frac{f(x)}{\alpha}\right|^{p(x)} |dx| \le 1\right\}$$

and

$$\|f(x)\|_{p,\pi}^* := \sup\left\{\int\limits_{T} |f(x)g(x)|\, dx : g \in L_{2\pi}^{p'(\cdot)},\ \int\limits_{T} |g(x)|^{p'(x)}\, dx \le 1\right\}$$

having the property[1]

$$\|f\|_{p,\pi} \asymp \|f\|_{p,\pi}^*\,, \tag{1}$$

where $p'(x) := p(x)/\left(p\,(x) - 1\right)$ is the conjugate exponent of $p(x)$.

The variable exponent $p(x)$ which is defined on $\boldsymbol{T}$ is said to be satisfy *Dini–Lipschitz property* DL_γ *of order* γ on $\boldsymbol{T}$ if

$$\sup_{x_1, x_2 \in \boldsymbol{T}} \Big\{|p\,(x_1) - p\,(x_2)| :\ |x_1 - x_2| \le \delta\Big\} \left(\ln\frac{1}{\delta}\right)^{\gamma} \le c, \quad 0 < \delta < 1.$$

Let $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ satisfy DL_1, $0 < h \le 1$ and let

$$\sigma_h f(x) := \frac{1}{h} \int\limits_{x-h/2}^{x+h/2} f(t)dt, \quad x \in \boldsymbol{T},$$

be Steklov's mean operator. In this case the operator σ_h is bounded [37] in $L_{2\pi}^{p(\cdot)}$. Using these facts and setting $x,\, t \in \boldsymbol{T}$, $0 \le \alpha < 1$ we define

$$\sigma_h^\alpha f(x) := \left(I - \sigma_h\right)^\alpha f(x) =$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{h^k} \int\limits_{-h/2}^{h/2} \ldots \int\limits_{-h/2}^{h/2} f\left(x + u_1 + \ldots + u_k\right) du_1 \ldots du_k, \tag{2}$$

[1] $X \asymp Y$ means that there exist constants $C,\, c > 0$ such that $cY \le X \le CY$ hold. Throughout this work by $c,\, C,\, c_1,\, c_2,\, \ldots,$ we denote the constants which are different in different places. $X_n = \mathcal{O}\left(Y_n\right)$, $n = 1, 2, \ldots,$ means that there exists a constant $C > 0$ such that $X_n \le CY_n$ holds for $n = 1, 2, \ldots.$

where $f \in L_{2\pi}^{p(\cdot)}$, $\binom{\alpha}{k} := \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ for $k > 1$, $\binom{\alpha}{1} := \alpha$, $\binom{\alpha}{0} := 1$ and I is the identity operator.

Since the Binomial coefficients $\binom{\alpha}{k}$ satisfy [34, p. 14]

$$\left|\binom{\alpha}{k}\right| \leq \frac{c(\alpha)}{k^{\alpha+1}}, \quad k \in \mathbb{Z}^+,$$

we get

$$C(\alpha) := \sum_{k=0}^{\infty} \left|\binom{\alpha}{k}\right| < \infty$$

and therefore

$$\|\sigma_h^\alpha f\|_{p,\pi} \leq c\,\|f\|_{p,\pi} < \infty \tag{3}$$

provided $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ satisfy DL_1 and $0 < h \leq 1$.

For $0 \leq \alpha < 1$ and $r = 1, 2, 3, \ldots$ we define the *fractional modulus of smoothness of index* $r + \alpha$ for $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$, satisfy DL_1 and $0 < h \leq 1$ as

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} := \sup_{0 \leq h_i, h \leq \delta} \left\| \prod_{i=1}^{r} (I - \sigma_{h_i})\, \sigma_h^\alpha f \right\|_{p,\pi}$$

and

$$\Omega_\alpha(f,\delta)_{p(\cdot)} := \sup_{0 \leq h \leq \delta} \|\sigma_h^\alpha f\|_{p,\pi}.$$

We have by (3) that

$$\Omega_{r+\alpha}(f,\delta)_{p(\cdot)} \leq c\,\|f\|_{p,\pi}$$

where $f \in L_{2\pi}^{p(\cdot)}$, $p \in \mathcal{P}$ satisfy DL_1, $0 < h \leq 1$ and the constant $c > 0$ dependent only on α, r and p.

***Remark* 1.** The modulus of smoothness $\Omega_\alpha(f,\delta)_{p(\cdot)}$, $\alpha \in \mathbb{R}^+$, has the following properties for $p \in \mathcal{P}$ satisfying DL_1: (i) $\Omega_\alpha(f,\delta)_{p(\cdot)}$ is non-negative and non-decreasing function of $\delta \geq 0$, (ii) $\Omega_\alpha(f_1 + f_2, \cdot)_{p(\cdot)} \leq \Omega_\alpha(f_1, \cdot)_{p(\cdot)} + \Omega_\alpha(f_2, \cdot)_{p(\cdot)}$, (iii) $\lim\limits_{\delta \to 0} \Omega_\alpha(f,\delta)_{p(\cdot)} = 0$.

Let

$$E_n(f)_{p(\cdot)} := \inf_{T \in \mathcal{T}_n} \|f - T\|_{p,\pi}, \quad n = 0, 1, 2, \ldots,$$

be the approximation error of function $f \in L_{2\pi}^{p(\cdot)}$ where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n.

For a given $f \in L^1$, assuming

$$\int_{\boldsymbol{T}} f(x)dx = 0, \tag{4}$$

we define *α-th fractional* ($\alpha \in \mathbb{R}^+$) *integral* of f as [42, v. 2, p. 134]

$$I_\alpha(x,f) := \sum_{k \in \mathbb{Z}^*} c_k\,(ik)^{-\alpha} e^{ikx},$$

where $c_k := \int_{\boldsymbol{T}} f(x) e^{-ikx} dx$ for $k \in \mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$ and

$$(ik)^{-\alpha} := |k|^{-\alpha}\, e^{(-1/2)\pi i\alpha \operatorname{sign} k}$$

as principal value.

Let $\alpha \in \mathbb{R}^+$ be given. We define *fractional derivative* of a function $f \in L^1$, satisfying (4), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{1+[\alpha]-\alpha}(x, f)$$

provided the right-hand side exists, where $[x]$ denotes the integer part of a real number x.

Let $W^{\alpha}_{p(\cdot)}$, $p \in \mathcal{P}$, $\alpha > 0$, be the class of functions $f \in L^{p(\cdot)}_{2\pi}$ such that $f^{(\alpha)} \in L^{p(\cdot)}_{2\pi}$. $W^{\alpha}_{p(\cdot)}$ becomes a Banach space with the norm

$$\|f\|_{W^{\alpha}_{p(\cdot)}} := \|f\|_{p,\pi} + \left\|f^{(\alpha)}\right\|_{p,\pi}.$$

Main results of this work are following.

Theorem 1. *Let $f \in W^{\alpha}_{p(\cdot)}$, $\alpha \in \mathbb{R}^+$, and $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$, then for every natural n there exists a constant $c > 0$ independent of n such that*

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} E_n(f^{(\alpha)})_{p(\cdot)}$$

holds.

***Corollary* 1.** *Under the conditions of Theorem* 1

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} \left\|f^{(\alpha)}\right\|_{p,\pi}$$

with a constant $c > 0$ independent of $n = 0, 1, 2, 3, \ldots$.

Theorem 2. *If $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$, then there exists a constant $c > 0$ dependent only on α and p such that for $n = 0, 1, 2, 3, \ldots$*

$$E_n(f)_{p(\cdot)} \leq c\, \Omega_\alpha\left(f, \frac{2\pi}{n+1}\right)_{p(\cdot)}$$

holds.

