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APPROXIMATION IN WEIGHTED ORLICZ SPACES

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ABSTRACT. We prove some direct and converse theorems of trigonometric approximation in weighted Orlicz spaces with weights satisfying so called Muckenhoupt's A_p condition.

1. Introduction

A function Φ is called *Young function* if Φ is even, continuous, nonnegative in $\mathbb{R}$, increasing on $(0,\infty)$ such that

$$\Phi(0)=0, \qquad \lim_{x\to\infty}\Phi(x)=\infty.$$

A Young function Φ said to satisfy Δ_2 *condition* ($\Phi\in\Delta_2$) if there is a constant $c_1>0$ such that

$$\Phi(2x)\le c_1\Phi(x) \qquad \text{for all} \quad x\in\mathbb{R}.$$

Two Young functions Φ and Φ_1 are said to be *equivalent* (we shall write $\Phi\sim\Phi_1$) if there are $c_2,c_3>0$ such that

$$\Phi_1(c_2x)\le\Phi(x)\le\Phi_1(c_3x), \qquad \text{for all} \quad x>0.$$

A nonnegative function $M\colon[0,\infty)\to[0,\infty)$ is said to be *quasiconvex* if there exist a convex Young function Φ and a constant $c_4\ge1$ such that

$$\Phi(x)\le M(x)\le\Phi(c_4x), \qquad \text{for all} \quad x\ge0$$

holds.

Let $\mathbb{T}:=[-\pi,\pi]$. A function $\omega\colon T\to[0,\infty]$ will be called *weight* if ω is measurable and almost everywhere (a.e.) positive.

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A 2π-periodic weight function ω belongs to the *Muckenhoupt class* A_p, $p>1$, if

$$\sup_J \left(\frac{1}{|J|}\int_J \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|J|}\int_J \omega^{-1/(p-1)}(x)\,\mathrm{d}x\right)^{p-1} \le C$$

with a finite constant C independent of J, where J is any subinterval of $\mathbb{T}$.

Let M be a quasiconvex Young function. We denote by $\tilde{L}_{M,\omega}(\mathbb{T})$ the class of Lebesgue measurable functions $f\colon \mathbb{T}\to\mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} M\left(|f(x)|\right)\omega(x)\,\mathrm{d}x < \infty.$$

The linear span of the *weighted Orlicz class* $\tilde{L}_{M,\omega}(\mathbb{T})$, denoted by $L_{M,\omega}(\mathbb{T})$, becomes a normed space with the *Orlicz* norm

$$\|f\|_{M,\omega} := \sup\left\{\int_{\mathbb{T}} |f(x)g(x)|\,\omega(x)\,\mathrm{d}x :\ \int_{\mathbb{T}} \tilde{M}\left(|g|\right)\omega(x)\,\mathrm{d}x \le 1\right\},$$

where $\tilde{M}(y) := \sup\limits_{x\ge 0}(xy - M(x))$, $y\ge 0$, is the *complementary function* of M.

If M is quasiconvex and $\tilde{M}$ is its complementary function, then *Young's inequality* holds

$$xy \le M(x) + \tilde{M}(y), \qquad x,y\ge 0. \tag{1.1}$$

For a quasiconvex function M we define the indice $p(M)$ of M as

$$\frac{1}{p(M)} := \inf\{p :\ p>0,\ \ M^p \text{ is quasiconvex}\}$$

and

$$p'(M) := \frac{p(M)}{p(M)-1}.$$

If $\omega\in A_{p(M)}$, then it can be easily seen that $L_{M,\omega}(\mathbb{T})\subset L^1(\mathbb{T})$ and $L_{M,\omega}(\mathbb{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_{M,\omega}(\mathbb{T})$ is called *weighted Orlicz space.*

We define the *Luxemburg* functional as

$$\|f\|_{(M),\omega} := \inf\left\{\tau>0 :\ \int_{\mathbb{T}} M\left(\frac{|f(x)|}{\tau}\right)\omega(x)\,\mathrm{d}x \le 1\right\}.$$

There exist [8, p. 23] constants c, $C>0$ such that

$$c\,\|f\|_{(M),\omega} \le \|f\|_{M,\omega} \le C\,\|f\|_{(M),\omega}\,.$$

Throughout this work by $c, C, c_1, c_2, \ldots$, we denote the constants which are different in different places.

Detailed information about Orlicz spaces, defined with respect to the convex Young function M, can be found in [17]. Orlicz spaces, considered in this work, are investigated in the books [8] and [25].

Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. For $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, we define the *shift operator*

$$\sigma_h(f) := (\sigma_h f)(x) := \frac{1}{2h}\int_{-h}^{h} f(x+t)\,\mathrm{d}t, \qquad 0 < h < \pi, \quad x \in \mathbb{T}$$

and the *modulus of smoothness*

$$\Omega^r_{M,\omega}(f,\delta) := \sup_{\substack{0<h_i\le\delta \\ i=1,2,\dots,r}} \left\| \prod_{i=1}^{r}\left(I-\sigma_{h_i}\right) f \right\|_{M,\omega}, \qquad \delta > 0$$

of order $r = 1, 2, \dots$.

Let

$$E_n(f)_{M,\omega} := \inf_{T\in\mathscr{T}_n} \|f-T\|_{M,\omega}, \qquad f \in L_{M,\omega}(\mathbb{T}), \quad 1<p<\infty, \quad n = 0,1,2,\dots,$$

where $\mathscr{T}_n$ is the class of trigonometrical polynomials of degree not greater than n.

