TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 4, pp. 1701-1719, August 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

WARPED PRODUCTS WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We find relations between the Levi-Civita connection and a semi-symmetric metric connection of the warped product $M = M_1 \times_f M_2$. We obtain some results of Einstein warped product manifolds with a semi-symmetric metric connection.

1. Introduction

The idea of a semi-symmetric linear connection on a Riemannian manifold was introduced by A. Friedmann and J. A. Schouten in [1]. Later, H. A. Hayden [3] gave the definition of a semi-symmetric metric connection. In 1970, K. Yano [8] considered semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold admitting the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. Then, the generalization of this result for vanishing Ricci tensor of the semi-symmetric metric connection was given by T. Imai ([4, 5]).

Motivated by the above studies, we study warped product manifolds with semi-symmetric metric connection and find relations between the Levi-Civita connection and the semi-symmetric metric connection.

Furthermore, in [2], A. Gebarowski studied Einstein warped product manifolds. As an application, in this study we consider Einstein warped product manifolds endowed with semi-symmetric metric connection.

2. Semi-symmetric Metric Connection

Let M be an n-dimensional Riemannian manifold with Riemannian metric g. A linear connection $\overset{\circ}{\nabla}$ on a Riemannian manifold M is called a *semi-symmetric connection* if the torsion tensor T of the connection $\overset{\circ}{\nabla}$

Received March 5, 2010, accepted April 6, 2010.
Communicated by Bang-Yen Chen.
2000 *Mathematics Subject Classification*: 53B05, 53B20, 53C25.
Key words and phrases: Warped product manifold, Semi-symmetric metric connection, Einstein manifold.

$$T(X,Y) = \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X,Y] \tag{1}$$

satisfies

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2}$$

where π is a 1-form associated with the vector field P on M defined by

$$\pi(X) = g(X,P). \tag{3}$$

$\overset{\circ}{\nabla}$ is called a *semi-symmetric metric connection* if it satisfies

$$\overset{\circ}{\nabla} g = 0.$$

If ∇ is the Levi-Civita connection of a Riemannian manifold M, the semi-symmetric metric connection $\overset{\circ}{\nabla}$ is given by

$$\overset{\circ}{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P, \tag{4}$$

(see [8]).

Let R and $\overset{\circ}{R}$ be curvature tensors of ∇ and $\overset{\circ}{\nabla}$ of a Riemannian manifold M, respectively. Then R and $\overset{\circ}{R}$ are related by

$$\begin{aligned} \overset{\circ}{R}(X,Y)Z = \; & R(X,Y)Z + g(Z,\nabla_X P)Y - g(Z,\nabla_Y P)X \\ & + g(X,Z)\nabla_Y P - g(Y,Z)\nabla_X P \\ & + \pi(P)[g(X,Z)Y - g(Y,Z)X] \\ & + [g(Y,Z)\pi(X) - g(X,Z)\pi(Y)]P \\ & + \pi(Z)[\pi(Y)X - \pi(X)Y], \end{aligned} \tag{5}$$

for any vector fields X, Y, Z on M [8]. For a general survey of different kinds of connections see also [7].

3. Warped Product Manifolds

Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be two Riemannian manifolds and f a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$. The *warped product* $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ with the Riemannian structure such that

$$\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p)) \|\sigma^*(X)\|^2 ,$$

for any tangent vector X on M. Thus we have

$$g = g_{M_1} + f^2 g_{M_2}. \tag{6}$$

The function f is called the *warping function* of the warped product [6].

We need the following three lemmas from [6], for later use :

Lemma 3.1. *Let us consider* $M = M_1 \times_f M_2$ *and denote by* ∇, ${}^{M_1}\nabla$ *and* ${}^{M_2}\nabla$ *the Riemannian connections on* M, M_1 *and* M_2, *respectively. If* X, Y *are vector fields on* M_1 *and* V, W *on* M_2, *then:*

(i) $\nabla_X Y$ *is the lift of* ${}^{M_1}\nabla_X Y$,

(ii) $\nabla_X V = \nabla_V X = (Xf/f)V$,

(iii) *The component of* $\nabla_V W$ *normal to the fibers is* $-(g(V,W)/f)\mathtt{grad}f$,

(iv) *The component of* $\nabla_V W$ *tangent to the fibers is the lift of* ${}^{M_2}\nabla_V W$.

Lemma 3.2. *Let* $M = M_1 \times_f M_2$ *be a warped product with Riemannian curvature* ${}^M R$. *Given fields* X, Y, Z *on* M_1 *and* U, V, W *on* M_2, *then:*

(i) ${}^M R(X,Y)Z$ *is the lift of* ${}^{M_1}R(X,Y)Z$,

(ii) ${}^M R(V,X)Y = -(H^f(X,Y)/f)V$, *where* H^f *is the Hessian of* f,

(iii) ${}^M R(X,Y)V = {}^M R(V,W)X = 0$,

(iv) ${}^M R(X,V)W = -(g(V,W)/f)\nabla_X(\mathtt{grad}f)$,

(v)
$$\begin{aligned} {}^M R(V,W)U &= {}^{M_2} R(V,W)U \\ &+ \|\mathtt{grad}f\|^2/f^2\{g(V,U)W - g(W,U)V\}. \end{aligned}$$

Lemma 3.3. *Let* $M = M_1 \times_f M_2$ *be a warped product with Ricci tensor* ${}^M S$. *Given fields* X, Y *on* M_1 *and* V, W *on* M_2, *then:*

(i) ${}^M S(X,Y) = {}^{M_1} S(X,Y) - \frac{d}{f}H^f(X,Y)$, *where* $d = \dim M_2$,

(ii) ${}^M S(X,V) = 0$,

(iii)
$${}^M S(V,W) = {}^{M_2} S(V,W) - g(V,W)\left[\frac{\Delta f}{f} + \frac{(d-1)}{f^2}\|\mathtt{grad}f\|^2\right],$$

where Δf *is the Laplacian of* f *on* M_1.