The following converse theorem of trigonometric approximation holds.

Theorem 3. *If $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$, then for $n = 0, 1, 2, 3, \ldots$*

$$\Omega_\alpha\left(f, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_{p(\cdot)}$$

hold where the constant $c > 0$ dependent only on α and p.

***Corollary* 2.** *Let $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$ and $f \in L^{p(\cdot)}_{2\pi}$. If*

$$E_n(f)_{p(\cdot)} = \mathcal{O}\left(n^{-\sigma}\right), \qquad \sigma > 0, \quad n = 1, 2, \ldots,$$

then

$$\Omega_\alpha\left(f, \delta\right)_{p(\cdot)} = \begin{cases} \mathcal{O}(\delta^\sigma), & \alpha > \sigma, \\ \mathcal{O}\left(\delta^\sigma \left|\log\left(1/\delta\right)\right|\right), & \alpha = \sigma, \\ \mathcal{O}(\delta^\alpha), & \alpha < \sigma, \end{cases}$$

hold.

Definition 1. *For $0<\sigma<\alpha$ we set*

$$\operatorname{Lip}\sigma\left(\alpha,p(\cdot)\right):=\Big\{f\in L_{2\pi}^{p(\cdot)}:\Omega_{\alpha}\left(f,\delta\right)_{p(\cdot)}=\mathcal{O}\left(\delta^{\sigma}\right),\ \delta>0\Big\}.$$

***Corollary* 3.** *Let $0<\sigma<\alpha$, $p\in\mathcal{P}$ satisfy DL_{γ} with $\gamma\geq 1$ and $f\in L_{2\pi}^{p(\cdot)}$ be fulfilled. Then the following conditions are equivalent*:

(a) $f\in\operatorname{Lip}\sigma\left(\alpha,p(\cdot)\right)$,

(b) $E_{n}(f)_{p(\cdot)}=\mathcal{O}\left(n^{-\sigma}\right)$, $n=1,2,\ldots$.

Theorem 4. *Let $p\in\mathcal{P}$ satisfy DL_{γ} with $\gamma\geq 1$ and $f\in L_{2\pi}^{p(\cdot)}$. If $\beta\in(0,\infty)$ and*

$$\sum_{\nu=1}^{\infty}\nu^{\beta-1}E_{\nu}(f)_{p,\pi}<\infty$$

then $f\in W_{p(\cdot)}^{\beta}$ and

$$E_{n}(f^{(\beta)})_{p(\cdot)}\leq c\left((n+1)^{\beta}E_{n}(f)_{p(\cdot)}+\sum_{\nu=n+1}^{\infty}\nu^{\beta-1}E_{\nu}\left(f\right)_{p(\cdot)}\right)$$

hold where the constant $c>0$ dependent only on β and p.

***Corollary* 4.** *Let $p\in\mathcal{P}$ satisfy DL_{γ} with $\gamma\geq 1$, $f\in L_{2\pi}^{p(\cdot)}$, $\beta\in(0,\infty)$ and*

$$\sum_{\nu=1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{p(\cdot)}<\infty$$

for some $\alpha>0$. In this case for $n=0,1,2,\ldots$ there exists a constant $c>0$ dependent only on α, β and p such that

$$\Omega_{\beta}\left(f^{(\alpha)},\frac{\pi}{n+1}\right)_{p(\cdot)}\leq\frac{c}{(n+1)^{\beta}}\sum_{\nu=0}^{n}(\nu+1)^{\alpha+\beta-1}E_{\nu}(f)_{p(\cdot)}+c\sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_{\nu}(f)_{p(\cdot)}$$

hold.

The following simultaneous approximation theorem holds.

Theorem 5. *Let $\beta\in[0,\infty)$, $p\in\mathcal{P}$ satisfy DL_{γ} with $\gamma\geq 1$ and $f\in L_{2\pi}^{p(\cdot)}$. Then there exist a $T\in\mathcal{T}_{n}$ and a constant $c>0$ depending only on α and p such that*

$$\left\|f^{(\beta)}-T^{(\beta)}\right\|_{p,\pi}\leq cE_{n}\left(f^{(\beta)}\right)_{p(\cdot)}$$

holds.

Definition 2 (Hardy space of variable exponent $H^{p(\cdot)}$ on the unit disc $\mathbb{D}$ with the boundary $\mathbb{T}:=\partial\mathbb{D}$) [21]**.** *Let $p(z)\colon\mathbb{T}\to(1,\infty)$, be measurable function. We say that a complex valued analytic function Φ in $\mathbb{D}$ belongs to the Hardy space $H^{p(\cdot)}$ if*

$$\sup_{0<r<1}\int_{0}^{2\pi}\left|\Phi\left(re^{i\vartheta}\right)\right|^{p(\vartheta)}d\vartheta<+\infty$$

where $p(\vartheta):=p\left(e^{i\vartheta}\right)$, $\vartheta\in[0,2\pi]$ (and therefore $p(\vartheta)$ is 2π-periodic function). Let $\underline{p}:=\inf_{z\in\mathbb{T}}p(z)$ and $\overline{p}:=\sup_{z\in\mathbb{T}}p(z)$. If $\underline{p}>0$, then it is obvious that $H^{\overline{p}}\subset H^{p(\cdot)}\subset\subset H^{\underline{p}}$. Therefore if $f\in H^{p(\cdot)}$ and $\underline{p}>0$, then there exist nontangential boundary-values

$f\left(e^{i\theta}\right)$ a.e. on $\mathbb{T}$ and $f\left(e^{i\theta}\right) \in L_{2\pi}^{p(\cdot)}(\mathbb{T})$. Under the conditions $1 < \underline{p}$ and $\overline{p} < \infty$, $H^{p(\cdot)}$ becomes a Banach space with the norm

$$\|f\|_{H^{p(\cdot)}} := \left\|f\left(e^{i\theta}\right)\right\|_{p,\pi,\mathbb{T}} = \inf\left\{\lambda > 0\colon \int\limits_{\mathbb{T}} \left|\frac{f\left(e^{i\theta}\right)}{\lambda}\right|^{p(\theta)} d\theta \le 1\right\}.$$

Theorem 6. *If $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \ge 1$, f belongs to Hardy space $H^{p(\cdot)}$ on $\mathbb{D}$ and $r \in \mathbb{R}^+$, then there exists a constant $c > 0$ independent of n such that*

$$\left\|f(z) - \sum_{k=0}^{n} a_k(f) z^k\right\|_{H^{p(\cdot)}} \le c\,\Omega_r\left(f\left(e^{i\theta}\right), \frac{1}{n+1}\right)_{p(\cdot)}, \quad n = 0, 1, 2, \ldots,$$

where $a_k(f)$, $k = 0, 1, 2, 3, \ldots$, are the Taylor coefficients of f at the origin.

2. Some auxiliary results. We begin with the following lemma.

Lemma A [20]. *For $r \in \mathbb{R}^+$ we suppose that*

(i) $a_1 + a_2 + \ldots + a_n + \ldots,$

(ii) $a_1 + 2^r a_2 + \ldots + n^r a_n + \ldots$

be two series in a Banach space $(B, \|\cdot\|)$. Let

$$R_n^{\langle r\rangle} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) a_k$$

and

$$R_n^{\langle r\rangle *} := \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1}\right)^r\right) k^r a_k$$

for $n = 1, 2, \ldots$. Then

$$\left\|R_n^{\langle r\rangle *}\right\| \le c, \quad n = 1, 2, \ldots,$$

for some $c > 0$ if and only if there exists a $R \in B$ such that

$$\left\|R_n^{\langle r\rangle} - R\right\| \le \frac{C}{n^r},$$

where c and C are constants depending only on one another.