Let also

$$f(x) \backsim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx} = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1.2}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

be the *Fourier* and the *conjugate Fourier series* of $f \in L^1(\mathbb{T})$, respectively. In addition, we put

$$S_n(x,f) := \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \qquad n = 0,1,2,\dots.$$

By $L^1_0(\mathbb{T})$ we denote the class of $L^1(\mathbb{T})$ functions f for which the constant term c_0 in (1.2) equals zero. If $\alpha > 0$, then αth integral of $f \in L^1_0(\mathbb{T})$ is defined as

$$I_\alpha(x,f) \sim \sum_{k\in\mathbb{Z}^*} c_k (\mathrm{i}k)^{-\alpha} \mathrm{e}^{\mathrm{i}kx},$$

where

$$(\mathrm{i}k)^{-\alpha} := |k|^{-\alpha} \mathrm{e}^{(-1/2)\pi\mathrm{i}\alpha\,\mathrm{sign}\,k} \qquad \text{and} \qquad \mathbb{Z}^* := \{\pm1, \pm2, \pm3, \dots\}.$$

It is known [28, V2, p. 134] that $I_\alpha(x,f) \in L^1(\mathbb{T})$ and

$$I_\alpha(x,f) = \sum_{k\in\mathbb{Z}^*} c_k (\mathrm{i}k)^{-\alpha} \mathrm{e}^{\mathrm{i}kx} \qquad \text{a.e. on} \quad \mathbb{T}.$$

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{\mathrm{d}}{\mathrm{d}x} I_{1-\alpha}(x,f),$$
$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{\mathrm{d}^{r+1}}{\mathrm{d}x^{r+1}} I_{1-\alpha}(x,f)$$

if the right hand sides exist, where $r \in \mathbb{Z}^+ := \{1,2,3,\dots\}$.

In this work we investigate the direct and inverse problems of approximation theory in the weighted Orlicz spaces $L_{M,\omega}(\mathbb{T})$. In the literature many results on such approximation problems have been obtained in weighted and nonweighted Lebesque spaces. The corresponding results in the nonweighted Lebesgue spaces $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, can be found in the books [3] and [27]. The best approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class $A_p(\mathbb{T})$ were investigated in [9] and [18]. Detailed information on weighted polynomial approximation can be found in the books [5] and [22].

For more general doubling weights, approximation by trigonometric polynomials and other related problems in the weighted Lebesgue spaces were studied in [2], [20], [19] and [21]. Some interesting results concerning best polynomial approximation in weighted Lebesgue spaces were also proved in [4] and [6].

Direct problems in nonweighted Orlicz spaces, defined with respect to the convex Young function M, were studied in [24], [7] and [26]. In the weighted case, when the weighted Orlicz classes are defined as the subclass of the measurable functions on $\mathbb{T}$ satisfying the condition

$$\int_{\mathbb{T}} M\big(|f(x)|\,\omega(x)\big)\,\mathrm{d}x < \infty,$$

some direct and inverse theorems of approximation theory were obtained in [13].

Some generalizations of these results to the weighted Lebesgue and Orlicz spaces defined on the various sets of complex plane, were proved in [15], [16], [10], [11] and [12].

Since every convex function is quasiconvex, the Orlicz spaces considered by us in this work are more general than the Orlicz spaces studied in the above mentioned works. Therefore, the results obtained in this paper are new also in the nonweighted cases.

2. Main results

Let $W^{\alpha}_{M,\omega}(\mathbb{T})$, $\alpha > 0$, be the class of functions $f \in L_{M,\omega}(\mathbb{T})$ such that $f^{(\alpha)} \in L_{M,\omega}(\mathbb{T})$. $W^{\alpha}_{M,\omega}(\mathbb{T})$ becomes a Banach space with the norm

$$\|f\|_{W^{\alpha}_{M,\omega}(\mathbb{T})} := \|f\|_{M,\omega} + \|f^{(\alpha)}\|_{M,\omega}.$$

Main results of this work are following.

Theorem 1. *Let $M \in \triangle_2$, M^{θ} is quasiconvex for some $\theta \in (0,1)$ and let $\omega \in A_{p(M)}$. Then for every $f \in W^{\alpha}_{M,\omega}(\mathbb{T})$, $\alpha > 0$, the inequality*

$$E_n(f)_{M,\omega} \le \frac{c_5}{(n+1)^{\alpha}} E_n\left(f^{(\alpha)}\right)_{M,\omega}, \qquad n = 0, 1, 2, \dots$$

holds with some constant $c_5 > 0$ independent of n.

Corollary 1. *Under the conditions of* Theorem 1 *the inequality*

$$E_n(f)_{M,\omega} \le \frac{c_6}{(n+1)^{\alpha}} \|f^{(\alpha)}\|_{M,\omega}, \qquad n = 0, 1, 2, 3, \dots$$

holds with a constant $c_6 > 0$ independent of n.

Theorem 2. *Let $M \in \triangle_2$, M^{θ} is quasiconvex for some $\theta \in (0,1)$ and let $\omega \in A_{p(M)}$. Then for $f \in L_{M,\omega}(\mathbb{T})$ and for every natural number n the estimate*

$$E_n(f)_{M,\omega} \le c_7 \Omega^r_{M,\omega}\left(f, \frac{1}{n+1}\right), \qquad r = 1, 2, 3, \dots$$

holds with a constant $c_7 > 0$ independent of n.

Definition 1. Let $\mathbb{D}$ be unit disc in the complex plane and let T be the unit circle. For a weight ω given on T, we set

$$H_{M,\omega}(\mathbb{D}) := \left\{ f \in H^1(\mathbb{D}) : f \in L_{M,\omega}(\mathsf{T}) \right\}.$$

The class of functions $H_{M,\omega}(\mathbb{D})$ will be called weighted Hardy-Orlicz space.

The direct theorem of polynomial approximation in the space $H_{M,\omega}(\mathbb{D})$ is formulated as following.

Theorem 3. *Let $M \in \triangle_2$, M^{θ} is quasiconvex for some $\theta \in (0,1)$ and let $\omega \in A_{p(M)}$. Then for $f \in H_{M,\omega}(\mathbb{D})$ and for every natural number n there exists a constant $c_8 > 0$ independent of n such that*

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{M,\omega} \le c_8 \Omega^r_{M,\omega}\left(f, \frac{1}{n+1}\right), \qquad r = 1, 2, \dots,$$

where $a_k(f)$, $k = 0, 1, 2, 3, \dots$, are the Taylor coefficients of f at the origin.

The following inverse results hold.

