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Gröbner–Shirshov bases of some monoids

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ABSTRACT

The main goal of this paper is to define Gröbner–Shirshov bases for some monoids. Therefore, after giving some preliminary material, we first give Gröbner–Shirshov bases for graphs and Schützenberger products of monoids in separate sections. In the final section, we further present a Gröbner–Shirshov basis for a Rees matrix semigroup.

1. Introduction

The Gröbner basis theory for commutative algebras was introduced by Buchberger [12] and provides a solution to the reduction problem for commutative algebras. In [1], Bergman generalized the Gröbner basis theory to associative algebras by proving the Diamond Lemma. On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [25]. The key ingredient of the theory is the so-called Composition Lemma which characterizes the leading terms of elements in the given ideal. In [2], Bokut noticed that Shirshov's method also works for associative algebras. Hence, for this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner–Shirshov basis* theory. Gröbner–Shirshov bases for finite dimensional simple Lie algebras were constructed explicitly in a series of papers by Bokut and Klein [8–10]. Moreover, in [11], Bokut et al. defined the Gröbner–Shirshov basis for some braid groups. In [16], Gröbner–Shirshov bases for HNN-extensions of groups and for alternating groups were considered. Furthermore, in [15,14], Gröbner–Shirshov bases for Schreier extensions of groups and for the Chinese monoid were defined, separately. Some other recent papers about Gröbner–Shirshov bases are, for instance, [3,4,7,6,22].

It is well known that the graph product is an operator which is mixing direct and free products. In fact the graph product between two monoids whether free or direct can be determined by a simplicial graph (a graph with no loops). Considering a monoid attached to each vertex of the graph, the associated graph product is the monoid generated by each of the vertex monoids with the added relations that elements of adjacent vertex monoids commute. For more details on it, we may refer to, for instance, [17,18].

One of the most useful tools for studying the concatenation product is the Schützenberger product of monoids which was originally defined by Schützenberger [24] for two monoids, and extended by Straubing [26] for any number of monoids.

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The other most useful and important construction is Rees matrix semigroups. After Rees matrix semigroups were introduced by Rees [23], they became a very important family of semigroups, especially in the study of structure theory of completely (0)-simple semigroups (see for example [19]).

In this paper, we find Gröbner–Shirshov bases for monoids and semigroups that are mentioned in above paragraphs. In the light of this aim, sections are organized by including details and Gröbner–Shirshov bases of these types of monoids and semigroups as follows. First of all, we provide some background material about the Gröbner–Shirshov basis and the Composition–Diamond Lemma. Then in Sections 3–5, we study Gröbner–Shirshov bases for graphs and Schützenberger products of monoids, and for Rees matrix semigroups, respectively.

Throughout this paper, $p_1 \cap p_2$ denotes the intersection compositions of p_1 and p_2 polynomials. Additionally also $\widetilde{u_i}$ and $\underline{u_i}$ denote the words which do not have the last generator and the first generator of the word u_i, respectively.

2. Gröbner–Shirshov bases and the Composition–Diamond Lemma

Let K be a field and $K\langle X\rangle$ be the free associative algebra over K generated by X. Denote X^* the free monoid generated by X, where the empty word is the identity which is denoted by 1. For a word $w \in X^*$, we denote the length of w by $|w|$. Let X^* be a well ordered set. Then every nonzero polynomial $f \in K\langle X\rangle$ has the leading word $\overline{f}$. If the coefficient of $\overline{f}$ in f is equal to 1, then f is called monic.

Definition 1. Let f and g be two monic polynomials in $K\langle X\rangle$. Then, there are two kinds of compositions.

1. If w is a word such that $w = \overline{f}b = a\overline{g}$ for some $a, b \in X^*$ with $|\overline{f}| + |\overline{g}| > |w|$, then the polynomial $(f, g)_w = fb - ag$ is called the *intersection composition* of f and g with respect to w. The word w is called an *ambiguity* of the intersection.
2. If $w = \overline{f} = a\overline{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = \overline{f} - a\overline{g}b$ is called the *inclusion composition* of f and g with respect to w. The word w is called an *ambiguity* of inclusion.

Definition 2. If g is monic, $\overline{f} = a\overline{g}b$ and α is the coefficient of the leading term $\overline{f}$, then transformation $f \mapsto f - \alpha agb$ is called an elimination of the leading word (ELW) of g in f.

Definition 3. Let $S \subseteq K\langle X\rangle$ with each $s \in S$ monic. Then the composition $(f, g)_w$ is called a trivial modulo (S, w) if $(f, g)_w = \sum \alpha a_i s_i b_i$, where each $\alpha_i \in K$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$. If this is the case, then we write

$$(f, g)_w \equiv 0 \; mod(S, w).$$

In general, for $p, q \in K\langle X\rangle$, we write

$$p \equiv q \; mod(S, w)$$

which means that $p - q = \sum \alpha a_i s_i b_i$, where each $\alpha_i \in K$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$.

Definition 4. We call the set S endowed with the well ordering $<$ a Gröbner–Shirshov basis for $K\langle X \mid S\rangle$ if any composition $(f, g)_w$ of polynomials in S is trivial in modulo S and the corresponding w.