Lemma B [38]. *If $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \ge 1$ and $f \in L_{2\pi}^{p(\cdot)}$ then there are constants c, $C > 0$ such that*

$$\left\|\tilde{f}\right\|_{p,\pi} \le c\,\|f\|_{p,\pi} \tag{5}$$

and

$$\left\|S_n(\cdot, f)\right\|_{p,\pi} \le C\,\|f\|_{p,\pi} \tag{6}$$

hold for $n = 1, 2, \ldots$.

***Remark* 2.** Under the conditions of Lemma B

(i) It can be easily seen from (5) and (6) that there exists constant $c > 0$ such that

$$\left\|f - S_n(\cdot, f)\right\|_{p,\pi} \le cE_n(f)_{p(\cdot)} \asymp E_n\left(\tilde{f}\right)_{p(\cdot)}.$$

(ii) From generalized Hölder inequality [24] (Theorem 2.1) we have

$$L_{2\pi}^{p(\cdot)} \subset L^1.$$

For a given $f \in L^1$ let

$$f(x) \backsim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{7}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

be the *Fourier* and the *conjugate Fourier series* of f, respectively. Putting $A_k(x) :=$ $:= c_k e^{ikx}$ in (7) we define

$$S_n(f) := S_n(x, f) := \sum_{k=0}^{n} (A_k(x) + A_{-k}(x)) =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 0, 1, 2, \ldots,$$

$$R_n^{\langle \alpha \rangle}(f, x) := \sum_{k=0}^{n} \left(1 - \left(\frac{k}{n+1}\right)^{\alpha}\right) (A_k(x) + A_{-k}(x))$$

and

$$\Theta_m^{\langle r \rangle} := \frac{1}{1 - \left(\dfrac{m+1}{2m+1}\right)^r} R_{2m}^{\langle r \rangle} - \frac{1}{\left(\dfrac{2m+1}{m+1}\right)^r - 1} R_m^{\langle r \rangle}, \quad \text{for} \quad m = 1, 2, 3, \ldots. \tag{8}$$

Under the conditions of Lemma B using (6) and Abel's transformation we get

$$\left\| R_n^{\langle \alpha \rangle}(f, x) \right\|_{p,\pi} \le c \, \|f\|_{p,\pi}, \quad n = 1, 2, 3, \ldots, \quad x \in \boldsymbol{T}, \quad f \in L_{2\pi}^{p(\cdot)}, \tag{9}$$

and therefore from (8) and (9)

$$\left\| \Theta_m^{\langle r \rangle}(f, x) \right\|_{p,\pi} \le c \, \|f\|_{p,\pi}, \quad m = 1, 2, 3, \ldots, \quad x \in \boldsymbol{T}, \quad f \in L_{2\pi}^{p(\cdot)}.$$

From the property [25] ((16))

$$\Theta_m^{\langle r \rangle}(f)(x) =$$

$$= \frac{1}{\sum_{k=m+1}^{2m} \left[(k+1)^r - k^r\right]} \sum_{k=m+1}^{2m} \left[(k+1)^r - k^r\right] S_k(x, f), \quad x \in \boldsymbol{T}, \quad f \in L^1,$$

it is known [25] ((18)) that

$$\Theta_m^{\langle r \rangle}(T_m) = T_m \tag{10}$$

for $T_m \in \mathcal{T}_m$, $m = 1, 2, 3, \ldots$.

Lemma 1. *Let $T_n \in \mathcal{T}_n$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$ and $r \in \mathbb{R}^+$. Then there exists a constant $c > 0$ independent of n such that*

$$\left\| T_n^{(r)} \right\|_{p,\pi} \leq c n^r \left\| T_n \right\|_{p,\pi}$$

holds.

Proof. Without loss of generality one can assume that $\|T_n\|_{p,\pi} = 1$. Since

$$T_n = \sum_{k=0}^{n} \left(A_k(x) + A_{-k}(x) \right)$$

we get

$$\frac{\tilde{T}_n}{n^r} = \sum_{k=1}^{n} \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big]$$

and

$$\frac{T_n^{(r)}}{n^r} = i^r \sum_{k=1}^{n} k^r \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big].$$

In this case we have by (9) and (5) that

$$\left\| R_n^{\langle r \rangle} \left(\frac{\tilde{T}_n}{n^r} \right) \right\|_{p,\pi} \leq \frac{c}{n^r} \|\tilde{T}_n\|_{p,\pi} \leq \frac{c}{n^r} \|T_n\|_{p,\pi} = \frac{c}{n^r}$$

and hence applying Lemma A (with $R = 0$) to the series

$$\sum_{k=1}^{n} \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big] + 0 + 0 + \ldots + 0 + \ldots,$$

$$\sum_{k=1}^{n} k^r \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big] + 0 + 0 + \ldots + 0 + \ldots,$$

we find

$$\left\| \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^r \right) k^r \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big] \right\|_{p,\pi} \leq c,$$

namely,

$$\left\| R_n^{\langle r \rangle} \left(\frac{T_n^{(r)}}{n^r} \right) \right\|_{p,\pi} = \left\| i^r \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^r \right) k^r \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big] \right\|_{p,\pi} =$$

$$= \left\| \sum_{k=1}^{n} \left(1 - \left(\frac{k}{n+1} \right)^r \right) k^r \Big[\left(A_k(x) - A_{-k}(x) \right) / n^r \Big] \right\|_{p,\pi} \leq c_*.$$

Since $R_n^{\langle r \rangle}(cf) = c R_n^{\langle r \rangle}(f)$ for every real c we obtain from (10) and the last inequality that

$$\left\| T_n^{(r)} \right\|_{p,\pi} = \left\| \Theta_n^{\langle r \rangle} \left(T_n^{(r)} \right) \right\|_{p,\pi} = n^r \left\| \frac{1}{n^r} \Theta_n^{\langle r \rangle} \left(T_n^{(r)} \right) \right\|_{p,\pi} =$$

$$= n^r \left\| \Theta_n^{\langle r \rangle} \left(\frac{T_n^{(r)}}{n^r} \right) \right\|_{p,\pi} \leq c_* n^r = c_* n^r \left\| T_n \right\|_{p,\pi}.$$

General case follows immediately from this.

Lemma 2. *If $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \geq 1$, $f \in W^2_{p(\cdot)}$ and $r = 1, 2, 3, \ldots$, then*

$$\Omega_r \left(f, \delta \right)_{p(\cdot)} \leq c\delta^2 \Omega_{r-1} \left(f'', \delta \right)_{p(\cdot)}, \quad \delta \geq 0,$$

with some constant $c > 0$.