Theorem 4. *Let $M \in \triangle_2$, M^θ is quasiconvex for some $\theta \in (0,1)$ and let $\omega \in A_{p(M)}$. Then for $f \in L_{M,\omega}(\mathbb{T})$ and for every natural number n the estimate*

$$\Omega^r_{M,\omega}\left(f,\frac{1}{n}\right) \leq \frac{c_9}{n^{2r}}\left\{E_0(f)_{M,\omega} + \sum_{k=1}^{n} k^{2r-1}E_k(f)_{M,\omega}\right\}, \qquad r = 1,2,\ldots$$

holds with a constant $c_9 > 0$ independent of n.

Theorem 5. *Under the conditions of* Theorem 4 *if*

$$\sum_{k=0}^{\infty} k^{2r-1}E_k(f)_{M,\omega} < \infty$$

for some $r = 1,2,3,\ldots$, then $f \in W^{2r}_{M,\omega}(\mathbb{T})$.

In particular we have the following corollary.

Corollary 2. *Under the conditions of* Theorem 4 *if*

$$E_n(f)_{M,\omega} = \mathscr{O}\left(n^{-\beta}\right), \qquad n = 1,2,3,\ldots,$$

for some $\beta > 0$, then for a given $r = 1,2,3,\ldots$, we have

$$\Omega^r_{M,\omega}(f,\delta) = \begin{cases} \mathscr{O}\left(\delta^\beta\right), & r > \beta/2; \\ \mathscr{O}\left(\delta^\beta \log\frac{1}{\delta}\right), & r = \beta/2; \\ \mathscr{O}\left(\delta^{2r}\right), & r < \beta/2. \end{cases}$$

Definition 2. Let $\beta > 0$, $r := [\beta/2] + 1$ and let

$$\operatorname{Lip}\beta(M,\omega) := \left\{f \in L_{M,\omega}(\mathbb{T}) :\ \Omega^r_{M,\omega}(f,\delta) = \mathscr{O}\left(\delta^\beta\right),\ \delta > 0\right\}.$$

Corollary 3. *Under the conditions of* Theorem 4 *if*

$$E_n(f)_{M,\omega} = \mathscr{O}\left(n^{-\beta}\right), \qquad n = 1,2,3,\ldots,$$

for some $\beta > 0$, then $f \in \operatorname{Lip}\beta(M,\omega)$.

The following result gives a constructive description of the classes $\operatorname{Lip}\beta(M,\omega)$.

Theorem 6. *Let $\beta > 0$. Under the conditions of* Theorem 4 *the following conditions are equivalent*

(i) $f \in \operatorname{Lip}\beta(M,\omega)$
(ii) $E_n(f)_{M,\omega} = \mathscr{O}\left(n^{-\beta}\right)$, $n = 1,2,3,\ldots$.

3. Some auxiliary results

We need the following interpolation lemma that was proved in the more general case in [8, Lemma 7.4.1, p. 310], for an additive opertor T, mapping the measure space (Y_0, S_0, υ_0) into measure space (Y_1, S_1, υ_1).

Lemma 1. *Let M be a quasiconvex Young function and let $1 \le r < p(M) \le p'(\tilde{M}) < s < \infty$. If there exist the constants $c_{10}, c_{11} > 0$ such that for all $f \in L^r + L^s$*

$$\int\limits_{\{y \in Y_1\colon\ |Tf(x)| > \lambda\}} \mathrm{d}\nu_1 \le c_{10} \lambda^{-r} \int\limits_{Y_0} |f(x)|^r \,\mathrm{d}\nu_0$$

and

$$\int\limits_{\{y \in Y_1\colon\ |Tf(x)| > \lambda\}} \mathrm{d}\nu_1 \le c_{11} \lambda^{-s} \int\limits_{Y_0} |f(x)|^s \,\mathrm{d}\nu_0,$$

then

$$\int\limits_{Y_1} M\left(Tf(x)\right) \mathrm{d}\nu_1 \le c_{12} \int\limits_{Y_0} M\left(f(x)\right) \mathrm{d}\nu_0$$

with a constant $c_{12} > 0$.

Lemma 2. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, then there is a constant $c_{13} > 0$ such that*

$$\|\sigma_h f\|_{M,\omega} \le c_{13} \|f\|_{M,\omega}.$$

P r o o f. Using [8, Lemma 6.1.6 (1), (3), p. 215]; [8, Lemma 6.1.1 (1), (3), p. 211] and $\tilde{\tilde{M}} \le M$ we have $\tilde{\tilde{M}} \sim M$. Then $\tilde{\tilde{M}} \in \triangle_2$ because $M \in \triangle_2$. On the other hand since M^θ is quasiconvex for some $\theta \in (0,1)$, by [8, Lemma 6.1.6, p. 215] we have that M is quasiconvex, which implies that $\widetilde{M}$ is also quasiconvex. This property together with the relation $\tilde{\tilde{M}} \in \triangle_2$ is equivalent to the quasiconvexity of $\widetilde{M}^\beta$ for some $\beta \in (0,1)$, by [8, Lemma 6.1.6, p. 215]. Therefore by definition we have $p'(M)$, $p'(\widetilde{M}) < \infty$. Hence, we can choose the numbers r, s such that $p'(\widetilde{M}) < s < \infty$, $r < p(M)$ and $\omega \in A_r$. On the other hand from the boundedness [23] of the operator σ_h in $L^p(\mathbb{T}, \omega)$, in case of $\omega \in A_p$ $(1 < p < \infty)$, implies that σ_h is weak types (r,r) and (s,s). Hence, choosing $Y_0 = Y_1 = \mathbb{T}$, $S_0 = S_1 = B$ (B is Borel σ-algebra), $\mathrm{d}\upsilon_0 = \mathrm{d}\upsilon_1 = \omega(x)\,\mathrm{d}x$ and applying Lemma 1 we have

$$\int\limits_{\mathbb{T}} M\left(|\sigma_h f(x)|\right) \omega(x)\,\mathrm{d}x \le c_{14} \int\limits_{\mathbb{T}} M\left(|f(x)|\right) \omega\left(x\right)\,\mathrm{d}x \tag{3.1}$$

for some constant $c_{14} > 1$.