Moreover, the scalar curvature ${}^M r$ of M satisfies the condition

$${}^M r = {}^{M_1} r + \frac{1}{f^2}{}^{M_2} r - \frac{2d}{f}\Delta f - \frac{d(d-1)}{f^2}\|\mathtt{grad}f\|^2, \tag{7}$$

where ${}^{M_1}r$ and ${}^{M_2}r$ are scalar curvatures of M_1 and M_2, respectively.

4. Warped Product Manifolds Endowed with a Semi-symmetric Metric Connection

In this section, we consider warped product manifolds with respect to the semi-symmetric metric connection and find new expressions concerning with curvature tensor, Ricci tensor and the scalar curvature admitting this connection where the associated vector field $P \in \chi(M_1)$ or $P \in \chi(M_2)$.

Now, let begin with the following lemma:

Lemma 4.1. *Let us consider* $M = M_1 \times_f M_2$ *and denote by* $\overset{\circ}{\nabla}$ *the semi-symmetric metric connection on* M, ${}^{M_1}\overset{\circ}{\nabla}$ *and* ${}^{M_2}\overset{\circ}{\nabla}$ *be connections on* M_1 *and* M_2*, respectively. If* $X, Y \in \chi(M_1)$, $V, W \in \chi(M_2)$ *and* $P \in \chi(M_1)$*, then:*

(i) $\overset{\circ}{\nabla}_X Y$ *is the lift of* ${}^{M_1}\overset{\circ}{\nabla}_X Y$,

(ii) $\overset{\circ}{\nabla}_X V = (Xf/f)V$ *and* $\overset{\circ}{\nabla}_V X = [(Xf/f) + \pi(X)]V$,

(iii) $nor\overset{\circ}{\nabla}_V W = -[g(V,W)/f]\mathtt{grad}f - g(V,W)P$,

(iv) $\tan\overset{\circ}{\nabla}_V W$ *is the lift of* $\overset{\circ}{\nabla}_V W$ *on* M_2.

Proof. From the Koszul formula we can write

$$
\begin{aligned}
2g(\nabla_X Y, Z) = {} & Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\
& -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),
\end{aligned} \tag{8}
$$

for all vector fields X, Y, Z on M, where ∇ is the Levi-Civita connection of M. By the use of (4) for the semi-symmetric metric connection, the equation (8) reduces to

$$
\begin{aligned}
2g(\overset{\circ}{\nabla}_X Y, V) = {} & Xg(Y,V) + Yg(X,V) - Vg(X,Y) \\
& -g(X,[Y,V]) - g(Y,[X,V]) + g(V,[X,Y]) \\
& +2\pi(Y)g(X,V) - 2\pi(V)g(X,Y),
\end{aligned} \tag{9}
$$

for any vector fields $X, Y \in \chi(M_1)$ and $V \in \chi(M_2)$.

Since X, Y and $[X,Y]$ are lifts from M_1 and V is vertical, we know from [6] that

$$g(Y,V) = g(X,V) = 0 \tag{10}$$

and

$$[X,V] = [Y,V] = 0. \tag{11}$$

Hence, the equation (9) can be written as

$$2g(\overset{\circ}{\nabla}_X Y, V) = -Vg(X,Y) - 2\pi(V)g(X,Y). \tag{12}$$

On the other hand, since X and Y are lifts from M_1 and V is vertical, $g(X,Y)$ is constant on fibers which means that

$$Vg(X,Y) = 0.$$

So the equation (12) turns into

$$g(\overset{\circ}{\nabla}_X Y, V) = -\pi(V)g(X,Y). \tag{13}$$

Since $P \in \chi(M_1)$, from the equation (13) we get

$$g(\overset{\circ}{\nabla}_X Y, V) = 0,$$

which gives us (i).

By the use of the definition of the covariant derivative with respect to the semi-symmetric metric connection, we can write

$$g(\overset{\circ}{\nabla}_X V, Y) = Xg(Y,V) - g(V, \overset{\circ}{\nabla}_X Y),$$

for all vector fields X, Y on M_1 and V on M_2. By making use of (10) and (13), the above equation turns into

$$g(\overset{\circ}{\nabla}_X V, Y) = \pi(V)g(X,Y). \tag{14}$$

Taking $P \in \chi(M_1)$, we get

$$g(\overset{\circ}{\nabla}_X V, Y) = 0. \tag{15}$$

On the other hand, from the definitions of Koszul formula and the semi-symmetric metric connection we can write

$$\begin{aligned} 2g(\overset{\circ}{\nabla}_X V, W) = \ & Xg(V,W) + Vg(X,W) - Wg(X,V) \\ & -g(X,[V,W]) - g(V,[X,W]) + g(W,[X,V]) \\ & +2\pi(V)g(X,W) - 2\pi(W)g(X,V), \end{aligned}$$

for any vector fields X on M_1 and V, W on M_2. In view of (10) and (11), the last equation reduces to

$$2g(\overset{\circ}{\nabla}_X V, W) = Xg(V,W) - g(X,[V,W]).$$

Since X is horizontal and $[V,W]$ is vertical, $g(X,[V,W]) = 0$ hence we find

$$2g(\overset{\circ}{\nabla}_X V, W) = Xg(V,W). \tag{16}$$

By the definition of the warped product metric from (6), we have

$$g(V,W)(p,q) = (f \circ \pi)^2(p,q) g_{M_2}(V_q, W_q).$$

Then by making use of f instead of $f \circ \pi$, we get

$$g(V,W) = f^2(g_{M_2}(V,W) \circ \sigma).$$

Hence, we can write

$$\begin{aligned} Xg(V,W) &= X[f^2(g_{M_2}(V,W) \circ \sigma)] \\ &= 2fXf(g_{M_2}(V,W) \circ \sigma) + f^2 X(g_{M_2}(V,W) \circ \sigma). \end{aligned}$$