Proof. Putting

$$g(x) := \prod_{i=2}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

we have

$$\left(I - \sigma_{h_1} \right) g(x) = \prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

and

$$\prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f(x) = \frac{1}{h_1} \int\limits_{-h_1/2}^{h_1/2} \left(g(x) - g\left(x + t \right) \right) dt =$$

$$= -\frac{1}{2h_1} \int\limits_0^{h_1/2} \int\limits_0^{2t} \int\limits_{-u/2}^{u/2} g''\left(x + s \right) ds du dt.$$

Therefore from (1)

$$\left\| \prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f\left(x \right) \right\|_{p,\pi} \leq$$

$$\leq \frac{c}{2h_1} \sup \left\{ \int\limits_T \left| \int\limits_0^{h_1/2} \int\limits_0^{2t} \int\limits_{-u/2}^{u/2} g''\left(x + s \right) ds du dt \right| \left| g_0\left(x \right) \right| dx : \right.$$

$$\left. g_0 \in L_{2\pi}^{p'(\cdot)} \text{ and } \int\limits_T \left| g_0(x) \right|^{p'(x)} dx \leq 1 \right\} \leq$$

$$\leq \frac{c}{2h_1} \int\limits_0^{h_1/2} \int\limits_0^{2t} u \left\| \frac{1}{u} \int\limits_{-u/2}^{u/2} g''\left(x + s \right) ds \right\|_{p,\pi} du dt \leq$$

$$\leq \frac{c}{2h_1} \int\limits_0^{h_1/2} \int\limits_0^{2t} u \left\| g'' \right\|_{p,\pi} du dt = c h_1^2 \left\| g'' \right\|_{p,\pi}.$$

Since

$$g''(x) = \prod_{i=2}^{r} (I - \sigma_{h_i}) f''(x),$$

we obtain that

$$\Omega_r (f, \delta)_{p(\cdot)} \le \sup_{\substack{0<h_i\le\delta \\ i=1,2,\ldots,r}} ch_1^2 \|g''\|_{p,\pi} = c\delta^2 \sup_{\substack{0<h_i\le\delta \\ i=2,\ldots,r}} \left\| \prod_{i=2}^{r} (I - \sigma_{h_i}) f''(x) \right\|_{p,\pi} =$$

$$= c\delta^2 \sup_{\substack{0<h_j\le\delta \\ j=2,\ldots,r-1}} \left\| \prod_{j=1}^{r-1} \left(I - \sigma_{h_j}\right) f''(x) \right\|_{p,\pi} = c\delta^2 \Omega_{r-1} (f'', \delta)_{p(\cdot)} .$$

Lemma 2 is proved.

***Corollary* 5.** *If $r = 1, 2, 3, \ldots$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \ge 1$ and $f \in W^{2r}_{p(\cdot)}$, then*

$$\Omega_r (f, \delta)_{p(\cdot)} \le c\delta^{2r} \left\| f^{(2r)} \right\|_{p,\pi}, \quad \delta \ge 0,$$

with some constant $c > 0$.

Lemma 3. *Let $\alpha \in \mathbb{R}^+$, $p \in \mathcal{P}$ satisfy DL_γ with $\gamma \ge 1$, $n = 0, 1, 2, \ldots$ and $T_n \in \mathcal{T}_n$. Then*

$$\Omega_\alpha \left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \le \frac{c}{(n+1)^\alpha} \left\| T_n^{(\alpha)} \right\|_{p,\pi}$$

hold where the constant $c > 0$ dependent only on α and p.

Proof. Firstly we prove that if $0 < \alpha < \beta$, $\alpha, \beta \in \mathbb{R}^+$ then

$$\Omega_\beta (f, \cdot)_{p(\cdot)} \le c\, \Omega_\alpha (f, \cdot)_{p(\cdot)} . \tag{11}$$

It is easily seen that if $\alpha \le \beta$, α, $\beta \in \mathbb{Z}^+$, then

$$\Omega_\beta (f, \cdot)_{p(\cdot)} \le c (\alpha, \beta, p)\, \Omega_\alpha (f, \cdot)_{p(\cdot)} . \tag{12}$$

Now, we assume that $0 < \alpha < \beta < 1$. In this case putting $\Phi(x) := \sigma_h^\alpha f(x)$ we have

$$\sigma_h^{\beta-\alpha} \Phi(x) = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} \Phi (x + u_1 + \ldots u_j)\, du_1 \ldots du_j =$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{\beta-\alpha}{j} \frac{1}{h^j} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} \left[\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \frac{1}{h^k} \int_{-h/2}^{h/2} \ldots \right.$$

$$\left. \ldots \int_{-h/2}^{h/2} f (x + u_1 + \ldots u_j + u_{j+1} + \ldots u_{j+k})\, du_1 \ldots du_j du_{j+1} \ldots du_{j+k} \right] =$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{\beta-\alpha}{j} \binom{\alpha}{k} \times$$

$$\times \left[\frac{1}{h^{j+k}} \int_{-h/2}^{h/2} \ldots \int_{-h/2}^{h/2} f(x + u_1 + \ldots u_{j+k}) du_1 \ldots du_{j+k} \right] =$$

$$= \sum_{\upsilon=0}^{\infty} (-1)^{\upsilon} \binom{\beta}{\upsilon} \frac{1}{h^{\upsilon}} \int\limits_{-h/2}^{h/2} \ldots \int\limits_{-h/2}^{h/2} f\left(x + u_1 + \ldots u_{\upsilon}\right) du_1 \ldots du_{\upsilon} = \sigma_h^{\beta} f\left(x\right) \quad \text{a.e.}$$

Then

$$\left\|\sigma_h^{\beta} f(x)\right\|_{p,\pi} = \left\|\sigma_h^{\beta-\alpha} \Phi(x)\right\|_{p,\pi} \leq c \left\|\sigma_h^{\alpha} f(x)\right\|_{p,\pi}$$

and

$$\Omega_{\beta}\left(f,\cdot\right)_{p(\cdot)} \leq c\,\Omega_{\alpha}\left(f,\cdot\right)_{p(\cdot)}. \tag{13}$$

We note that if $r_1, r_2 \in \mathbb{Z}^+$, $\alpha_1, \beta_1 \in (0,1)$ taking $\alpha := r_1 + \alpha_1$, $\beta := r_2 + \beta_1$ for the remaining cases $r_1 = r_2$, $\alpha_1 < \beta_1$ or $r_1 < r_2$, $\alpha_1 = \beta_1$ or $r_1 < r_2$, $\alpha_1 < \beta_1$ it can easily be obtained from (12) and (13) that the required inequality (11) holds.

Using (11), Corollary 5 and Lemma 1 we get

$$\Omega_{\alpha}\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq c\,\Omega_{[\alpha]}\left(T_n, \frac{\pi}{n+1}\right)_{p(\cdot)} \leq c\left(\frac{\pi}{n+1}\right)^{2[\alpha]} \left\|T_n^{(2[\alpha])}\right\|_{p,\pi} \leq$$

$$\leq \frac{c}{(n+1)^{2[\alpha]}} (n+1)^{[\alpha]-(\alpha-[\alpha])} \left\|T_n^{(\alpha)}\right\|_{p,\pi} = \frac{c}{(n+1)^{\alpha}} \left\|T_n^{(\alpha)}\right\|_{p,\pi}$$

the required result.

Definition 3. *For $p \in \mathcal{P}$, $f \in L_{2\pi}^{p(\cdot)}$, $\delta > 0$ and $r = 1, 2, 3, \ldots$ the Peetre K-functional is defined as*

$$K\left(\delta, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^r\right) := \inf_{g \in W_{p(\cdot)}^r} \left\{ \|f - g\|_{p,\pi} + \delta \left\|g^{(r)}\right\|_{p,\pi} \right\}. \tag{14}$$

Theorem 7. *If $p \in \mathcal{P}$ satisfy DL_{γ} with $\gamma \geq 1$ and $f \in L_{2\pi}^{p(\cdot)}$, then the K-functional $K\left(\delta^{2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right)$ in* (14) *and the modulus $\Omega_r\left(f,\delta\right)_{p(\cdot)}$, $r = 1, 2, 3, \ldots$ are equivalent.*

Proof. If $h \in W_{p(\cdot)}^{2r}$, then we have by Corollary 5 and (14) that

$$\Omega_r\left(f,\delta\right)_{p(\cdot)} \leq c\,\|f - h\|_{p,\pi} + c\delta^{2r} \left\|h^{(2r)}\right\|_{p,\pi} \leq cK\left(\delta^{2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right).$$