Since M is quasiconvex, we have

$$M(\alpha x) \leq \Phi(\alpha c_3 x) \leq \alpha \Phi(c_3 x) \leq \alpha M(c_3 x), \qquad \alpha \in (0,1),$$

for some convex Young function Φ and constant $c_3 \geq 1$. Using this inequality in (3.1) for $f := f/\lambda$, $\lambda > 0$, we obtain the inequality

$$\int_{\mathbb{T}} M\left(\frac{\left|\sigma_h\left(\frac{f}{c_3 c_{14}}\right)(x)\right|}{\lambda}\right)\omega(x)\,\mathrm{d}x \leq \int_{\mathbb{T}} M\left(\frac{|f(x)|}{\lambda}\right)\omega(x)\,\mathrm{d}x,$$

which implies that

$$\|\sigma_h f\|_{(M),\omega} \leq c_3 c_{14} \|f\|_{(M),\omega}.$$

The last relation is equivalent to the required inequality

$$\|\sigma_h f\|_{M,\omega} \leq c_{15} \|f\|_{M,\omega}$$

with some constant $c_{15} > 0$. □

By the similar way we have the following result.

Lemma 3. *Under the conditions of* Lemma 2 *we have*

(i) $\|\tilde{f}\|_{M,\omega} \leq c_{16} \|f\|_{M,\omega}$,
(ii) $\|S_n(\cdot, f)\|_{M,\omega} \leq c_{17} \|f\|_{M,\omega}$.

From this Lemma we obtain the estimations:

$$\|f - S_n(\cdot, f)\|_{M,\omega} \leq c_{18} E_n(f)_{M,\omega}, \tag{3.2}$$
$$E_n(\tilde{f})_{M,\omega} \leq c_{19} E_n(f)_{M,\omega}.$$

Corollary 4. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, then*

$$\lim_{h \to 0^+} \|f - \sigma_h f\|_{M,\omega} = 0,$$
$$\lim_{\delta \to 0^+} \Omega^r_{M,\omega}(f, \delta) = 0, \qquad r = 1, 2, 3, \ldots,$$

and

$$\Omega^r_{M,\omega}(f, \delta) \leq c_{20} \|f\|_{M,\omega}$$

with some constant $c_{20} > 0$ independent of f.

Remark 1. The modulus of smoothness $\Omega^r_{M,\omega}(f, \delta)$ has the following properties:

(i) $\Omega^r_{M,\omega}(f, \delta)$ is non-negative and non-decreasing function of $\delta \geq 0$.
(ii) $\Omega^r_{M,\omega}(f_1 + f_2, \cdot) \leq \Omega^r_{M,\omega}(f_1, \cdot) + \Omega^r_{M,\omega}(f_2, \cdot)$.

The proof of the following result is similar to that of Lemma 2.

Lemma 4. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, then*

$$\left\| K_n(\cdot, f) \right\|_{M,\omega} \leq c_{21} \left\| f \right\|_{M,\omega}, \qquad n = 0,1,2,3,\ldots,$$

where

$$K_n(x,f) := \frac{1}{n+1} \big\{ S_0(x,f) + S_1(x,f) + \cdots + S_n(x,f) \big\}.$$

Lemma 5. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $\omega \in A_{p(M)}$, then*

$$\big\| T_n^{(r)} \big\|_{M,\omega} \leq c_{22} n^r \left\| T_n \right\|_{M,\omega}, \qquad T_n \in \mathscr{T}_n, \ \ r = 1,2,3,\ldots$$

P r o o f. Since [1, p. 99]

$$T_n'(x) = \frac{1}{\pi} \int_{\mathbb{T}} T_n(x+u)\, n \sin nu F_{n-1}(u) \, \mathrm{d}u,$$

where

$$F_n(u) := \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{1}{2} + \sum_{j=1}^{k} \cos ju \right)$$

being Fejer's kernel, we get

$$\left| T_n'(x) \right| \leq 2n K_{n-1}(x, |T_n|).$$

Now, taking into account Lemma 4 we conclude the required result. □

Lemma 6. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, then*

$$\Omega_{M,\omega}^r(f,\delta) \leq c_{23} \delta^2 \Omega_{M,\omega}^{r-1}(f'', \delta), \qquad r = 1,2,3,\ldots$$

with some constant $c_{23} > 0$.

P r o o f. We follow the procedure used in the proof of [13, Lemma 5]. Putting

$$g(x) := \prod_{i=2}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

we have

$$\left(I - \sigma_{h_1} \right) g(x) = \prod_{i=1}^{r} \left(I - \sigma_{h_i} \right) f(x)$$

and

$$\prod_{i=1}^{r}\left(I-\sigma_{h_i}\right)f(x)=\frac{1}{2h_1}\int_{-h_1}^{h_1}\big(g(x)-g\left(x+t\right)\big)\,\mathrm{d}t$$
$$=-\frac{1}{8h_1}\int_{0}^{h_1}\int_{0}^{t}\int_{-u}^{u}g''\left(x+s\right)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t.$$

Therefore

$$\left\|\prod_{i=1}^{r}(I-\sigma_{h_i})\,f\left(x\right)\right\|_{M,\omega}=\frac{1}{8h_1}\sup\left\{\int_{\mathbb{T}}\left|\int_0^{h_1}\int_0^{t}\int_{-u}^{u}g''\left(x+s\right)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t\right|\left|v(x)\right|\omega(x)\,\mathrm{d}x:\right.$$
$$\left.\int_{\mathbb{T}}\tilde{M}\left(\left|v(x)\right|\right)\omega(x)\,\mathrm{d}x\leq 1\right\}$$
$$\leq\frac{1}{8h_1}\int_0^{h_1}\int_0^{t}2u\left\|\frac{1}{2u}\int_{-u}^{u}g''\left(x+s\right)\,\mathrm{d}s\right\|_{M,\omega}\mathrm{d}u\,\mathrm{d}t$$
$$\leq\frac{c_{13}}{8h_1}\int_0^{h_1}\int_0^{t}2u\left\|g''\right\|_{M,\omega}\,\mathrm{d}u\,\mathrm{d}t=c_{24}h_1^2\left\|g''\right\|_{M,\omega}.$$