Since the term $(g_{M_2}(V,W) \circ \sigma)$ is constant on leaves, by the use of (6), the above equation turns into

$$Xg(V,W) = 2(Xf/f)g(V,W). \tag{17}$$

By making use of (17) in (16), we obtain

$$g(\overset{\circ}{\nabla}_X V, W) = (Xf/f)g(V,W). \tag{18}$$

Taking $P \in \chi(M_1)$, in view of the equations (15) and (18), we have

$$\overset{\circ}{\nabla}_X V = (Xf/f)V.$$

On the other hand, by the use of (1) we can write

$$g(\overset{\circ}{\nabla}_X V, W) = g(\overset{\circ}{\nabla}_V X, W) + g([X,V], W) + g(T(X,V), W).$$

Using (2) and (11), the above equation reduces to

$$g(\overset{\circ}{\nabla}_X V, W) = g(\overset{\circ}{\nabla}_V X, W) - \pi(X)g(V,W), \tag{19}$$

which means that

$$g(\overset{\circ}{\nabla}_V X, W) = [(Xf/f) + \pi(X)]g(V,W).$$

Then we get

$$\overset{\circ}{\nabla}_V X = [(Xf/f) + \pi(X)]V, \tag{20}$$

so we have (ii). By the definition of the covariant derivative with respect to the semi-symmetric metric connection, we can write

$$Vg(X,W) = g(\overset{\circ}{\nabla}_V X, W) + g(\overset{\circ}{\nabla}_V W, X),$$

for any vector fields X on M_1 and V, W on M_2. From (10), the above equation reduces to

$$g(\overset{\circ}{\nabla}_V W, X) = -g(\overset{\circ}{\nabla}_V X, W). \tag{21}$$

Taking $P \in \chi(M_1)$, by the use of (20), we get

$$g(\overset{\circ}{\nabla}_V W, X) = -[(Xf/f) + \pi(X)]g(V, W),$$

which implies that

$$nor\overset{\circ}{\nabla}_V W = -[g(V, W)/f]\mathtt{grad}f - g(V, W)P,$$

where $Xf = g(\mathtt{grad}f, X)$ for any vector field X on M_1. Thus, the proof of the lemma is completed. ■

Lemma 4.2. *Let us consider $M = M_1 \times_f M_2$ and denote by $\overset{\circ}{\nabla}$ the semi-symmetric metric connection on M, ${}^{M_1}\overset{\circ}{\nabla}$ and ${}^{M_2}\overset{\circ}{\nabla}$ be connections on M_1 and M_2, respectively. If $X, Y \in \chi(M_1)$, $V, W \in \chi(M_2)$ and $P \in \chi(M_2)$, then:*

(i) $nor\ \overset{\circ}{\nabla}_X Y$ *is the lift of* $\overset{\circ}{\nabla}_X Y$ *on* M_1,

(ii) $\tan \overset{\circ}{\nabla}_X Y = -g(X, Y)P$,

(iii) $\tan \overset{\circ}{\nabla}_X V = (Xf/f)V$ *and* $nor\overset{\circ}{\nabla}_X V = \pi(V)X$,

(iv) $\overset{\circ}{\nabla}_V X = (Xf/f)V$,

(v) $nor\overset{\circ}{\nabla}_V W = -[g(V, W)/f]\mathtt{grad}f$,

(vi) $\tan\overset{\circ}{\nabla}_V W$ *is the lift of* $\overset{\circ}{\nabla}_V W$ *on* M_2.

Proof. Since $P \in \chi(M_2)$, in view of the equation (13), we find

$$g(\overset{\circ}{\nabla}_X Y, V) = -\pi(V)g(X, Y),$$

which gives us the proof of (i) and (ii).

Similarly from the equation (14) we obtain

$$g(\overset{\circ}{\nabla}_X V, Y) = \pi(V)g(X, Y). \tag{22}$$

Then by the use of (18) for $P \in \chi(M_2)$ and in view of (22), we get

$$\overset{\circ}{\nabla}_X V = (Xf/f)V + \pi(V)X, \tag{23}$$

which implies that

$$\tan \overset{\circ}{\nabla}_X V = (Xf/f)V \quad \text{and} \quad nor\overset{\circ}{\nabla}_X V = \pi(V)X.$$

Hence we have (iii).

Moreover, in view of (1) and (11) we have

$$\overset{\circ}{\nabla}_V X = \overset{\circ}{\nabla}_X V - T(X,V).$$

Then by making use of the equations (2) and (23), the last equation gives us

$$\overset{\circ}{\nabla}_V X = (Xf/f)V, \tag{24}$$

which completes the proof of (iv).