A well ordered $<$ on X^* is monomial if for $u, v \in X^*$, we have

$$u < v \Rightarrow w_1 u w_2 < w_1 v w_2,$$

for all $w_1, w_2 \in X^*$.

The following lemma was proved by Shirshov [25] for free Lie algebras (with deg-lex ordering) in 1962 (see also [5]). In 1976, Bokut [2] specialized the Shirshov's approach to associative algebras (see also [1]). Meanwhile, for commutative polynomials, this lemma is known as the Buchberger's Theorem (see [12,13]).

Lemma 5 (*Composition–Diamond Lemma*)**.** *Let K be a field,*

$$A = K\langle X \mid S\rangle = K\langle X\rangle / Id(S)$$

and $<$ a monomial ordering on X^*, where $Id(S)$ is the ideal of $K\langle X\rangle$ generated by S.Then the following statements are equivalent:

1. *S is a Gröbner–Shirshov basis.*
2. *$f \in Id(S) \Rightarrow \overline{f} = a\overline{s}b$ for some $s \in S$ and $a, b \in X^*$.*
3. *$Irr(S) = \{u \in X^* \mid u \neq a\overline{s}b, s \in S, a, b \in X^*\}$ is a basis of the algebra $A = K\langle X \mid S\rangle$.*

If a subset S of $K\langle X\rangle$ is not a Gröbner–Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner–Shirshov basis S^{comp}. Such a process is called the *Shirshov algorithm*.

If S is a set of "semigroup relations" (that is, the polynomials of the form $u - v$, where $u, v \in X^*$), then any nontrivial composition will have the same form. As a result, the set S^{comp} also consists of semigroup relations.

Let $M = sgp\langle X \mid S\rangle$ be a semigroup presentation. Then S is a subset of $K\langle X\rangle$ and hence one can find a Gröbner–Shirshov basis S^{comp}. The last set does not depend on K, and as mentioned before, it consists of semigroup relations. We will call S^{comp} a Gröbner–Shirshov basis of M. This is the same as a Gröbner–Shirshov basis of the semigroup algebra $KM = K\langle X \mid S\rangle$. If S is a Gröbner–Shirshov basis of the semigroup $M = sgp\langle X \mid S\rangle$, then $Irr(S)$ is a normal form for M.

3. Gröbner–Shirshov basis for the graph product of monoids

Let $M_1, M_2, \ldots, M_j$ ($j \geq 4$) be monoids presented by generators and relations

$$\wp_{M_1} = \langle X_1 \mid R_1 \rangle, \qquad \wp_{M_2} = \langle X_2 \mid R_2 \rangle, \ldots, \wp_{M_j} = \langle X_j \mid R_j \rangle,$$

respectively, where $R_1, R_2, \ldots, R_j$ are Gröbner–Shirshov bases for $M_1, M_2, \ldots, M_j$ with the deg-lex orders $<_{M_i}$ on X_i^* ($1 \leq i \leq j$). Here, we assume that the sets $X_1, X_2, \ldots, X_j$ are disjoint and each X_i is a well-ordered set.

Let

$$R_1 = \{u_{1_1} = v_{1_1}, u_{1_2} = v_{1_2}, \ldots, u_{1_{m_1}} = v_{1_{m_1}}\},$$
$$R_2 = \{u_{2_1} = v_{2_1}, u_{2_2} = v_{2_2}, \ldots, u_{2_{m_2}} = v_{2_{m_2}}\},$$
$$\ldots$$
$$R_j = \{u_{j_1} = v_{j_1}, u_{j_2} = v_{j_2}, \ldots, u_{j_{m_j}} = v_{j_{m_j}}\},$$

where $m_1, m_2, \ldots, m_j$ are positive integers and u_{i_r} ($i \leq j$ and $r \leq m_i$) are the leading terms of polynomials $f_{u_{i_r}} = u_{i_r} - v_{i_r}$ in $k\langle X_i \rangle$.

Then we have the graph product of monoids M_i ($1 \leq i \leq j$), say M, presented by

$$\wp_M = \langle X_1, X_2, \ldots, X_j \mid R_1, R_2, \ldots, R_j, S' \rangle, \tag{1}$$

where $S' = \{x_i x_{i+1} - x_{i+1} x_i, x_1 x_j - x_j x_1\}$ ($1 \leq i < j$), and M_i, M_{i+1} are adjacent vertices of Γ, which is a simplicial graph (a graph with no loops) with vertices labeled $M_1, M_2, \ldots, M_j$ (see [17]).

Now let us order the set $(X_1 \cup X_2 \cup \cdots \cup X_j)^*$ with degree lexicographically by using the order

- $x_i > x_k$ if $i < k$ ($x_i \in X_i, x_k \in X_k$).

Now we give the main result of this section.

Theorem 6. *A Gröbner–Shirshov basis for M consists of the following relations:*

$$u_{i_r} = v_{i_r} \quad (1 \leq i \leq j), \tag{2}$$
$$x_i x_{i+1} = x_{i+1} x_i, \qquad x_1 x_j = x_j x_1 \quad (1 \leq i \leq j-1), \tag{3}$$
$$x_i w_{i+2} x_{i+1} = x_{i+1} x_i w_{i+2} \quad (1 \leq i \leq j-2), \tag{4}$$

where $w_{i+2} \in X_{i+2}^*$.