Since

$$g''(x)=\prod_{i=2}^{r}\left(I-\sigma_{h_i}\right)f''(x),$$

we obtain that

$$\begin{aligned}\Omega_{M,\omega}^{r}\left(f,\delta\right) &\leq \sup_{\substack{0<h_i\leq\delta\\ i=1,2,\dots,r}} c_{24}h_1^2\left\|g''\right\|_{M,\omega}\\ &\leq c_{24}\delta^2\sup_{\substack{0<h_i\leq\delta\\ i=2,\dots,r}}\left\|\prod_{i=2}^{r}\left(I-\sigma_{h_i}\right)f''(x)\right\|_{M,\omega}\\ &= c_{24}\delta^2\Omega_{M,\omega}^{r-1}\left(f'',\delta\right)\end{aligned}$$

and Lemma is proved. □

Corollary 5. *Let $M\in\triangle_2$ and M^{θ} is quasiconvex for some $\theta\in(0,1)$. If $f\in L_{M,\omega}\left(\mathbb{T}\right)$ with $\omega\in A_{p(M)}$, then, with some constant $c_{25}>0$,*

$$\Omega_{M,\omega}^{r}\left(f,\delta\right)\leq c_{25}\delta^{2r}\left\|f^{(2r)}\right\|_{M,\omega},\qquad r=1,2,3,\dots.$$

Definition 3. For $f \in L_{M,\omega}(\mathbb{T})$, $\delta > 0$ and $r = 1, 2, 3, \dots$, the Peetre K-functional is defined as

$$K\left(\delta, f; L_{M,\omega}(\mathbb{T}), W^r_{M,\omega}(\mathbb{T})\right) := \inf_{g \in W^r_{M,\omega}(\mathbb{T})} \left\{ \|f - g\|_{M,\omega} + \delta \left\|g^{(r)}\right\|_{M,\omega} \right\}. \quad (3.3)$$

Theorem 7. *Let $M \in \triangle_2$ and M^θ is quasiconvex for some $\theta \in (0,1)$. If $f \in L_{M,\omega}(\mathbb{T})$ with $\omega \in A_{p(M)}$, then the K-functional $K\left(\delta^{2r}, f; L_{M,\omega}(\mathbb{T}), W^{2r}_{M,\omega}(\mathbb{T})\right)$ and the modulus $\Omega^r_{M,\omega}(f, \delta)$, $r = 1, 2, 3, \dots$, are equivalent.*

Proof. If $h \in W^{2r}_{M,\omega}(\mathbb{T})$, then

$$\begin{aligned} \Omega^r_{M,\omega}(f,\delta) &\le c_{20}\|f-h\|_{M,\omega} + c_{25}\delta^{2r}\left\|h^{(2r)}\right\|_{M,\omega} \\ &\le c_{26} K\left(\delta^{2r}, f; L_{M,\omega}(\mathbb{T}), W^{2r}_{M,\omega}(\mathbb{T})\right). \end{aligned}$$

Putting

$$(L_\delta f)(x) := 3\delta^{-3} \int_0^\delta \int_0^u \int_{-t}^t f(x+s)\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}u, \qquad x \in \mathbb{T},$$

we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} L_\delta f = \frac{c_{27}}{\delta^2}(I - \sigma_\delta) f$$

and hence

$$\frac{\mathrm{d}^{2r}}{\mathrm{d}x^{2r}} L^r_\delta f = \frac{c_{28}}{\delta^{2r}}(I - \sigma_\delta)^r, \qquad r = 1, 2, 3, \dots.$$

On the other hand we find

$$\|L_\delta f\|_{M,\omega} \le 3\delta^{-3} \int_0^\delta \int_0^u 2t \|\sigma_t f\|_{M,\omega}\, \mathrm{d}t\, \mathrm{d}u \le c_{29}\|f\|_{M,\omega}.$$

Now let $A^r_\delta := I - (I - L^r_\delta)^r$. Then $A^r_\delta f \in W^{2r}_{M,\omega}(\mathbb{T})$ and

$$\left\|\frac{\mathrm{d}^{2r}}{\mathrm{d}x^{2r}} A^r_\delta f\right\|_{M,\omega} \le c_{30}\left\|\frac{\mathrm{d}}{\mathrm{d}x^{2r}} L^r_\delta f\right\|_{M,\omega} = \frac{c_{31}}{\delta^{2r}}\|(I-\sigma_\delta)^r\|_{M,\omega} \le \frac{c_{31}}{\delta^{2r}}\Omega^r_{M,\omega}(f,\delta).$$

Since $I - L^r_\delta = (I - L_\delta)\sum_{j=0}^{r-1} L^j_\delta$, we get

$$\begin{aligned} \|(I - L^r_\delta) g\|_{M,\omega} &\le c_{32}\|(I - L_\delta) g\|_{M,\omega} \\ &\le 3c_{33}\delta^{-3}\int_0^\delta\int_0^u 2t\|(I-\sigma_t) g\|_{M,\omega}\, \mathrm{d}t\, \mathrm{d}u \\ &\le c_{34}\sup_{0<t\le\delta}\|(I-\sigma_t) g\|_{M,\omega}. \end{aligned}$$