Similarly taking $P \in \chi(M_2)$ in the equation (21) and by making use of (24), we obtain

$$g(\overset{\circ}{\nabla}_V W, X) = -(Xf/f)g(V,W),$$

which gives us

$$nor\overset{\circ}{\nabla}_V W = -[g(V,W)/f]\mathtt{grad}f.$$

Hence, we complete the proof of the lemma. ■

Lemma 4.3. *Let $M = M_1 \times_f M_2$ be a warped product, R and $\overset{\circ}{R}$ denote the Riemannian curvature tensors of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. If $X, Y, Z \in \chi(M_1)$, $U, V, W \in \chi(M_2)$ and $P \in \chi(M_1)$, then:*

(*i*) $\overset{\circ}{R}(X,Y)Z \in \chi(M_1)$ *is the lift of* ${}^{M_1}\overset{\circ}{R}(X,Y)Z$ *on* M_1,

(*ii*)
$$\overset{\circ}{R}(V,X)Y = -[H^f(X,Y)/f + (Pf/f)g(X,Y) + \pi(P)g(X,Y) + g(Y,\nabla_X P) - \pi(X)\pi(Y)]V,$$

(*iii*) $\overset{\circ}{R}(X,Y)V = 0$,

(*iv*) $\overset{\circ}{R}(V,W)X = 0$,

(*v*)
$$\overset{\circ}{R}(X,V)W = g(V,W)[-(\nabla_X \mathtt{grad}f)/f - (Pf/f)X - \nabla_X P - \pi(P)X + \pi(X)P],$$

(*vi*)
$$\overset{\circ}{R}(U,V)W = {}^{M_2}R(U,V)W - \{\|\mathtt{grad}f\|^2/f^2 + 2(Pf/f) + \pi(P)\}[g(V,W)U - g(U,W)V].$$

Proof. Assume that $M = M_1 \times_f M_2$ is a warped product, R and $\overset{\circ}{R}$ denote the curvature tensors of the Levi-Civita connection and the semi-symmetric metric connection, respectively.

(i) Since $\overset{\circ}{\nabla}_X Y$ is the lift of ${}^{M_1}\overset{\circ}{\nabla}_X Y$, for $X, Y, P \in \chi(M_1)$, then by the definition of $\overset{\circ}{R}$ it is easy to see that $\overset{\circ}{R}(X,Y)Z \in \chi(M_1)$ is the lift of ${}^{M_1}\overset{\circ}{R}(X,Y)Z$ on M_1, for the vector field Z on M_1 and $P \in \chi(M_1)$.

(ii) In view of the equation (5), we can write

$$\begin{aligned} \overset{\circ}{R}(V,X)Y = \; & R(V,X)Y + g(Y, \nabla_V P)X - g(Y, \nabla_X P)V \\ & -g(X,Y)[\nabla_V P + \pi(P)V - \pi(V)P] \\ & +\pi(Y)[\pi(X)V - \pi(V)X], \end{aligned} \tag{25}$$

for all vector fields X, Y on M_1 and V on M_2, respectively.

Since $P \in \chi(M_1)$, by making use of Lemma 3.2, we get

$$\begin{aligned} \overset{\circ}{R}(V,X)Y = \; & -[H^f(X,Y)/f + (Pf/f)g(X,Y) + \pi(P)g(X,Y) \\ & +g(Y, \nabla_X P) - \pi(X)\pi(Y)]V. \end{aligned}$$

(iii) Putting $Z = V$ in equation (5), where $V \in \chi(M_2)$, we get

$$\begin{aligned} \overset{\circ}{R}(X,Y)V = \; & g(V, \nabla_X P)Y - g(V, \nabla_Y P)X \\ & +\pi(V)[\pi(Y)X - \pi(X)Y], \end{aligned} \tag{26}$$

which shows us

$$\overset{\circ}{R}(X,Y)V = 0,$$

for $P \in \chi(M_1)$.

(iv) By making use of (5) and Lemma 3.2, we can write

$$\begin{aligned} \overset{\circ}{R}(V,W)X = \; & g(X, \nabla_V P)W - g(X, \nabla_W P)V \\ & +\pi(X)[\pi(W)V - \pi(V)W], \end{aligned} \tag{27}$$

for any vector fields X on M_1 and V, W on M_2, respectively. Taking $P \in \chi(M_1)$, we get

$$\overset{\circ}{R}(V,W)X = 0.$$

(v) From the equation (5), we find

$$\begin{aligned} \overset{\circ}{R}(X,V)W = \; & R(X,V)W + g(W, \nabla_X P)V - g(W, \nabla_V P)X \\ & -g(V,W)[\nabla_X P + \pi(P)X - \pi(X)P] \\ & +\pi(W)[\pi(V)X - \pi(X)V], \end{aligned} \tag{28}$$

for all vector fields $X \in \chi(M_1)$ and $V, W \in \chi(M_2)$.

If $P \in \chi(M_1)$, then by making use of Lemma 3.2 in (28), we have

$$\overset{\circ}{R}(X,V)W = g(V,W)[-(\nabla_X \mathtt{grad} f)/f - (Pf/f)X - \nabla_X P - \pi(P)X + \pi(X)P].$$

(vi) In view of the equation (5), we have

$$\begin{aligned} \overset{\circ}{R}(U,V)W &= R(U,V)W + g(W,\nabla_U P)V - g(W,\nabla_V P)U \\ &+ g(U,W)\nabla_V P - g(V,W)\nabla_U P \\ &+ \pi(P)[g(U,W)V - g(V,W)U] \\ &+ [g(U,W)\pi(U) - g(V,W)\pi(V)]P \\ &+ \pi(W)[\pi(V)U - \pi(U)V], \end{aligned} \tag{29}$$

for any vector fields U, V, W on M_2.