We estimate the reverse of the last inequality. The operator L_{δ} defined by

$$\left(L_{\delta} f\right)(x) := 3\delta^{-3} \int\limits_0^{\delta/2} \int\limits_0^{2t} \int\limits_{-u/2}^{u/2} f\left(x+s\right) ds\, du\, dt, \quad x \in \boldsymbol{T},$$

is bounded in $L_{2\pi}^{p(\cdot)}$ because

$$\|L_{\delta} f\|_{p,\pi} \leq 3\delta^{-3} \int\limits_0^{\delta/2} \int\limits_0^{2t} u \left\|\sigma_u f\right\|_{p,\pi} du\, dt \leq c\,\|f\|_{p,\pi}.$$

We prove

$$\frac{d^2}{dx^2} L_{\delta} f = \frac{c}{\delta^2}\left(I - \sigma_{\delta}\right) f$$

with a real constant c. Since

$$\left(L_\delta f\right)(x)=3\delta^{-3}\int\limits_0^{\delta/2}\int\limits_0^{2t}\int\limits_{-u/2}^{u/2} f\left(x+s\right)ds\,du\,dt=$$

$$=3\delta^{-3}\int\limits_0^{\delta/2}\int\limits_0^{2t}\left[\int\limits_0^{x+u/2} f\left(s\right)ds-\int\limits_0^{x-u/2} f\left(s\right)ds\right]du\,dt$$

using Lebesgue Differentiation Theorem

$$\frac{d}{dx}\left(L_\delta f\right)(x)=3\delta^{-3}\int\limits_0^{\delta/2}\int\limits_0^{2t}\left[\frac{d}{dx}\int\limits_0^{x+u/2} f\left(s\right)ds-\frac{d}{dx}\int\limits_0^{x-u/2} f\left(s\right)ds\right]du\,dt=$$

$$=3\delta^{-3}\int\limits_0^{\delta/2}\int\limits_0^{2t}\Big[f\left(x+u/2\right)-f\left(x-u/2\right)\Big]du\,dt=$$

$$=6\delta^{-3}\int\limits_0^{\delta/2}\left[\int\limits_x^{x+t} f(u)du+\int\limits_x^{x-t} f(u)du\right]dt\quad \text{a.e.}$$

Using Lebesgue Differentiation Theorem once more

$$\frac{d^2}{dx^2}\left(L_\delta f\right)(x)=6\delta^{-3}\int\limits_0^{\delta/2}\left[\frac{d}{dx}\int\limits_x^{x+t} f(u)du+\frac{d}{dx}\int\limits_0^{x-t} f(u)du\right]dt=$$

$$=6\delta^{-3}\int\limits_0^{\delta/2}\left[f\left(x+t\right)-f\left(x\right)+f\left(x-t\right)-f(x)\right]dt=$$

$$=\frac{6}{\delta^3}\left[\int\limits_0^{\delta/2} f\left(x+t\right)dt+\int\limits_0^{\delta/2} f\left(x-t\right)dt-\delta f(x)\right]=$$

$$=\frac{6}{\delta^2}\left[\frac{1}{\delta}\int\limits_0^{\delta/2} f\left(x+t\right)dt+\frac{1}{\delta}\int\limits_{-\delta/2}^{0} f\left(x+t\right)dt-f(x)\right]=$$

$$=\frac{6}{\delta^2}\left[\frac{1}{\delta}\int\limits_{-\delta/2}^{\delta/2} f\left(x+t\right)dt-f(x)\right]=$$

$$=\frac{-6}{\delta^2}\left[f(x)-\frac{1}{\delta}\int\limits_{-\delta/2}^{\delta/2} f\left(x+t\right)dt\right]=\frac{-6}{\delta^2}\left(I-\sigma_\delta\right)f(x)\quad \text{a.e.}$$

The last equality implies by induction on r that

$$\frac{d^{2r}}{dx^{2r}}L_\delta^r f=\frac{c}{\delta^{2r}}\left(I-\sigma_\delta\right)^r f,\quad r=1,2,3,\ldots \text{ a.e.}$$

Indeed, for $r=2$

$$\frac{d^4}{dx^4}L_\delta^2 f=\frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta^2 f\right)=\frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta\left(L_\delta f=:u\right)\right)=$$

$$=\frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}L_\delta u\right)=\frac{d^2}{dx^2}\left(\frac{-6}{\delta^2}\left(I-\sigma_\delta\right)u\right)=$$

$$=\frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}\left(I-\sigma_\delta\right)u\right)=\frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}\left(I-\sigma_\delta\right)L_\delta f\right)\quad \text{a.e.}$$

Since $\frac{d^2}{dx^2}\sigma_\delta\left(L_\delta f\right)=\sigma_\delta\left(\frac{d^2}{dx^2}L_\delta f\right)$ we get

$$\frac{d^2}{dx^2}\left(I-\sigma_\delta\right)L_\delta f=\frac{d^2}{dx^2}L_\delta f-\frac{d^2}{dx^2}\sigma_\delta\left(L_\delta f\right)=$$

$$=\frac{d^2}{dx^2}L_\delta f-\sigma_\delta\left(\frac{d^2}{dx^2}L_\delta f\right)=\left(I-\sigma_\delta\right)\left[\frac{d^2}{dx^2}L_\delta f\right]\quad \text{a.e.}$$

and therefore

$$\frac{d^4}{dx^4}L_\delta^2 f=\frac{-6}{\delta^2}\left(\frac{d^2}{dx^2}\left(I-\sigma_\delta\right)L_\delta f\right)=\frac{-6}{\delta^2}\left(I-\sigma_\delta\right)\left[\frac{d^2}{dx^2}L_\delta f\right]=$$

$$=\frac{-6}{\delta^2}\left(I-\sigma_\delta\right)\left[\frac{-6}{\delta^2}\left(I-\sigma_\delta\right)f\right]=\frac{c}{\delta^4}\left(I-\sigma_\delta\right)^2 f\quad \text{a.e.}$$

Now let be $\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}f=\frac{c}{\delta^{2(r-1)}}\left(I-\sigma_\delta\right)^{(r-1)}f$ a.e. Then

$$\frac{d^{2r}}{dx^{2r}}L_\delta^r f=\frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}\left(L_\delta f:=u\right)\right]=\frac{d^2}{dx^2}\left[\frac{d^{2(r-1)}}{dx^{2(r-1)}}L_\delta^{(r-1)}u\right]=$$

$$=\frac{d^2}{dx^2}\left[\frac{c}{\delta^{2(r-1)}}\left(I-\sigma_\delta\right)^{(r-1)}u\right]=\frac{d^2}{dx^2}\left[\frac{c}{\delta^{2(r-1)}}\left(I-\sigma_\delta\right)^{(r-1)}L_\delta f\right]=$$

$$=\frac{c}{\delta^{2(r-1)}}\left(I-\sigma_\delta\right)^{(r-1)}\left[\frac{d^2}{dx^2}L_\delta f\right]=\frac{c}{\delta^{2r}}\left(I-\sigma_\delta\right)^r f\quad \text{a.e.}$$