Taking into account the equality

$$\|f - A_\delta^r f\|_{M,\omega} = \|(I - L_\delta^r)^r f\|_{M,\omega},$$

by a recursive procedure we obtain

$$\begin{aligned}\|f - A_\delta^r f\|_{M,\omega} &\le c_{34} \sup_{0<t_1\le\delta} \left\|(I-\sigma_{t_1})(I-L_\delta^r)^{r-1} f\right\|_{M,\omega} \\ &\le c_{35} \sup_{0<t_1\le\delta} \sup_{0<t_2\le\delta} \left\|(I-\sigma_{t_1})(I-\sigma_{t_2})(I-L_\delta^r)^{r-2} f\right\|_{M,\omega} \\ &\le \cdots \le c_{36} \sup_{\substack{0<t_i\le\delta \\ i=1,2,\ldots,r}} \left\| \prod_{i=1}^{r} (I-\sigma_{t_i}) f(x)\right\|_{M,\omega} = c_{36}\Omega_{M,\omega}^r(f,\delta)\end{aligned}$$

and the proof is completed. □

4. Proofs of the main results

Proof of Theorem 1. We set

$$A_k(x,f) := a_k \cos kx + b_k \sin kx.$$

For given $f \in L_{M,\omega}(\mathbb{T})$ and $\varepsilon > 0$, by [14, Lemma 3] there exists a trigonometric polynomial T such that

$$\int_{\mathbb{T}} M\big(|f(x) - T(x)|\big)\omega(x)\,\mathrm{d}x < \varepsilon$$

which by (1.1) implies that

$$\|f - T\|_{M,\omega} < \varepsilon$$

and hence we obtain

$$E_n(f)_{M,\omega} \to 0, \qquad \text{as} \quad n \to \infty.$$

On the other hand, from (3.2) we have

$$\left\| f(x) - \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)\right\|_{M,\omega} \le c_{18} E_n(f)_{M,\omega}$$

and therefore,

$$f(x) = \sum_{k=0}^{\infty} A_k(x,f)$$

in $\|\cdot\|_{M,\omega}$ norm. For $k = 1,2,3,\ldots$ we find that

$$A_k(x,f) \;=\; a_k \cos k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right) + b_k \sin k\left(x + \frac{\alpha\pi}{2} - \frac{\alpha\pi}{2}\right)$$

$$\begin{aligned} &= \cos\frac{\alpha\pi}{2}\left[a_k\cos k\left(x+\frac{\alpha\pi}{2k}\right)+b_k\sin k\left(x+\frac{\alpha\pi}{2k}\right)\right] \\ &\quad +\sin\frac{\alpha\pi}{2}\left[a_k\sin k\left(x+\frac{\alpha\pi}{2k}\right)-b_k\cos k\left(x+\frac{\alpha\pi}{2k}\right)\right] \\ &= A_k\left(x+\frac{\alpha\pi}{2k},f\right)\cos\frac{\alpha\pi}{2}+A_k\left(x+\frac{\alpha\pi}{2k},\tilde{f}\right)\sin\frac{\alpha\pi}{2}. \end{aligned}$$

After simple computations we have

$$A_k\left(x,f^{(\alpha)}\right)=k^\alpha A_k\left(x+\frac{\alpha\pi}{2k},f\right)$$

and then

$$\begin{aligned} &\sum_{k=0}^{\infty}A_k\left(x,f\right) \\ &=A_0\left(x,f\right)+\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}A_k\left(x+\frac{\alpha\pi}{2k},f\right)+\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}A_k\left(x+\frac{\alpha\pi}{2k},\tilde{f}\right) \\ &=A_0\left(x,f\right)+\cos\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{-\alpha}A_k\left(x,f^{(\alpha)}\right)+\sin\frac{\alpha\pi}{2}\sum_{k=1}^{\infty}k^{-\alpha}A_k\left(x,\tilde{f}^{(\alpha)}\right). \end{aligned}$$

Therefore,

$$f(x)-S_n\left(x,f\right)=\cos\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha}A_k\left(x,f^{(\alpha)}\right)+\sin\frac{\alpha\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha}A_k\left(x,\tilde{f}^{(\alpha)}\right).$$

Since

$$\begin{aligned} &\sum_{k=n+1}^{\infty}k^{-\alpha}A_k\left(x,f^{(\alpha)}\right) \\ &=\sum_{k=n+1}^{\infty}k^{-\alpha}\left[\left(S_k\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}\left(\cdot\right)\right)-\left(S_{k-1}\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}\left(\cdot\right)\right)\right] \\ &=\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left(S_k\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}\left(\cdot\right)\right) \\ &\qquad -(n+1)^{-\alpha}\left(S_n\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}\left(\cdot\right)\right) \end{aligned}$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty}k^{-\alpha}A_k\left(x,\tilde{f}^{(\alpha)}\right)&=\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left(S_k\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}\left(\cdot\right)\right) \\ &\qquad -(n+1)^{-\alpha}\left(S_n\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}\left(\cdot\right)\right), \end{aligned}$$

we obtain

$$
\begin{aligned}
&\|f(x)-S_n(x,f)\|_{M,\omega} \\
&\le \sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left\|S_k\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+(n+1)^{-\alpha}\left\|S_n\left(\cdot,f^{(\alpha)}\right)-f^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)\left\|S_k\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\quad+(n+1)^{-\alpha}\left\|S_n\left(\cdot,\tilde{f}^{(\alpha)}\right)-\tilde{f}^{(\alpha)}(\cdot)\right\|_{M,\omega} \\
&\le c_{18}\Bigg[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)E_k(f)_{M,\omega}+(n+1)^{-\alpha}E_n\left(f^{(\alpha)}\right)_{M,\omega}\Bigg] \\
&\quad+c_{18}\Bigg[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)E_k(f)_{M,\omega}+(n+1)^{-\alpha}E_n\left(\tilde{f}^{(\alpha)}\right)_{M,\omega}\Bigg].
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\|f(x)-S_n(x,f)\|_{M,\omega} \\
&\le c_{37}E_k\left(f^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\qquad+c_{38}E_n\left(\tilde{f}^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\le c_{39}E_n\left(f^{(\alpha)}\right)_{M,\omega}\left[\sum_{k=n+1}^{\infty}\left(k^{-\alpha}-(k+1)^{-\alpha}\right)+(n+1)^{-\alpha}\right] \\
&\le \frac{c_{40}}{(n+1)^{\alpha}}E_n\left(f^{(\alpha)}\right)_{M,\omega}
\end{aligned}
$$

and Theorem is proved. □

Proof of Theorem 2. For $g \in W^{2r}_{M,\omega}(\mathbb{T})$ we have by Corollary 1, (3.3) and Theorem 7