Sketch of the proof. We need to prove that all compositions of relations (2)–(4) are trivial. To do that we must check all the ambiguities in S, where S is the set of relations at $\wp_M$ (see (1)), by considering the following cases;

1. Ambiguities which are from the leading words of polynomials in R_i and R_k for $1 \leq i, k \leq j$ and $i \neq k$,
2. Ambiguities which are from the leading words of polynomials in S', by this process we get the relation (4),
3. Ambiguities which are from the leading words of polynomials in S' and R_i for $1 \leq i \leq j$.

Proof. 1. If we check leading words from R_i and R_k for $1 \leq i, k \leq j$ and $i \neq k$, then we see that there are no any ambiguities since the generator sets of these relation sets are different from each other. So we do not need to check the ambiguities obtained by intersection compositions of leadings terms of polynomilas in R_i and R_k.

2. We examine the intersection compositions of polynomials in the set S' with each other. To do that, let

$$g_1 = x_i x_{i+1} - x_{i+1} x_i \quad \text{and} \quad g_2 = x_{i+1} x_{i+2} - x_{i+2} x_{i+1} \in S'.$$

Then we have the ambiguity $w = x_i x_{i+1} x_{i+2}$. Here $a = x_i$ and $b = x_{i+2}$. Then we get

$$\begin{aligned}(g_1, g_2)_w &= g_1 b - a g_2 \\ &= (x_i x_{i+1} - x_{i+1} x_i) x_{i+2} - x_i (x_{i+1} x_{i+2} - x_{i+2} x_{i+1}) \\ &= x_i x_{i+1} x_{i+2} - x_{i+1} x_i x_{i+2} - x_i x_{i+1} x_{i+2} + x_i x_{i+2} x_{i+1} \\ &= x_i x_{i+2} x_{i+1} - x_{i+1} x_i x_{i+2}\end{aligned}$$

which is not trivial modulo S.

Now let $h_1 = x_i x_{i+2} x_{i+1} - x_{i+1} x_i x_{i+2}$. If we consider the intersection composition of h_1 with g_2, then we get the polynomial $h_2 = x_i x_{i+2}^2 x_{i+1} - x_{i+1} x_i x_{i+2}^2$. By continuing this procedure, we obtain the following non-trivial polynomial

$$h = x_i w_{i+2} x_{i+1} - x_{i+1} x_i w_{i+2} \quad (i \in \{1, 2, \ldots, (j-2)\}),$$

where $w_{i+2} \in X_{i+2}^*$. Now let us consider the intersection composition of h with itself. Hence we obtain the ambiguity $w = x_i w_{i+2} x_{i+1} w_{i+3} x_{i+2}$ and thus we get

$$\begin{aligned}(h, h)_w &= (x_i w_{i+2} x_{i+1} - x_{i+1} x_i w_{i+2}) w_{i+3} x_{i+2} - x_i w_{i+2} (x_{i+1} w_{i+3} x_{i+2} - x_{i+2} x_{i+1} w_{i+3}) \\ &= x_i w_{i+2} x_{i+1} w_{i+3} x_{i+2} - x_{i+1} x_i w_{i+2} w_{i+3} x_{i+2} - x_i w_{i+2} x_{i+1} w_{i+3} x_{i+2} + x_i w_{i+2} x_{i+2} x_{i+1} w_{i+3} \\ &= x_i w_{i+2} x_{i+2} x_{i+1} w_{i+3} - x_{i+1} x_i w_{i+2} w_{i+3} x_{i+2} \\ &= x_{i+1} x_i w_{i+2} x_{i+2} w_{i+3} - x_{i+1} x_i w_{i+2} w_{i+3} x_{i+2} \\ &= x_{i+1} x_i w_{i+2} w_{i+3} x_{i+2} - x_{i+1} x_i w_{i+2} w_{i+3} x_{i+2} \equiv 0.\end{aligned}$$

At this stage, it remains to check intersection composition of g_1 with h, $f_{u_{i_r}}$ with h and h with $f_{u_{i_r}}$.

$$\begin{aligned}&g_1 \cap h : w = x_i x_{i+1} w_{i+3} x_{i+2}, \\ &(g_1, h)_w = (x_i x_{i+1} - x_{i+1} x_i) w_{i+3} x_{i+2} - x_i (x_{i+1} w_{i+3} x_{i+2} - x_{i+2} x_{i+1} w_{i+3}) \\ &\quad = x_i x_{i+1} w_{i+3} x_{i+2} - x_{i+1} x_i w_{i+3} x_{i+2} - x_i x_{i+1} w_{i+3} x_{i+2} + x_i x_{i+2} x_{i+1} w_{i+3} \\ &\quad = x_i x_{i+2} x_{i+1} w_{i+3} - x_{i+1} x_i w_{i+3} x_{i+2} = x_{i+1} x_i x_{i+2} w_{i+3} - x_{i+1} x_i w_{i+3} x_{i+2} \\ &\quad = x_{i+1} x_i w_{i+3} x_{i+2} - x_{i+1} x_i w_{i+3} x_{i+2} \equiv 0.\end{aligned}$$