Letting $A_\delta^r:=I-\left(I-L_\delta^r\right)^r$ we prove that $\left\|\frac{d^{2r}}{dx^{2r}}A_\delta^r f\right\|_{p,\pi}\leq c\left\|\frac{d^{2r}}{dx^{2r}}L_\delta^r f\right\|_{p,\pi}$ and $A_\delta^r f\in W_{p(\cdot)}^{2r}$. For $r=1$ we have $A_\delta^1 f:=I-\left(I-L_\delta^1 f\right)^1=L_\delta^1 f$ and $\left\|\frac{d^2}{dx^2}A_\delta^1 f\right\|_{p,\pi}=$ $=\left\|\frac{d^2}{dx^2}L_\delta^1 f\right\|_{p,\pi}$. Since $\frac{d^2}{dx^2}L_\delta f=\frac{c}{\delta^2}\left(I-\sigma_\delta\right)f$ we get $A_\delta^1 f\in W_{p(\cdot)}^2$. For $r=$ $=2,3,\ldots$ using

$$A_\delta^r:=I-\left(I-L_\delta^r\right)^r=\sum_{j=0}^{r-1}(-1)^{r-j+1}\binom{r}{j}L_\delta^{r(r-j)}$$

we obtain

$$\left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{p,\pi} \le \sum_{j=0}^{r-1} \binom{r}{j} \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f \right\|_{p,\pi}.$$

We estimate $\left\| \frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f \right\|_{p,\pi}$ as the following

$$\left\| \frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f \right\|_{p,\pi} = \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r \left(L_\delta^{(r-j)} f =: u \right) \right\|_{p,\pi} =$$

$$= \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r u \right\|_{p,\pi} = \left\| \frac{c}{\delta^{2r}} \left(I - \sigma_\delta \right)^r u \right\|_{p,\pi} =$$

$$= \left\| \frac{c}{\delta^{2r}} \left(I - \sigma_\delta \right)^r \left[L_\delta^{(r-j)} f \right] \right\|_{p,\pi} = \frac{c}{\delta^{2r}} \left\| \left(I - \sigma_\delta \right)^r \left[L_\delta^{(r-j)} f \right] \right\|_{p,\pi} \le$$

$$\le \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i \left[L_\delta^{(r-j)} f \right] \right\|_{p,\pi}.$$

Since $\sigma_\delta \left(L_\delta f \right) = L_\delta \left(\sigma_\delta f \right)$ we have $\sigma_\delta^i \left[L_\delta^{(r-j)} f \right] = L_\delta^{(r-j)} \left(\sigma_\delta^i f \right)$ and hence

$$\left\| \frac{d^{2r}}{dx^{2r}} L_\delta^{r(r-j)} f \right\|_{p,\pi} \le \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i \left[L_\delta^{(r-j)} f \right] \right\|_{p,\pi} \le$$

$$\le \frac{c}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} L_\delta^{(r-j)} \left(\sigma_\delta^i f \right) \right\|_{p,\pi} =$$

$$= \frac{c}{\delta^{2r}} \left\| L_\delta^{(r-j)} \left[\sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i f \right] \right\|_{p,\pi} \le \frac{C}{\delta^{2r}} \left\| \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sigma_\delta^i f \right\|_{p,\pi} =$$

$$= \frac{C}{\delta^{2r}} \left\| \left(I - \sigma_\delta \right)^r f \right\|_{p,\pi} = \left\| \frac{C}{\delta^{2r}} \left(I - \sigma_\delta \right)^r f \right\|_{p,\pi} = c_1 \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi}.$$

From the last inequality

$$\left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{p,\pi} \le c \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi} \quad \text{and} \quad A_\delta^r f \in W_{p(\cdot)}^{2r}.$$

Therefore we find

$$\left\| \frac{d^{2r}}{dx^{2r}} A_\delta^r f \right\|_{p,\pi} \le c \left\| \frac{d^{2r}}{dx^{2r}} L_\delta^r f \right\|_{p,\pi} = \frac{c}{\delta^{2r}} \left\| \left(I - \sigma_\delta \right)^r \right\|_{p,\pi} \le \frac{c}{\delta^{2r}} \Omega_r (f, \delta)_{p(\cdot)}.$$

Since

$$I - L_\delta^r = \left(I - L_\delta \right) \sum_{j=0}^{r-1} L_\delta^j$$

we get

$$\left\| \left(I - L_\delta^r \right) g \right\|_{p,\pi} \le c \left\| \left(I - L_\delta \right) g \right\|_{p,\pi} \le$$

$$\leq 3c\delta^{-3}\int\limits_{0}^{\delta/2}\int\limits_{0}^{2t} u\left\|(I-\sigma_u)\,g\right\|_{p,\pi} dudt \leq c\sup_{0<u\leq\delta}\left\|(I-\sigma_u)\,g\right\|_{p,\pi}.$$

Taking into account

$$\left\|f-A_{\delta}^{r}f\right\|_{p,\pi}=\left\|(I-L_{\delta}^{r})^{r}f\right\|_{p,\pi}$$

by a recursive procedure we obtain

$$\left\|f-A_{\delta}^{r}f\right\|_{p,\pi}\leq c\sup_{0<t_1\leq\delta}\left\|(I-\sigma_{t_1})\,(I-L_{\delta}^{r})^{r-1}f\right\|_{p,\pi}\leq$$

$$\leq c\sup_{0<t_1\leq\delta}\sup_{0<t_2\leq\delta}\left\|(I-\sigma_{t_1})\,(I-\sigma_{t_2})\,(I-L_{\delta}^{r})^{r-2}f\right\|_{p,\pi}\leq\ldots$$

$$\ldots\leq c\sup_{\substack{0<t_i\leq\delta\\ i=1,2,\ldots,r}}\left\|\prod_{i=1}^{r}(I-\sigma_{t_i})\,f(x)\right\|_{p,\pi}=c\,\Omega_r\,(f,\delta)_{p(\cdot)}.$$

Theorem 7 is proved.

3. Proofs of the main results. ***Proof of Theorem 1.*** We set $A_k(x,f):=a_k\cos kx+$ $+\,b_k\sin kx$. Since the set of trigonometric polynomials is dense [22] in $L_{2\pi}^{p(\cdot)}$ for given $f\in L_{2\pi}^{p(\cdot)}$ we have $E_n(f)_{p(\cdot)}\to 0$ as $n\to\infty$. From the first inequality in Remark 2, we have $f(x)=\sum_{k=0}^{\infty}A_k(x,f)$ in $\|\cdot\|_{p,\pi}$ norm. For $k=1,2,3,\ldots$ we can find

$$A_k(x,f)=a_k\cos k\left(x+\frac{\alpha\pi}{2k}-\frac{\alpha\pi}{2k}\right)+b_k\sin k\left(x+\frac{\alpha\pi}{2k}-\frac{\alpha\pi}{2k}\right)=$$

$$=A_k\Big(x+\frac{\alpha\pi}{2k},f\Big)\cos\frac{\alpha\pi}{2}+A_k\Big(x+\frac{\alpha\pi}{2k},\tilde f\Big)\sin\frac{\alpha\pi}{2}$$

and

$$A_k\Big(x,f^{(\alpha)}\Big)=k^{\alpha}A_k\Big(x+\frac{\alpha\pi}{2k},f\Big).$$

Therefore

$$\sum_{k=0}^{\infty}A_k(x,f)=$$

$$=A_0(x,f)+\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}A_k\Big(x+\frac{\alpha\pi}{2k},f\Big)+\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}A_k\Big(x+\frac{\alpha\pi}{2k},\tilde f\Big)=$$

$$=A_0(x,f)+\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{-\alpha}A_k\Big(x,f^{(\alpha)}\Big)+\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{-\alpha}A_k\Big(x,\tilde f^{(\alpha)}\Big)$$

and hence

$$f(x)-S_n(x,f)=\cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^{\alpha}}A_k\Big(x,f^{(\alpha)}\Big)+\sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^{\alpha}}A_k\Big(x,\tilde f^{(\alpha)}\Big).$$