Taking $P \in \chi(M_1)$ and by making use of Lemma 3.2 in the above equation, we obtain

$$\begin{aligned} \overset{\circ}{R}(U,V)W &= {}^{M_2}R(U,V)W \\ &- \{\|\mathtt{grad} f\|^2/f^2 + 2(Pf/f) \\ &+ \pi(P)\}[g(V,W)U - g(U,W)V]. \end{aligned}$$

Hence, the proof of the lemma is completed. ∎

Lemma 4.4. *Let $M = M_1 \times_f M_2$ be a warped product, R and $\overset{\circ}{R}$ denote the Riemannian curvature tensors of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively. If $X, Y, Z \in \chi(M_1)$, $U, V, W \in \chi(M_2)$ and $P \in \chi(M_2)$, then:*

(*i*) ${}^{M_1}\overset{\circ}{R}(X,Y)Z = {}^{M_1}R(X,Y)Z + \pi(P)[g(X,Z)Y - g(Y,Z)X]$,

(*ii*) ${}^{M_2}\overset{\circ}{R}(X,Y)Z = [g(X,Z)(Yf/f) - g(Y,Z)(Xf/f)]P$,

(*iii*) ${}^{M_1}\overset{\circ}{R}(V,X)Y = -g((\pi(V)/f)\mathtt{grad} f, Y)X + g(X,Y)[\pi(V)/f]\mathtt{grad} f$,

(*iv*) $$\begin{aligned} {}^{M_2}\overset{\circ}{R}(V,X)Y &= -[H^f(X,Y)/f]V - g(X,Y)(\tan \nabla_V P) \\ &- \pi(P)g(X,Y)V + \pi(V)g(X,Y)P, \end{aligned}$$

(*v*) $\overset{\circ}{R}(X,Y)V = \pi(V)[(Xf/f)Y - (Yf/f)X]$, (*vi*) $\overset{\circ}{R}(V,W)X = (Xf/f)[\pi(W)V - \pi(V)W]$,

(*vii*)
$$ {}^{M_1}\overset{\circ}{R}(X,V)W = -g(V,W)[(\nabla_X \mathtt{grad} f)/f + \pi(P)X] - g(W,\nabla_V P)X + \pi(V)\pi(W)X, $$

(*viii*) ${}^{M_2}\overset{\circ}{R}(X,V)W = (Xf/f)[\pi(W)V - g(V,W)P]$,

(*ix*)
$$ \begin{aligned} \overset{\circ}{R}(U,V)W &= {}^{M_2}R(U,V)W \\ &\quad -[\|\mathtt{grad} f\|^2/f^2]\{g(V,W)U - g(U,W)V\} \\ &\quad +g(W,\nabla_U P)V - g(W,\nabla_V P)U \\ &\quad +g(U,W)\nabla_V P - g(V,W)\nabla_U P \\ &\quad +\pi(P)[g(U,W)V - g(V,W)U] \\ &\quad +[g(V,W)\pi(U) - g(U,W)\pi(V)]P \\ &\quad +\pi(W)[\pi(V)U - \pi(U)V]. \end{aligned} $$

Proof. Assume that the associated vector field $P \in \chi(M_2)$. Then the equation (5) can be written as

$$ \begin{aligned} \overset{\circ}{R}(X,Y)Z &= R(X,Y)Z + [g(X,Z)(Yf/f) - g(Y,Z)(Xf/f)]P \\ &\quad +\pi(P)[g(X,Z)Y - g(Y,Z)X], \end{aligned} $$

for any vector fields $X, Y, Z \in \chi(M_1)$. By the use of Lemma 3.2, the above equation gives us

$$ {}^{M_1}\overset{\circ}{R}(X,Y)Z = {}^{M_1}R(X,Y)Z + \pi(P)[g(X,Z)Y - g(Y,Z)X] $$

and

$$ {}^{M_2}\overset{\circ}{R}(X,Y)Z = [g(X,Z)(Yf/f) - g(Y,Z)(Xf/f)]P, $$

which finishes the proof of (i) and (ii).

Similarly taking $P \in \chi(M_2)$ in (25) and using Lemma 3.2, we obtain

$$ \begin{aligned} \overset{\circ}{R}(V,X)Y &= -[H^f(X,Y)/f]V - g([\pi(V)/f]\mathtt{grad} f, Y)X \\ &\quad -g(X,Y)[\nabla_V P + \pi(P)V - \pi(V)P], \end{aligned} $$

which implies that

$$ {}^{M_1}\overset{\circ}{R}(V,X)Y = -g([\pi(V)/f]\mathtt{grad} f, Y)X + g(X,Y)[\pi(V)/f]\mathtt{grad} f $$

and

$$
\begin{aligned}
{}^{M_2}\overset{\circ}{R}(V,X)Y = & -[H^f(X,Y)/f]V - g(X,Y)(\tan \nabla_V P) \\
& -g(X,Y)[\pi(P)V - \pi(V)P],
\end{aligned}
$$

which completes the proof of (iii) and (iv).

Taking $P \in \chi(M_2)$ in the equation (26), we get

$$
\overset{\circ}{R}(X,Y)V = \pi(V)[(Xf/f)Y - (Yf/f)X],
$$

which gives us (v).

From the equation (27) and by the use of Lemma 3.1 for $P \in \chi(M_2)$ it can be easily seen that

$$
\overset{\circ}{R}(V,W)X = (Xf/f)[\pi(W)V - \pi(V)W],
$$

which proves (vi).