$$\begin{aligned}&f_{u_{i_r}} \cap h : w = \widetilde{u_{i_r}} x_i w_{i+2} x_{i+1}, \\ &(f_{u_{i_r}}, h)_w = (u_{i_r} - v_{i_r}) w_{i+2} x_{i+1} - \widetilde{u_{i_r}} (x_i w_{i+2} x_{i+1} - x_{i+1} x_i w_{i+2}) \\ &\quad = u_{i_r} w_{i+2} x_{i+1} - v_{i_r} w_{i+2} x_{i+1} - u_{i_r} w_{i+2} x_{i+1} + \widetilde{u_{i_r}} x_{i+1} x_i w_{i+2} \\ &\quad = \widetilde{u_{i_r}} x_{i+1} x_i w_{i+2} - v_{i_r} w_{i+2} x_{i+1} = x_{i+1} \widetilde{u_{i_r}} x_i w_{i+2} - v_{i_r} w_{i+2} x_{i+1} \\ &\quad = x_{i+1} u_{i_r} w_{i+2} - v_{i_r} w_{i+2} x_{i+1} = x_{i+1} v_{i_r} w_{i+2} - x_{i+1} v_{i_r} w_{i+2} \equiv 0.\end{aligned}$$

$$\begin{aligned}&h \cap f_{u_{i_r}} : w = x_i w_{i+2} x_{i+1} \underline{u_{i+1_r}} \\ &(h, f_{u_{i_r}})_w = (x_i w_{i+1} x_{i+1} - x_{i+1} x_i w_{i+2}) \underline{u_{i+1_r}} - x_i w_{i+2} (u_{i+1_r} - v_{i+1_r}) \\ &\quad = x_i w_{i+1} x_{i+1} \underline{u_{i+1_r}} - x_{i+1} x_i w_{i+2} \underline{u_{i+1_r}} - x_i w_{i+2} u_{i+1_r} + x_i w_{i+2} v_{i+1_r} \\ &\quad = x_i w_{i+2} v_{i+1_r} - x_{i+1} x_i w_{i+2} \underline{u_{i+1_r}} = v_{i+1_r} x_i w_{i+2} - x_{i+1} \underline{u_{i+1_r}} x_i w_{i+2} \\ &\quad = v_{i+1_r} x_i w_{i+2} - v_{i+1_r} x_i w_{i+2} \equiv 0.\end{aligned}$$

3. In this part of the proof we check the ambiguities obtained by intersection compositions of leading terms of polynomials in S' and R_i $(1 \le i \le j)$. To do that let us suppose that $g = x_i x_{i+1} - x_{i+1} x_i \in S'$ and $f_{u_{i_r}} = u_{i_r} - v_{i_r} \in R_i$, $(1 \le i \le j)$. So the ambiguity obtained by the intersection composition of $f_{u_{i_r}}$ with g is $w = \widetilde{u_{i_r}} x_i x_{i+1}$. Then we get

$$\begin{aligned}(f_{u_{i_r}}, g)_w &= (u_{i_r} - v_{i_r}) x_{i+1} - \widetilde{u_{i_r}} (x_i x_{i+1} - x_{i+1} x_i) \\ &= u_{i_r} x_{i+1} - v_{i_r} x_{i+1} - \widetilde{u_{i_q}} x_i x_{i+1} + \widetilde{u_{i_q}} x_{i+1} x_i \\ &= u_{i_r} x_{i+1} - v_{i_r} x_{i+1} - u_{i_r} x_{i+1} + \widetilde{u_{i_r}} x_{i+1} x_i \\ &= \widetilde{u_{i_r}} x_{i+1} x_i - v_{i_r} x_{i+1} \\ &= x_{i+1} \widetilde{u_{i_r}} x_i - x_{i+1} v_{i_r} \\ &= x_{i+1} u_{i_r} - x_{i+1} v_{i_r} \equiv 0.\end{aligned}$$

Similarly, by checking the intersection composition of g by $f_{u_{i_r}}$, we obtain the triviality again.

The above procedure shows that there are no new polynomials by considering the relations R_j and S' to obtain a Gröbner–Shirshov basis for the graph product of monoids.

Finally, it remains to check compositions of including of polynomials (2)–(4). But it is clear.

Hence the proof. □

Remark 7. At the beginning of the Section 3, we take $j \ge 4$. The reason for this is that for the graph product of less than four monoids, we get a direct product of monoids. So one can find a Gröbner–Shirshov basis for this monoid consists of the relations (2) and (3).

By using the Composition–Diamond Lemma, the normal form for the graph product of monoids can be given by the following result.