Since

$$\sum_{k=n+1}^{\infty}k^{-\alpha}A_k\Big(x,f^{(\alpha)}\Big)=$$

$$= \sum_{k=n+1}^{\infty} k^{-\alpha}\left[\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) - \left(S_{k-1}\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)\right] =$$

$$= \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right) -$$

$$-(n+1)^{-\alpha}\left(S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right)$$

and

$$\sum_{k=n+1}^{\infty} k^{-\alpha} A_k\left(x, \tilde{f}^{(\alpha)}\right) = \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left(S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right) -$$

$$-(n+1)^{-\alpha}\left(S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right)$$

we obtain

$$\|f(\cdot) - S_n(\cdot, f)\|_{p,\pi} \le \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left\|S_k\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi} +$$

$$+(n+1)^{-\alpha}\left\|S_n\left(\cdot, f^{(\alpha)}\right) - f^{(\alpha)}(\cdot)\right\|_{p,\pi} +$$

$$+ \sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right)\left\|S_k\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p,\pi} +$$

$$+(n+1)^{-\alpha}\left\|S_n\left(\cdot, \tilde{f}^{(\alpha)}\right) - \tilde{f}^{(\alpha)}(\cdot)\right\|_{p,\pi} \le$$

$$\le c\left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}\right] +$$

$$+c\left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) E_k\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)} + (n+1)^{-\alpha} E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)}\right].$$

Consequently from equivalence in Remark 2 (i) we have

$$\|f(x) - S_n(x, f)\|_{p,\pi} \le$$

$$\le c\left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) + (n+1)^{-\alpha}\right]\left\{E_k\left(f^{(\alpha)}\right)_{p(\cdot)} + E_n\left(\tilde{f}^{(\alpha)}\right)_{p(\cdot)}\right\} \le$$

$$\le cE_n\left(f^{(\alpha)}\right)_{p(\cdot)}\left[\sum_{k=n+1}^{\infty} \left(k^{-\alpha} - (k+1)^{-\alpha}\right) + (n+1)^{-\alpha}\right] \le \frac{c}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{p(\cdot)}.$$

Theorem 1 is proved.

Proof of Theorem 2. We put $r - 1 < \alpha < r$, $r \in \mathbb{Z}^+$. For $g \in W_{p(\cdot)}^{2r}$ we have by Corollary 1, (14) and Theorem 7 that

$$E_n(f)_{p(\cdot)} \le E_n(f-g)_{p(\cdot)} + E_n(g)_{p(\cdot)} \le c\left[\|f-g\|_{p,\pi} + (n+1)^{-2r}\left\|g^{(2r)}\right\|_{p,\pi}\right] \le$$

$$\leq cK\left((n+1)^{-2r}, f; L_{2\pi}^{p(\cdot)}, W_{p(\cdot)}^{2r}\right) \leq c\,\Omega_r\left(f, \frac{1}{n+1}\right)_{p(\cdot)}$$

as required for $r \in \mathbb{Z}^+$. Therefore by the last inequality

$$E_n(f)_{p(\cdot)} \leq c\,\Omega_r\left(f, 1/(n+1)\right)_{p(\cdot)} \leq c\,\Omega_r\left(f, 2\pi/(n+1)\right)_{p(\cdot)}, \quad n = 0, 1, 2, 3, \ldots,$$

and by (11) we get

$$E_n(f)_{p(\cdot)} \leq c\,\Omega_r\left(f, 2\pi/(n+1)\right)_{p(\cdot)} \leq c\,\Omega_\alpha\left(f, 2\pi/(n+1)\right)_{p(\cdot)}$$

and the assertion follows.

***Proof of Theorem* 3.** Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial of $f \in L_{2\pi}^{p(\cdot)}$ and let $m \in \mathbb{Z}^+$. Then by Remark 1 (ii)

$$\Omega_\alpha\left(f, \pi/n+1\right)_{p(\cdot)} \leq \Omega_\alpha\left(f - T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} + \Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \leq$$
$$\leq cE_{2^m}(f)_{p(\cdot)} + \Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)}.$$

Since

$$T_{2^m}^{(\alpha)}(x) = T_1^{(\alpha)}(x) + \sum_{\nu=0}^{m-1}\left\{T_{2^{\nu+1}}^{(\alpha)}(x) - T_{2^\nu}^{(\alpha)}(x)\right\}$$

we get by Lemma 3 that

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha}\left\{\left\|T_1^{(\alpha)}\right\|_{p,\pi} + \sum_{\nu=0}^{m-1}\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi}\right\}.$$

Lemma 1 gives

$$\left\|T_{2^{\nu+1}}^{(\alpha)} - T_{2^\nu}^{(\alpha)}\right\|_{p,\pi} \leq c2^{\nu\alpha}\left\|T_{2^{\nu+1}} - T_{2^\nu}\right\|_{p,\pi} \leq c2^{\nu\alpha+1}E_{2^\nu}(f)_{p(\cdot)}$$

and

$$\left\|T_1^{(\alpha)}\right\|_{p,\pi} = \left\|T_1^{(\alpha)} - T_0^{(\alpha)}\right\|_{p,\pi} \leq cE_0(f)_{p(\cdot)}.$$

Hence

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + \sum_{\nu=0}^{m-1} 2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)}\right\}.$$

Using

$$2^{(\nu+1)\alpha}E_{2^\nu}(f)_{p(\cdot)} \leq c^* \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1}E_\mu(f)_{p(\cdot)}, \quad \nu = 1, 2, 3, \ldots,$$

we obtain

$$\Omega_\alpha\left(T_{2^m}, \pi/(n+1)\right)_{p(\cdot)} \leq$$
$$\leq \frac{c}{(n+1)^\alpha}\left\{E_0(f)_{p(\cdot)} + 2^\alpha E_1(f)_{p(\cdot)} + c\sum_{\nu=1}^{m}\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\alpha-1}E_\mu(f)_{p(\cdot)}\right\} \leq$$

$$\leq \frac{c}{(n+1)^{\alpha}}\left\{E_0(f)_{p(\cdot)}+\sum_{\mu=1}^{2^m}\mu^{\alpha-1}E_\mu(f)_{p(\cdot)}\right\}\leq \frac{c}{(n+1)^{\alpha}}\sum_{\nu=0}^{2^m-1}(\nu+1)^{\alpha-1}E_\nu(f)_{p(\cdot)}.$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_\alpha\left(T_{2^m},\pi/(n+1)\right)_{p(\cdot)} \leq \frac{c}{(n+1)^{\alpha}}\sum_{\nu=0}^{n}(\nu+1)^{\alpha-1}E_\nu(f)_{p(\cdot)},$$

$$E_{2^m}(f)_{p(\cdot)} \leq E_{2^{m-1}}(f)_{p(\cdot)} \leq \frac{c}{(n+1)^{\alpha}}\sum_{\nu=0}^{n}(\nu+1)^{\alpha-1}E_\nu(f)_{p(\cdot)}.$$

Last two inequalities complete the proof.