Similarly, from the equation (28) if $P \in \chi(M_2)$, then we obtain

$$
\begin{aligned}
{}^{M_1}\overset{\circ}{R}(X,V)W = & -g(V,W)[(\nabla_X \mathtt{grad} f)/f + \pi(P)X] \\
& -g(W,\nabla_V P)X + \pi(V)\pi(W)X
\end{aligned}
$$

and

$$
{}^{M_2}\overset{\circ}{R}(X,V)W = (Xf/f)[\pi(W)V - g(V,W)P].
$$

So we prove (vii) and (viii). Taking $P \in \chi(M_2)$ in (29) and by the use of Lemma 3.2, we obtain

$$
\begin{aligned}
\overset{\circ}{R}(U,V)W = & {}^{M_2}R(U,V)W \\
& -[\|\mathtt{grad} f\|^2/f^2]\{g(V,W)U - g(U,W)V\} \\
& +g(W,\nabla_U P)V - g(W,\nabla_V P)U \\
& +g(U,W)\nabla_V P - g(V,W)\nabla_U P \\
& +\pi(P)[g(U,W)V - g(V,W)U] \\
& +[g(U,W)\pi(U) - g(V,W)\pi(V)]P \\
& +\pi(W)[\pi(V)U - \pi(U)V],
\end{aligned}
$$

for any vector fields U, V, W on M_2, hence the last equation gives us (ix). Thus, we complete the proof of the lemma. ∎

As a consequence of Lemma 4.3 and Lemma 4.4, by a contraction of the curvature tensors we obtain the Ricci tensors of the warped product with respect to the semi-symmetric metric connection as follows:

Corollary 4.5. *Let* $M = M_1 \times_f M_2$ *be a warped product,* S *and* $\overset{\circ}{S}$ *denote the Ricci tensors of* M *with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively, where* $\dim M_1 = n_1$ *and* $\dim M_2 = n_2$*. If* $X, Y \in \chi(M_1)$*,* $V, W \in \chi(M_2)$ *and* $P \in \chi(M_1)$*, then:*

(*i*)
$$\begin{aligned}\overset{\circ}{S}(X,Y) &= ^{M_1} \overset{\circ}{S}(X,Y) - n_2[H^f(X,Y)/f + (Pf/f)g(X,Y)\\ &\quad +\pi(P)g(X,Y) + g(Y, \nabla_X P) - \pi(X)\pi(Y)],\end{aligned}$$

(*ii*) $\overset{\circ}{S}(X,V) = \overset{\circ}{S}(V,X) = 0,$

(*iii*)
$$\begin{aligned}\overset{\circ}{S}(V,W) &= ^{M_2} S(V,W) - \sum_{i=1}^{n_1} g(\nabla_{e_i}P, e_i)g(V,W)\\ &\quad -[(n_2-1)\left\|\mathtt{grad}f\right\|^2/f^2 + (n_1+2n_2-2)(Pf/f)\\ &\quad +(n-2)\pi(P) + \frac{\Delta f}{f}]g(V,W).\end{aligned}$$

Corollary 4.6. *Let* $M = M_1 \times_f M_2$ *be a warped product,* S *and* $\overset{\circ}{S}$ *denote the Ricci tensors of* M *with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively, where* $\dim M_1 = n_1$ *and* $\dim M_2 = n_2$*. If* $X, Y \in \chi(M_1)$*,* $V, W \in \chi(M_2)$ *and* $P \in \chi(M_2)$ *, then:*

(*i*)
$$\begin{aligned}\overset{\circ}{S}(X,Y) &= ^{M_1} S(X,Y) - (n-2)\pi(P)g(X,Y)\\ &\quad -n_2[H^f(X,Y)/f] - \sum_{i=n_1+1}^{n} g(\nabla_{e_i}P, e_i)g(X,Y),\end{aligned}$$

(*ii*) $\overset{\circ}{S}(X,V) = (2-n)\pi(V)(Xf/f)$ *and* $\overset{\circ}{S}(V,X) = (n-2)\pi(V)(Xf/f),$

(*iii*)
$$\begin{aligned}\overset{\circ}{S}(V,W) &= ^{M_2} S(V,W) + \sum_{i=n_1+1}^{n}\{g(W,\nabla_{e_i}P)g(V,e_i) - g(\nabla_{e_i}P, e_i)g(V,W)\}\\ &\quad -[(n_2-1)\left\|\mathtt{grad}f\right\|^2/f^2 + \frac{\Delta f}{f} + (n-2)\pi(P)]g(V,W)\\ &\quad -(n-1)g(W,\nabla_V P) + (n-2)\pi(V)\pi(W).\end{aligned}$$

As a consequence of Corollary 4.5 and Corollary 4.6, by a contraction of the Ricci tensors we get scalar curvatures of the warped product with respect to the semi-symmetric metric connection as follows:

Corollary 4.7. *Let $M = M_1 \times_f M_2$ be a warped product, r and $\overset{\circ}{r}$ denote the scalar curvatures of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively and $P \in \chi(M_1)$. Then we have*

$$\overset{\circ}{r} = {}^{M_1}\overset{\circ}{r} + \frac{{}^{M_2}r}{f^2} - n_2(n_2-1)\left\|\mathtt{grad}f\right\|^2/f^2 - 2n_2(n-1)(Pf/f)$$
$$-2n_2\frac{\Delta f}{f} - n_2[2n_1+n_2-3]\pi(P) - 2n_2\sum_{i=1}^{n_1} g(\nabla_{e_i}P, e_i).$$