Corollary 8 ([21]). *Every element w of M has one of the normal forms $w_1 w_2 \cdots w_n$ where each of w_i is an element of some vertex monoid M_k $(1 \le k \le j)$. Here we have the following:*

1. *Remove $w_i = 1$.*
2. *Replace consecutive elements w_i and w_{i+1} in the same vertex monoid M_k with the single element $w_i w_{i+1}$.*
3. *For consecutive elements $w_i \in M_i$, $w_{i+1} \in M_{i+1}$ and $w_1 \in M_1$, $w_j \in M_j$ such that M_i, M_{i+1} and M_1, M_j are adjacent monoids, exchange w_i, w_{i+1} and w_1, w_j.*

4. Gröbner–Shirshov basis for the Schützenberger product of monoids

Let A and B be monoids. For $P \subseteq A \times B$, $a \in A$, $b \in B$, we define

$$aP = \{(ac, d) \mid (c, d) \in P\}, \qquad Pb = \{(c, db) \mid (c, d) \in P\}.$$

The *Schützenberger product* of A and B, denoted by $A \diamond B$, is the set $A \times \mathcal{P}(A \times B) \times B$ with multiplication $(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1a_2, P_1b_2 \cup a_1P_2, b_1b_2)$.

Let M_1 and M_2 be monoids presented by $\wp_{M_1} = \langle X_1 \mid R_1\rangle$ and $\wp_{M_2} = \langle X_2 \mid R_2\rangle$, respectively, where R_1 and R_2 are Gröbner–Shirshov bases for M_1 and M_2 with the deg-lex order $<_{M_i}$ on X_i^* $(i = 1, 2)$.

The Schützenberger product of M_1 and M_2 is presented by

$$\wp_{M_1 \diamond M_2} = \langle Z \mid R_1, R_2, z^2_{w_1,w_2} = z_{w_1,w_2}, z_{w_1,w_2}z_{w'_1,w'_2} = z_{w'_1,w'_2}z_{w_1,w_2}, \\ x_1z_{w_1,w_2} = z_{x_1w_1,w_2}x_1, z_{w_1,w_2}x_2 = x_2z_{w_1,w_2x_2}, x_1x_2 = x_2x_1\rangle,$$

where $x_i \in X_i$, $w_i, w'_i \in M_i$ $(i \in \{1, 2\})$ and $Z = X_1 \cup X_2 \cup \{z_{w_1,w_2} \mid w_1 \in M_1, w_2 \in M_2\}$ (see [20]).

Now we order the set Z^* with degree lexicographically by using the following orders:

- $x_1 > x_2$ by the order $<_{M_i}$, $x_i \in X_i$ $(1 \le i \le 2)$,
- $x_1 > z_{w_1,w_2} > x_2$ for all $w_i \in M_i$ $(1 \le i \le 2)$,
- $(w_1, w_2) > (w'_1, w'_2)$ if $w_1 > w'_1$ or $w_1 = w'_1$ and $w_2 > w'_2$,
- $z_{w_1,w_2} > z_{w'_1,w'_2}$ if $(w_1, w_2) > (w'_1, w'_2)$, $w_i, w'_i \in M_i$ $(1 \le i \le 2)$.

Now we can give the following theorem as another main result of this paper.

Theorem 9. *A Gröbner–Shirshov basis for $M_1 \diamond M_2$ consists of the following polynomials:*

1. $u_1 - v_1$, 2. $u_2 - v_2$,
3. $z^2_{w_1,w_2} - z_{w_1,w_2}$, 4. $z_{w_1,w_2}z_{w'_1,w'_2} - z_{w'_1,w'_2}z_{w_1,w_2}$,
5. $x_1z_{w_1,w_2} - z_{x_1w_1,w_2}x_1$, 6. $z_{w_1,w_2}x_2 - x_2z_{w_1,w_2x_2}$,
7. $x_1x_2 - x_2x_1$,

where $u_i - v_i \in R_i$ $(1 \le i \le 2)$.

Proof. Let us consider all intersection compositions of 1–7 with each other. We need to prove that all these compositions are trivial. These compositions are summarized in the following table.

$i \cap j$	w: ambiguity	$i \cap j$	w: ambiguity
$1 \cap 5$	$\widetilde{u_1}x_1z_{w_1,w_2}$	$4 \cap 6$	$z_{w_1,w_2}z_{w'_1,w'_2}x_2$
$1 \cap 7$	$\widetilde{u_1}x_1x_2$	$5 \cap 3$	$x_1z^2_{w_1,w_2}$
$3 \cap 4$	$z^2_{w_1,w_2}z_{w'_1,w'_2}$	$5 \cap 4$	$x_1z_{w_1,w_2}z_{w'_1,w'_2}$
$3 \cap 6$	$z^2_{w_1,w_2}x_2$	$5 \cap 6$	$x_1z_{w_1,w_2}x_2$
$4 \cap 3$	$z_{w_1,w_2}z^2_{w'_1,w'_2}$	$6 \cap 2$	$z_{w_1,w_2}x_2\underline{u_2}$
$4 \cap 4$	$z_{w_1,w_2}z_{w'_1,w'_2}z_{w''_1,w''_2}$	$7 \cap 2$	$x_1x_2\underline{u_2}$

It is seen that these compositions are trivial. Let us check one of them as follows.

$$1 \cap 5 : w = \widetilde{u_1}x_1z_{w_1,w_2},$$

$$\begin{aligned}(f, g)_w &= (u_1 - v_1)z_{w_1,w_2} - \widetilde{u_1}(x_1z_{w_1,w_2} - z_{x_1w_1,w_2}x_1)\\ &= u_1z_{w_1,w_2} - v_1z_{w_1,w_2} - \widetilde{u_1}x_1z_{w_1,w_2} + \widetilde{u_1}z_{x_1w_1,w_2}x_1\\ &= \widetilde{u_1}z_{x_1w_1,w_2}x_1 - v_1z_{w_1,w_2} = z_{\widetilde{u_1}x_1w_1,w_2}\widetilde{u_1}x_1 - z_{v_1w_1,w_2}v_1\\ &= z_{u_1w_1,w_2}u_1 - z_{v_1w_1,w_2}v_1 \equiv 0.\end{aligned}$$

Finally, it remains to check compositions of including of polynomials 1–7. But it is clear that there are no any compositions of this type.