$$
\begin{aligned}
E_n(f)_{M,\omega} &\le E_n(f-g)_{M,\omega}+E_n(g)_{M,\omega} \\
&\le c_{41}\left[\|f-g\|_{M,\omega}+(n+1)^{-2k}\left\|g^{(2k)}\right\|_{M,\omega}\right]
\end{aligned}
$$

$$\leq c_{42} K\left((n+1)^{-2k}, f; L_{M,\omega}(\mathbb{T}), W^{2r}_{M,\omega}(\mathbb{T})\right)$$
$$\leq c_{43} \Omega^r_{M,\omega}\left(f, \frac{1}{n+1}\right)$$

as required. □

Proof of Theorem 3. Let $\sum\limits_{k=-\infty}^{\infty} \beta_k e^{ik\theta}$ be the Fourier series of the function $g \in H_{M,\omega}(\mathbb{D})$, and $S_n(g,\theta) := \sum\limits_{k=-n}^{n} \beta_k e^{ik\theta}$ be its nth partial sum. Since $g \in H^1(\mathbb{D})$, we have

$$\beta_k = \begin{cases} 0, & \text{for } k < 0; \\ \alpha_k(f), & \text{for } k \geq 0. \end{cases}$$

Therefore,

$$\left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{M,\omega} = \left\| f - S_n(f,\cdot) \right\|_{M,\omega}. \tag{4.1}$$

If t_n^* is the best approximating trigonometric polynomial for f in $L_{M,\omega}(\mathbb{T})$, then from (4.1) we get

$$\begin{aligned} \left\| f(z) - \sum_{k=0}^{n} a_k(f) z^k \right\|_{M,\omega} &\leq \left\| f - t_n^* \right\|_{M,\omega} + \left\| S_n(f - t_n^*, \cdot) \right\|_{M,\omega} \\ &\leq c_{44} E_n(f)_{M,\omega} \leq c_{45} \Omega^r_{M,\omega}\left(f, \frac{1}{n+1}\right) \end{aligned}$$

and the proof of theorem is completed. □

Proof of Theorem 4. By Remark 1 (ii), Corollary 4 and (3.2) we have

$$\Omega^r_{M,\omega}(f,\delta) \leq \Omega^r_{M,\omega}(f - T_{2^{m+1}}, \delta) + \Omega^r_{M,\omega}(T_{2^m+1}, \delta) \tag{4.2}$$

and

$$\Omega^r_{M,\omega}(f - T_{2^{m+1}}, \delta) \leq c_{20} \left\| f - T_{2^{m+1}} \right\|_{M,\omega} \leq c_{46} E_{2^{m+1}}(f)_{M,\omega}. \tag{4.3}$$

By Corollary 5

$$\begin{aligned} \Omega^r_{M,\omega}(T_{2^m+1}, \delta) &\leq c_{25} \delta^{2r} \left\| T^{(2r)}_{2^{m+1}} \right\|_{M,\omega} \\ &\leq c_{25} \delta^{2r} \left\{ \left\| T_1^{(2r)} - T_0^{(2r)} \right\|_{M,\omega} + \sum_{i=1}^{m} \left\| T^{(2r)}_{2^{i+1}} - T^{(2r)}_{2^i} \right\|_{M,\omega} \right\} \\ &\leq c_{47} \delta^{2r} \left\{ E_0(f)_{M,\omega} + \sum_{i=1}^{m} 2^{(i+1)2r} E_{2^i}(f)_{M,\omega} \right\} \end{aligned}$$

$$\leq c_{47}\delta^{2r}\left\{E_0(f)_{M,\omega}+2^{2r}E_1(f)_{M,\omega}+\sum_{i=1}^{m}2^{(i+1)2r}E_{2^i}(f)_{M,\omega}\right\}.$$

Applying here the inequality

$$2^{(i+1)2r}E_{2^i}(f)_{M,\omega}\leq 2^{4r}\sum_{k=2^{i-1}+1}^{2^m}k^{2r-1}E_k(f)_{M,\omega},\qquad i\geq 1 \tag{4.4}$$

we get

$$\Omega^r_{M,\omega}\left(T_{2^m+1},\delta\right)\leq c_{48}\delta^{2r}\left\{E_0(f)_{M,\omega}+2^{2r}E_1(f)_{M,\omega}+2^{4r}\sum_{k=2}^{2^m}k^{2r-1}E_k(f)_{M,\omega}\right\}$$
$$\leq c_{49}\delta^{2r}\left\{E_0(f)_{M,\omega}+\sum_{k=1}^{2^m}k^{2r-1}E_k(f)_{M,\omega}\right\}. \tag{4.5}$$

Since

$$E_{2^{m+1}}(f)_{M,\omega}\leq\frac{2^{4r}}{n^{2r}}\sum_{k=2^{m-1}+1}^{2^m}k^{2r-1}E_k(f)_{M,\omega},$$

choosing m as $2^m\leq n\leq 2^{m+1}$, from (4.2)-(4.5) we get the required relation. □

P r o o f o f T h e o r e m 5. If T_n is the best approximating trigonometric polynomial of f, then we have

$$\begin{aligned}\|T_{2^{m+1}}-T_{2^m}\|_{M,\omega} &\leq E_{2^{m+1}}(f)_{M,\omega}+E_{2^m}(f)_{M,\omega}\\ &\leq 2E_{2^m}(f)_{M,\omega}\leq 2^{(m+1)2r}E_{2^m}(f)_{M,\omega},\end{aligned}$$

which by Lemma 5 implies that

$$\left\|T^{(2r)}_{2^{m+1}}-T^{(2r)}_{2^m}\right\|_{M,\omega}\leq c_{22}2^{(m+1)2r}E_{2^m}\left(f\right)_{M,\omega}$$

and hence by the inequality (4.4)

$$\begin{aligned}&\sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{W^{2r}_{M,\omega}(\mathbb{T})}\\ &=\sum_{m=1}^{\infty}\|T_{2^{m+1}}-T_{2^m}\|_{M,\omega}+\sum_{m=1}^{\infty}\left\|T^{(2r)}_{2^{m+1}}-T^{(2r)}_{2^m}\right\|_{M,\omega}\\ &\leq\sum_{m=1}^{\infty}2^{(m+1)2r}E_{2^m}\left(f\right)_{M,\omega}+c_{22}\sum_{m=1}^{\infty}2^{(m+1)2r}E_{2^m}(f)_{M,\omega}\\ &=c_{50}\sum_{m=1}^{\infty}2^{(m+1)2r}E_{2^m}\left(f\right)_{M,\omega}\end{aligned}$$

$$\leq c_{51} 2^{4r} \sum_{m=1}^{\infty} \sum_{j=2^{m-1}+1}^{2^m} j^{2r-1} E_j(f)_{M,\omega}$$
$$= c_{52} \sum_{m=2}^{\infty} j^{2r-1} E_j(f)_{M,\omega} < \infty.$$