Corollary 4.8. *Let $M = M_1 \times_f M_2$ be a warped product, r and $\overset{\circ}{r}$ denote the scalar curvatures of M with respect to the Levi-Civita connection and the semi-symmetric metric connection, respectively and $P \in \chi(M_2)$. Then we have*

$$\overset{\circ}{r} = {}^{M_1}r + \frac{{}^{M_2}r}{f^2} - \sum_{i=n_1+1}^{n} 2(n-1)g(\nabla_{e_i}P, e_i)$$
$$-(n-1)(n-2)\pi(P) - n_2[(n_2-1)\left\|\mathtt{grad}f\right\|^2/f^2 + 2\frac{\Delta f}{f}].$$

5. Einstein Warped Product Manifolds Endowed with the Semi-symmetric Metric Connection

In this section, we consider Einstein warped products endowed with the semi-symmetric metric connection.

Now, let begin with the following theorem:

Theorem 5.1. *Let (M, g) be a warped product $I \times_f M_2$, where $\dim I = 1$ and $\dim M_2 = n-1$ $(n \geq 3)$. Then (M, g) is an Einstein manifold with respect to the semi-symmetric metric connection if and only if M_2 is Einstein for $P \in \chi(M_1)$ with respect to the Levi-Civita connection or the warping function f is a constant on I for $P \in \chi(M_2)$.*

Proof. Assume that $P \in \chi(M_1)$ and denote by g_I the metric on I. Taking $f = \exp\{\frac{q}{2}\}$ and by making use of Corollary 4.5, we can write

$$\begin{aligned} \overset{\circ}{S}(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}) &= \left(-\frac{(n-1)}{4}[2q'' + (q')^2] + \frac{q'}{2}\right) g_I(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}), \\ \overset{\circ}{S}(\tfrac{\partial}{\partial t}, V) &= 0 \end{aligned} \tag{30}$$

and

$$\overset{\circ}{S}(V,W) = {}^{M_2}S(V,W) - e^q\left[\frac{(n-1)}{4}(q')^2 + \frac{(2n-3)}{2}q' + (n-2)\right] g_{M_2}(V,W), \tag{31}$$

for any vector fields V, W on M_2.

Since M is an Einstein manifold with respect to the semi-symmetric metric connection, we have

$$\overset{\circ}{S}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = \alpha g(\frac{\partial}{\partial t},\frac{\partial}{\partial t})$$

and

$$\overset{\circ}{S}(V,W) = \alpha g(V,W).$$

Then by making use of (6), the last two equations reduce to

(32) $$\overset{\circ}{S}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = \alpha g_{I}(\frac{\partial}{\partial t},\frac{\partial}{\partial t})$$

and

(33) $$\overset{\circ}{S}(V,W) = \alpha e^{q} g_{M_2}(V,W).$$

Comparing the right hand sides of the equations (30) and (32) we get

(34) $$\alpha = \left(-\frac{(n-1)}{4}[2q'' + (q')^2] + \frac{q'}{2}\right).$$

Similarly, comparing the right hand sides of (31) and (33) and by the use of (34), we obtain

$${}^{M_2}S(V,W) = -e^{q}\left(\frac{(n-2)}{2}q'' + (n-1)q' + (n-2)\right) g_{M_2}(V,W),$$

which implies that M_2 is an Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(M_1)$.

Taking $P \in \chi(M_2)$ and by the use of Corollary 4.6, we have

(35) $$\overset{\circ}{S}(\frac{\partial}{\partial t},V) = (2-n)\frac{q'}{2}\pi(V) g_{I}(\frac{\partial}{\partial t},\frac{\partial}{\partial t})$$

and

(36) $$\overset{\circ}{S}(V,\frac{\partial}{\partial t}) = (n-2)\frac{q'}{2}\pi(V) g_{I}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}),$$

for any vector field $V \in \chi(M_2)$.

Since M is an Einstein manifold, we can write

$$\overset{\circ}{S}(\frac{\partial}{\partial t},V) = \overset{\circ}{S}(V,\frac{\partial}{\partial t}) = \alpha g(V,\frac{\partial}{\partial t}),$$

where $g(V,\frac{\partial}{\partial t}) = 0$ for $\frac{\partial}{\partial t} \in \chi(M_1)$ and $V \in \chi(M_2)$. Hence, the last equation turns into

(37) $$\overset{\circ}{S}(\frac{\partial}{\partial t},V) = \overset{\circ}{S}(V,\frac{\partial}{\partial t}) = 0.$$

Comparing the right hand sides of the equations (35), (36) and (37), we obtain

$$q' = 0,$$

which means that q is a constant on I. Since the warping function $f = \exp\{\frac{q}{2}\}$, then f is a constant on I. Thus, the proof of the theorem is completed. ∎

Theorem 5.2. *Let (M, g) be a warped product $M_1 \times_f I$, where $\dim I = 1$ and $\dim M_1 = n - 1$ $(n \geq 3)$.*

(i) *If (M, g) is an Einstein manifold with respect to the semi-symmetric metric connection, $P \in \chi(M_1)$ is parallel on M_1 with respect to the Levi-Civita connection on M_1 and f is a constant on M_1, then:*

$$^{M_1}\overset{\circ}{r} = -(n-2)^2\pi(P).$$

(ii) *If (M, g) is an Einstein manifold with respect to the semi-symmetric metric connection for $P \in \chi(M_2)$, then f is a constant on M_1.*