Hence the result. □

So under the relations which are actually Gröbner–Shirshov bases for the Schützenberger product of monoids, we give a normal form of words as follows:

Corollary 10 ([20]). *Every element w of $M_1 \diamond M_2$ has a unique representation $u_2z_{m_1,m_2}u_1$, where $z_{m_1,m_2} \in \{z_{w_1,w_2} \mid w_1 \in M_1, w_2 \in M_2\}^*$, $u_2 \in X_2^*$ and $u_1 \in X_1^*$ are irreducible words.*

5. Gröbner–Shirshov basis for Rees matrix semigroup

Let A be a monoid, 0 be an element not belonging to A, and let I and Λ be index sets. Also let $P=(p_{\lambda i})_{\lambda\in\Lambda,i\in I}$ be a $|\Lambda|\times|I|$ matrix with entries from the set $\Lambda\cup\{0\}$. Then the *Rees matrix semigroup* $M^0[A;I,\Lambda;P]$ is the set $(I\times A\times\Lambda)\cup\{0\}$ with the multiplication

$$(i_1,a_1,\lambda_1)(i_2,a_2,\lambda_2)=\begin{cases}(i_1,a_1p_{\lambda_1 i_2}a_2,\lambda_2) & \text{if } p_{\lambda_1 i_2}\neq 0\\ 0 & \text{if } p_{\lambda_1 i_2}=0\end{cases}$$

such that

$$0(i,a,\lambda)=(i,a,\lambda)0=00=0.$$

We may refer the reader to [20] for more details about Rees matrix semigroups.

Theorem 11 *([20]). For a monoid A, let $S=M^0[A;I,\Lambda;P]$ be a Rees matrix semigroup, where P is a $|\Lambda|\times|I|$ matrix with entries from A and $p_{11}=1_A$. Also let $\langle X|R\rangle$ be a semigroup presentation for A, $e\in X^*$ be a non-empty word representing the identity 1_A of A, and let $Y=X\cup\{y_i:i\in I-\{1\}\}\cup\{z_\lambda:\lambda\in\Lambda-\{1\}\}$. Then the presentation*

$$\langle Y\mid R, y_ie=y_i, ey_i=p_{1i}, z_\lambda e=p_{\lambda 1}, ez_\lambda=z_\lambda, z_\lambda y_i=p_{\lambda i}(i\in I-\{1\},\lambda\in\Lambda-\{1\})\rangle \tag{5}$$

defines S as a semigroup with zero.

We remark that, for the following result, we will assume $|p_{\lambda 1}|=|p_{\lambda' 1}|=|p_{1i}|=|p_{1j}|=1$ and $|p_{\lambda i}|,|p_{\lambda' i}|\le 2$, where $i,j\in I-\{1\}$, $\lambda,\lambda'\in\Lambda-\{1\}$. Additionally we will suppose that R is a Gröbner–Shirshov basis for A with the deg-lex order $<_A$ on X^*. We will order the set Y^* with degree lexicographically by using the orders $z_\lambda,z_{\lambda'}>x$ and $y_i,y_j>x$ $(x\in X)$.

Theorem 12. *A Gröbner–Shirshov basis for $S=M^0[A;I,\Lambda;P]$ consists of the relations given in the presentation (5) and the following relations:*

$$y_iy_j=y_ip_{1j},\quad z_\lambda z_{\lambda'}=p_{\lambda 1}z_{\lambda'},\quad p_{1i}e=p_{1i},\quad ep_{\lambda 1}=p_{\lambda 1}, \tag{6}$$
$$z_\lambda p_{1i}=p_{\lambda 1}y_i,\quad ep_{\lambda i}=p_{\lambda i},\quad p_{\lambda i}e=p_{\lambda i},\quad p_{1i}y_j=p_{1i}p_{1j}, \tag{7}$$
$$z_\lambda p_{\lambda' 1}=p_{\lambda 1}p_{\lambda' 1},\quad z_\lambda p_{\lambda' i}=p_{\lambda 1}p_{\lambda' i},\quad p_{\lambda i}y_j=p_{\lambda i}p_{1j}. \tag{8}$$

Proof. As a usual way, we need to show that all compositions of relations in presentation (5) and equations from (6)–(8) are trivial. To do that let us consider the following polynomials:

1. $u-v$, 2. y_ie-y_i, 3. ey_i-p_{1i},
4. $z_\lambda e-p_{\lambda 1}$, 5. $ez_\lambda-z_\lambda$, 6. $z_\lambda y_i-p_{\lambda i}$,

where $u=v\in R$. Now we can check intersection compositions of these polynomials by the following table. In this table we get new polynomials which are not trivial.