Therefore,

$$\sum_{m=1}^{\infty} \|T_{2^{m+1}} - T_{2^m}\|_{W^{2r}_{M,\omega}(\mathbb{T})} < \infty,$$

which implies that $\{T_{2^m}\}$ is a Cauchy sequence in $W^{2r}_{M,\omega}(\mathbb{T})$. Since $T_{2^m} \to f$ in $L_{M,\omega}(\mathbb{T})$, we have $f \in W^{2r}_{M,\omega}(\mathbb{T})$. □

REFERENCES

[1] BUTZER, P. L.—NESSEL, R. J.: *Fourier Analysis and Approximation. Vol. 1: One-dimensional Theory*, Birkhauser, Basel-Stuttgart, 1971.

[2] DE BONIS, M. C.—MASTROIANNI, G.—RUSSO, M. G.: *Polynomial approximation with special doubling weights*, Acta Sci. Math. (Szeged) **69** (2003), 159–184.

[3] DE VORE, R. A.—LORENTZ, G. G.: *Constructive approximation*. Grundlehren Math. Wiss. 303, Springer, Berlin, 1993.

[4] DITZIAN, Z.—TOTIK, V.: *K-functionals and best polynomial approximation in weighted $L^p(R)$*, J. Approx. Theory **46** (1986), 38–41.

[5] DITZIAN, Z.—TOTIK, V.: *Moduli of Smoothness*. Springer Ser. Comput. Math. 9, Springer, New York, 1987.

[6] DITZIAN, Z.—TOTIK, V.: *K-functionals and weighted moduli of smoothness*, J. Approx. Theory 63 (1990), 3–29.

[7] GARIDI, W.: *On approximation by polynomials in Orlicz spaces*, Approx. Theory Appl. (N.S.) **7** (1991), 97–110.

[8] GENEBASHVILI, I.—GOGATISHVILI, A.—KOKILASHVILI, V. M.—KRBEC, M.: *Weight theory for integral transforms on spaces of homogeneous type*, Addison Wesley Longman, Harlow, 1998.

[9] HACIYEVA, E. A.: *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces*. Author's summary of Dissertation, Tbilisi, 1986 (Russian).

[10] ISRAFILOV, D. M.: *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials*, Constr. Approx. **17** (2001), 335–351.

[11] ISRAFILOV, D. M.: *Approximation by p-Faber-Laurent rational functions in the weighted Lebesgue spaces*, Czechoslovak Math. J. **54** (2004), 751–765.

[12] ISRAFILOV, D. M.—AKGÜN, R.: *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46** (2006), 755–770.

[13] ISRAFILOV, D. M.—GUVEN, A.: *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Math. **174** (2006), 147–168.

[14] KHABAZI, M.: *The mean convergence of trigonometric Fourier series in weighted orlicz classes*, Proc. A. Razmadze Math. Inst. **129** (2002), 65–75.

[15] KOKILASHVILI, V. M.: *On analytic functions of Smirnov-Orlicz classes*, Studia Math. **31** (1968), 43–59.
[16] KOKILASHVILI, V. M.: *A direct theorem on mean approximation of analytic functions by polynomials*, Dokl. Akad. Nauk SSSR **10** (1969), 411–414.
[17] KRASNOSELSKII, M. A.—RUTICKII, YA. B.: *Convex Functions and Orlicz Spaces*, P. Noordhoff Ltd., Groningen, 1961.
[18] KY, N. X.: *On approximation by trigonometric polynomials in L^p_u-spaces*, Studia Sci. Math. Hungar. **28** (1993), 183–188.
[19] MASTROIANNI, G.—TOTIK, V.: *Jackson type inequalities for doubling and A_p weights.* In: Proc. Third International Conference on Functional Analysis and Approximation Theory, Vol. 1 (Acquafredda di Maratea, 1996). Rend. Circ. Mat. Palermo (2) Suppl. **52** (1998), 83–99.
[20] MASTROIANNI, G.—TOTIK, V.: *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx. **16** (2000), 37–71.
[21] MASTROIANNI, G.—TOTIK, V.: *Best approximation and moduli of smoothness for doubling weights*, J. Approx. Theory **110** (2001), 180–199.
[22] MHASKAR, H. N.: *Introduction to the theory of weighted polynomial approximation.* Series in Approximation and Decompositions 7, World Sci., River Edge, NJ, 1996.
[23] MUCKENHOUPT, B.: *Two weight function norm inequalities for the Poisson integral*, Trans. Amer. Math. Soc. **210** (1975), 225–231.
[24] RAMAZANOV, A. R-K.: *On approximation by polynomials and rational functions in Orlicz spaces*, Anal. Math. **10** (1984), 117–132.
[25] RAO, M. M.—REN, Z. D.: *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.
[26] RUNOVSKI, K.: *On Jackson type inequality in Orlicz classes*, Rev. Mat. Complut. **14** (2001), 395–404.
[27] TIMAN, A. F.: *Theory of approximation of functions of a real variable.* International Series of Monographs on Pure and Applied Mathematics 34, Pergamon Press and MacMillan, Oxford, 1963 [Russian original published by Fizmatgiz, Moscow, 1960].
[28] ZYGMUND, A.: *Trigonometric series, Vol. I and II*, Cambridge University Press, Cambridge, 1959.

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