***Proof of Theorem* 4.** For the polynomial T_n of the best approximation to f we have by Lemma 1 that

$$\left\|T_{2^{i+1}}^{(\beta)}-T_{2^i}^{(\beta)}\right\|_{p,\pi} \leq C(\beta)2^{(i+1)\beta}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p,\pi} \leq 2C(\beta)2^{(i+1)\beta}E_{2^i}(f)_{p(\cdot)}.$$

Hence

$$\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{W_{p(\cdot)}^{\beta}}=\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}^{(\beta)}-T_{2^i}^{(\beta)}\right\|_{p,\pi}+\sum_{i=1}^{\infty}\left\|T_{2^{i+1}}-T_{2^i}\right\|_{p,\pi}\leq$$

$$\leq c\sum_{m=2}^{\infty}m^{\beta-1}E_m(f)_{p(\cdot)}<\infty.$$

Therefore

$$\left\|T_{2^{i+1}}-T_{2^i}\right\|_{W_{p(\cdot)}^{\beta}}\to 0 \quad \text{as} \quad i\to\infty.$$

This means that $\{T_{2^i}\}$ is a Cauchy sequence in $L_{2\pi}^{p(\cdot)}$. Since $T_{2^i}\to f$ in $L_{2\pi}^{p(\cdot)}$ and $W_{p(\cdot)}^{\beta}$ is a Banach space we obtain $f\in W_{p(\cdot)}^{\beta}$.

On the other hand since

$$\left\|f^{(\beta)}-S_n(f^{(\beta)})\right\|_{p,\pi}\leq$$

$$\leq\left\|S_{2^{m+2}}(f^{(\beta)})-S_n(f^{(\beta)})\right\|_{p,\pi}+\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}(f^{(\beta)})-S_{2^k}(f^{(\beta)})\right\|_{p,\pi}$$

we have for $2^m<n<2^{m+1}$

$$\left\|S_{2^{m+2}}(f^{(\beta)})-S_n(f^{(\beta)})\right\|_{p,\pi}\leq c2^{(m+2)\beta}E_n(f)_{p(\cdot)}\leq c\,(n+1)^{\beta}E_n(f)_{p(\cdot)}.$$

On the other hand we find

$$\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}(f^{(\beta)})-S_{2^k}(f^{(\beta)})\right\|_{p,\pi}\leq c\sum_{k=m+2}^{\infty}2^{(k+1)\beta}E_{2^k}(f)_{p(\cdot)}\leq$$

$$\leq c\sum_{k=m+2}^{\infty}\sum_{\mu=2^{k-1}+1}^{2^k}\mu^{\beta-1}E_\mu(f)_{p(\cdot)}=$$

$$= c \sum_{\nu=2^{m+1}+1}^{\infty} \nu^{\beta-1} E_{\nu}\left(f\right)_{p(\cdot)} \leq c \sum_{\nu=n+1}^{\infty} \nu^{\beta-1} E_{\nu}(f)_{p(\cdot)}.$$

Theorem 4 is proved.

***Proof of Theorem* 5.** We set $W_n(f) := W_n(x,f) := \dfrac{1}{n+1}\sum_{\nu=n}^{2n} S_\nu(x,f)$, $n = 0,1,2,\ldots$. Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f)$$

we have

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\pi} \leq \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)})\right\|_{p,\pi} +$$

$$+ \left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\pi} + \left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, W_n(f))\right\|_{p,\pi} :=$$

$$:= I_1 + I_2 + I_3.$$

We denote by $T_n^*(x,f)$ the best approximating polynomial of degree at most n to f in $L_{2\pi}^{p(\cdot)}$. In this case, from the boundedness of the operator S_n in $L_{2\pi}^{p(\cdot)}$ we obtain the boundedness of operator W_n in $L_{2\pi}^{p(\cdot)}$ and there holds

$$I_1 \leq \left\| f^{(\alpha)}(\cdot) - T_n^*(\cdot, f^{(\alpha)})\right\|_{p,\pi} + \left\| T_n^*(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)})\right\|_{p,\pi} \leq$$

$$\leq cE_n(f^{(\alpha)})_{p(\cdot)} + \left\| W_n(\cdot, T_n^*(f^{(\alpha)}) - f^{(\alpha)})\right\|_{p,\pi} \leq cE_n(f^{(\alpha)})_{p(\cdot)}.$$

From Lemma 1 we get

$$I_2 \leq cn^{\alpha} \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f)\right\|_{p,\pi}$$

and

$$I_3 \leq c\,(2n)^{\alpha} \left\| W_n(\cdot, f) - T_n(\cdot, W_n(f))\right\|_{p,\pi} \leq c\,(2n)^{\alpha}\, E_n\left(W_n(f)\right)_{p(\cdot)}.$$

Now we have

$$\left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f)\right\|_{p,\pi} \leq$$

$$\leq \left\| T_n(\cdot, W_n(f)) - W_n(\cdot, f)\right\|_{p,\pi} + \left\| W_n(\cdot, f) - f(\cdot)\right\|_{p,\pi} + \left\| f(\cdot) - T_n(\cdot, f)\right\|_{p,\pi} \leq$$

$$\leq cE_n\left(W_n(f)\right)_{p(\cdot)} + cE_n(f)_{p(\cdot)} + cE_n(f)_{p(\cdot)}.$$

Since

$$E_n\left(W_n(f)\right)_{p(\cdot)} \leq cE_n(f)_{p(\cdot)}$$

we get

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f)\right\|_{p,\pi} \leq cE_n(f^{(\alpha)})_{p(\cdot)} + cn^{\alpha} E_n\left(W_n(f)\right)_{p(\cdot)} +$$

$$+ cn^{\alpha} E_n(f)_{p(\cdot)} + c\,(2n)^{\alpha}\, E_n\left(W_n(f)\right)_{p(\cdot)} \leq cE_n(f^{(\alpha)})_{p(\cdot)} + cn^{\alpha} E_n(f)_{p(\cdot)}.$$

Since by Theorem 1

$$E_n(f)_{p(\cdot)} \leq \frac{c}{(n+1)^\alpha} E_n(f^{(\alpha)})_{p(\cdot)}$$

we obtain

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{p,\pi} \leq c E_n(f^{(\alpha)})_{p(\cdot)}.$$

Theorem 5 is proved.

***Proof of Theorem* 6.** Let $f \in H^{p(\cdot)}(\mathbb{D})$. First of all if $p(x)$, defined on $\boldsymbol{T}$, satisfy *Dini–Lipschitz property* DL_γ for $\gamma \geq 1$ on $\boldsymbol{T}$, then $p\left(e^{ix}\right)$, $x \in \boldsymbol{T}$, defined on $\mathbb{T}$, satisfy *Dini–Lipschitz property* DL_γ for $\gamma \geq 1$ on $\mathbb{T}$. Since $H^{p(\cdot)} \subset H^1(\mathbb{D})$ for $1 < \underline{p}$, let $\sum_{k=-\infty}^{\infty} \beta_k e^{ik\theta}$ be the Fourier series of the function $f\left(e^{i\theta}\right)$, and $S_n(f,\theta) :=$ $:= \sum_{k=-n}^{n} \beta_k e^{ik\theta}$ be its nth partial sum. From $f\left(e^{i\theta}\right) \in H^1(\mathbb{D})$, we have [11, p. 38]

$$\beta_k = \begin{cases} 0, & \text{for } k < 0; \\ a_k(f), & \text{for } k \geq 0. \end{cases}$$

Therefore

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} = \left\| f - S_n(f, \cdot) \right\|_{p,\pi}. \tag{15}$$

If t_n^* is the best approximating trigonometric polynomial for $f(e^{i\theta})$ in $L_{2\pi}^{p(\cdot)}$, then from (6), (15) and Theorem 2 we get

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{H^{p(\cdot)}} \leq \left\| f\left(e^{i\theta}\right) - t_n^*(\theta) \right\|_{p,\pi} + \left\| S_n(f - t_n^*, \theta) \right\|_{p,\pi} \leq$$

$$\leq c E_n\left(f\left(e^{i\theta}\right)\right)_{p(\cdot)} \leq c\,\Omega_r\left(f\left(e^{i\theta}\right), \frac{1}{n+1}\right)_{p(\cdot)}.$$

Theorem 6 is proved.

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