$i\cap j$	w: ambiguity	New polynomial	$i\cap j$	w: ambiguity	New polynomial
$2\cap 3$	y_iey_j	7. $y_iy_j-y_ip_{1j}$	$4\cap 5$	$z_\lambda ez_{\lambda'}$	10. $z_\lambda z_{\lambda'}-p_{\lambda 1}z_{\lambda'}$
$2\cap 5$	y_iez_λ	*trivial*	$5\cap 4$	$ez_\lambda e$	11. $ep_{\lambda 1}-p_{\lambda 1}$
$3\cap 2$	ey_ie	8. $p_{1i}e-p_{1i}$	$5\cap 6$	$ez_\lambda y_i$	12. $ep_{\lambda i}-p_{\lambda i}$
$4\cap 3$	$z_\lambda ey_i$	9. $z_\lambda p_{1i}-p_{\lambda 1}y_i$	$6\cap 2$	$z_\lambda y_ie$	13. $p_{\lambda i}e-p_{\lambda i}$

Let us check one of the above compositions:

$$2\cap 3: w=y_iey_j,$$
$$\begin{aligned}(f,g)_w &= (y_ie-y_i)y_j-y_i(ey_j-p_{1j})\\ &= y_iey_j-y_iy_j-y_iey_j+y_ip_{1j}=y_ip_{1j}-y_iy_j.\end{aligned}$$

Since we have the order $y_j>x$ $(x\in X)$ we get the polynomial $y_iy_j-y_ip_{1j}$.

Now we check intersection compositions of polynomials 7–13 with each other and 7–13 with 1–6. These compositions which are trivial are summarized in the following tables, respectively.

$i \cap j$	w: ambiguity	$i \cap j$	w: ambiguity
$7 \cap 7$	$y_i y_j y_{j'}$	$10 \cap 9$	$z_\lambda z_{\lambda'} p_{1i}$
$8 \cap 11$	$p_{1i} e p_{\lambda 1}$	$12 \cap 13$	$e p_{\lambda i} e$
$8 \cap 12$	$p_{1i} e p_{\lambda i'}$	$13 \cap 11$	$p_{\lambda i} e p_{\lambda' 1}$
$9 \cap 8$	$z_\lambda p_{1i} e$	$13 \cap 12$	$p_{\lambda i} e p_{\lambda' i'}$
$7 \cap 2$	$y_i y_j e$	$11 \cap 1$	$e p_{\lambda 1} \underline{u}$
$8 \cap 3$	$p_{1i} e y_j$	$12 \cap 1$	$e p_{\lambda i} \underline{u}$
$8 \cap 5$	$p_{1i} e z_\lambda$	$13 \cap 3$	$p_{\lambda i} e y_j$
$9 \cap 1$	$z_\lambda p_{1i} \underline{u}$	$13 \cap 5$	$p_{\lambda i} e z_{\lambda'}$
$10 \cap 4$	$z_\lambda z_{\lambda'} e$	$10 \cap 6$	$z_\lambda z_{\lambda'} y_i$

Let us check any two of these above compositions:

$$8 \cap 11 : w = p_{1i} e p_{\lambda 1},$$
$$\begin{aligned}(f, g)_w &= (p_{1i} e - p_{1i}) p_{\lambda 1} - p_{1i}(e p_{\lambda 1} - p_{\lambda 1}) \\ &= p_{1i} e p_{\lambda 1} - p_{1i} p_{\lambda 1} - p_{1i} e p_{\lambda 1} + p_{1i} p_{\lambda 1} \equiv 0\end{aligned}$$
$$13 \cap 5 : w = p_{\lambda i} e z_{\lambda'},$$
$$\begin{aligned}(f, g)_w &= (p_{\lambda i} e - p_{\lambda i}) z_{\lambda'} - p_{\lambda i}(e z_{\lambda'} - z_{\lambda'}) \\ &= p_{\lambda i} e z_{\lambda'} - p_{\lambda i} z_{\lambda'} - p_{\lambda i} e z_{\lambda'} + p_{\lambda i} z_{\lambda'} \equiv 0.\end{aligned}$$

Now we check intersection compositions of 1–6 with 7–13 with the following table.