(iii) *If f is a constant on M_1 and M_1 is an Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(M_2)$, then M is an Einstein manifold with respect to the semi-symmetric metric connection.*

Proof. (i) Assume that (M, g) is an Einstein manifold with respect to the semi-symmetric metric connection. Then we can write

$$\overset{\circ}{S}(X, Y) = \frac{\overset{\circ}{r}}{n} g(X, Y), \tag{38}$$

for any vector fields $X, Y \in \chi(M_1)$. Taking $P \in \chi(M_1)$ and by the use of the equation (6) and Corollary 4.7, the equation (38) reduces to

$$\overset{\circ}{S}(X, Y) = \frac{1}{n}\Bigg[{}^{M_1}\overset{\circ}{r} - 2\sum_{i=1}^{n-1} g(\nabla_{e_i} P, e_i) - 2\frac{\Delta f}{f} \\ -2(n-1)(Pf/f) - 2(n-2)\pi(P)\Bigg] g_{M_1}(X, Y).$$

By a contraction from the above equation over X and Y, we get

$$\overset{\circ}{r} = \frac{(n-1)}{n}\Bigg[{}^{M_1}\overset{\circ}{r} - 2\sum_{i=1}^{n-1} g(\nabla_{e_i} P, e_i) - 2\frac{\Delta f}{f} \\ -2(n-1)(Pf/f) - 2(n-2)\pi(P)\Bigg]. \tag{39}$$

On the other hand, since the vector field $P \in \chi(M_1)$, then by the use of Corollary 4.5 we can write

$$\overset{\circ}{S}(X,Y) = {}^{M_1}\overset{\circ}{S}(X,Y) - [H^f(X,Y)/f + (Pf/f)g(X,Y) + \pi(P)g(X,Y) + g(Y,\nabla_X P) - \pi(X)\pi(Y)].$$

Similarly, by a contraction from the last equation over X and Y, it can be easily seen that

$$\overset{\circ}{r} = {}^{M_1}\overset{\circ}{r} - \frac{\Delta f}{f} - (n-1)(Pf/f) - (n-2)\pi(P) - \sum_{i=1}^{n-1} g(\nabla_{e_i}P, e_i). \tag{40}$$

Comparing the right hand sides of the equations (39) and (40), we can write

$$\frac{(n-1)}{n}\left[{}^{M_1}\overset{\circ}{r} - 2\sum_{i=1}^{n-1} g(\nabla_{e_i}P, e_i) - 2\frac{\Delta f}{f} - 2(n-1)(Pf/f) - 2(n-2)\pi(P)\right]$$
$$= {}^{M_1}\overset{\circ}{r} - \frac{\Delta f}{f} - (n-1)(Pf/f) - (n-2)\pi(P) - \sum_{i=1}^{n-1} g(\nabla_{e_i}P, e_i).$$

Since $P \in \chi(M_1)$ is parallel and f is a constant on M_1, then we get ${}^{M_1}\overset{\circ}{r} = -(n-2)^2\pi(P)$.

(ii) Let $P \in \chi(M_2)$. By the use of Corollary 4.6, we have

$$\overset{\circ}{S}(X,V) = (2-n)g([\pi(V)/f]\mathtt{grad}f, X)$$

and

$$\overset{\circ}{S}(V,X) = (n-2)g([\pi(V)/f]\mathtt{grad}f, X),$$

for any vector fields $X \in \chi(M_1)$ and $V \in \chi(M_2)$. Since $M_2 = I$, then taking $V = P$ and using the equality $g(\mathtt{grad}f, X) = Xf$ from the last equation we obtain

$$\overset{\circ}{S}(X,P) = (2-n)(Xf/f)\pi(P) \tag{41}$$

and

$$\overset{\circ}{S}(P,X) = (n-2)(Xf/f)\pi(P). \tag{42}$$

Since M is an Einstein manifold, we can write

$$\overset{\circ}{S}(X,P) = \overset{\circ}{S}(P,X) = \alpha g(P,X),$$

where $g(P, X) = 0$ for $X \in \chi(M_1)$ and $P \in \chi(M_2)$. Hence, the last equation turns into

$$\overset{\circ}{S}(X, P) = \overset{\circ}{S}(P, X) = 0. \tag{43}$$

Comparing the right hand sides of the equations (41), (42) and (43) we get

$$Xf = 0,$$

which gives us the warping function f is a constant on M_1.

(iii) Assume that M_1 is an Einstein manifold with respect to the Levi-Civita connection. Then we have

$$^{M_1}S(X, Y) = \alpha g(X, Y), \tag{44}$$

for any vector fields X, Y tangent to M_1.

On the other hand, in view of Corollary 4.6, we can write

$$\overset{\circ}{S}(X, Y) =^{M_1} S(X, Y) - (n-2)\pi(P)g(X, Y) - [H^f(X, Y)/f],$$

for $P \in \chi(M_2)$. Since f is a constant on M_1, then $H^f(X, Y) = 0$ for all $X, Y \in \chi(M_1)$. Thus, the above equation reduces to

$$\overset{\circ}{S}(X, Y) =^{M_1} S(X, Y) - (n-2)\pi(P)g(X, Y). \tag{45}$$

By the use of (44) in (45), we obtain

$$\overset{\circ}{S}(X, Y) = [\alpha - (n-2)\pi(P)]g(X, Y),$$

which shows us $M_1 \times_f I$ is an Einstein manifold with respect to the semi-symmetric metric connection. Therefore, we complete the proof of the theorem. ■

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