$i \cap j$	w: ambiguity	New polynomial	$i \cap j$	w: ambiguity	New polynomial
$1 \cap 8$	$\widetilde{u} p_{1i} e$	*trivial*	$4 \cap 12$	$z_\lambda e p_{\lambda' i}$	16. $z_\lambda p_{\lambda' i} - p_{\lambda 1} p_{\lambda' i}$
$2 \cap 11$	$y_i e p_{\lambda 1}$	*trivial*	$5 \cap 9$	$e z_\lambda p_{1i}$	*trivial*
$2 \cap 12$	$y_i e p_{\lambda i'}$	*trivial*	$5 \cap 10$	$e z_\lambda z_{\lambda'}$	*trivial*
$3 \cap 7$	$e y_i y_j$	14. $p_{1i} y_j - p_{1i} p_{1j}$	$6 \cap 7$	$z_\lambda y_i y_j$	17. $p_{\lambda i} y_j - p_{\lambda i} p_{1j}$
$4 \cap 11$	$z_\lambda e p_{\lambda' 1}$	15. $z_\lambda p_{\lambda' 1} - p_{\lambda 1} p_{\lambda' 1}$	$1 \cap 13$	$\widetilde{u} p_{\lambda i} e$	*trivial*

Let us check one of the compositions given above:

$$4 \cap 11 : w = z_\lambda e p_{\lambda' 1},$$
$$\begin{aligned}(f, g)_w &= (z_\lambda e - p_{\lambda 1}) p_{\lambda' 1} - z_\lambda (e p_{\lambda' 1} - p_{\lambda' 1}) \\ &= z_\lambda e p_{\lambda' 1} - p_{\lambda 1} p_{\lambda' 1} - z_\lambda e p_{\lambda' 1} + z_\lambda p_{\lambda' 1} = z_\lambda p_{\lambda' 1} - p_{\lambda 1} p_{\lambda' 1}.\end{aligned}$$

Now let us consider the polynomials 14–17 given in the above table and check their intersection compositions with each other, with the polynomials 7–13 and with the polynomials 1–6. Among these compositions those which are trivial are summarized in the following table.

$i \cap j$	w: ambiguity	$i \cap j$	w: ambiguity
$1 \cap 14$	$\widetilde{u} p_{1i} y_j$	$14 \cap 7$	$p_{1i} y_j y_{j'}$
$1 \cap 17$	$\widetilde{u} p_{\lambda i} y_j$	$15 \cap 1$	$z_\lambda p_{\lambda' 1} \underline{u}$
$5 \cap 15$	$e z_\lambda p_{\lambda' 1}$	$16 \cap 1$	$z_\lambda p_{\lambda' i} \underline{u}$
$5 \cap 16$	$e z_\lambda p_{\lambda' i}$	$14 \cap 2$	$p_{1i} y_j e$
$9 \cap 14$	$z_\lambda p_{1i} y_j$	$16 \cap 13$	$z_\lambda p_{\lambda' i} e$
$10 \cap 15$	$z_\lambda z_{\lambda'} p_{\lambda'' 1}$	$16 \cap 17$	$z_\lambda p_{\lambda' i} y_j$
$10 \cap 16$	$z_\lambda z_{\lambda'} p_{\lambda'' i}$	$17 \cap 2$	$p_{\lambda i} y_j e$
$12 \cap 17$	$e p_{\lambda i} y_j$	$17 \cap 7$	$p_{\lambda i} y_j y_{j'}$

Let us check one of the above compositions:

$$1 \cap 14 : w = \widetilde{u} p_{1i} y_j,$$
$$\begin{aligned}(f, g)_w &= (u - v) y_j - \widetilde{u}(p_{1i} y_j - p_{1i} p_{1j}) \\ &= u y_j - v y_j - \widetilde{u} p_{1i} y_j + \widetilde{u} p_{1i} p_{1j} = \widetilde{u} p_{1i} p_{1j} - v y_j \\ &= u p_{1j} - v y_j = v p_{1j} - v p_{1j} \equiv 0.\end{aligned}$$

Finally, it remains to check compositions of including of polynomials (5)–(8). But it is clear since there are no compositions of this type.

Hence the result. □

In Theorem 12, we assumed that $|p_{\lambda 1}| = |p_{\lambda' 1}| = |p_{1i}| = |p_{1j}| = 1$ and $|p_{\lambda i}|, |p_{\lambda' i}| \leq 2$. But, if we extend the inequalities given for the lengths of the words $p_{\lambda i}, p_{\lambda' i}$, then we obtain a similar result (such that the its proof can be made quite similar to the proof of Theorem 12) for a Gröbner–Shirshov basis of $S = M^0[A; I, \Lambda; P]$ as in the following.

Theorem 13. *Let $S = M^0[A; I, \Lambda; P]$ be a Rees matrix semigroup, where A is a monoid, P is a $|\Lambda| \times |I|$ matrix with entries from A (as given in Theorem 11). Let $|p_{\lambda 1}| = |p_{\lambda' 1}| = |p_{1i}| = |p_{1j}| = 1$ and $|p_{\lambda i}|, |p_{\lambda' i}| > 2$. Then a Gröbner–Shirshov basis of $S = M^0[A; I, \Lambda; P]$ consists of the relations given in the presentation (5) and the relations:*

$$y_i y_j = y_i p_{1j}, \qquad z_\lambda z_{\lambda'} = p_{\lambda 1} z_{\lambda'}, \qquad p_{1i} e = p_{1i}, \qquad e p_{\lambda 1} = p_{\lambda 1},$$
$$z_\lambda p_{1i} = p_{\lambda 1} y_i, \qquad p_{1i} y_j = p_{1i} p_{1j}, \qquad z_\lambda p_{\lambda' 1} = p_{\lambda 1} p_{\lambda' 1}, \qquad p_{\lambda i} y_j = p_{\lambda i} p_{1j}.$$

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