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On the some normal subgroups of the extended modular group

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ABSTRACT

In this paper, we give the group structures and the signatures of some normal subgroups of the extended modular group Π containing the principal congruence subgroup $\Gamma(12)$.

1. Introduction

The modular group Γ is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$t(z) = -\frac{1}{z} \quad \text{and } s(z) = -\frac{1}{z+1}.$$

Then modular group Γ has a presentation

$$\Gamma = < t, s | t^2 = s^3 = I > \cong C_2 * C_3.$$

The signature of Γ is $(0; +; [2, 3, \infty]; \{(-)\})$. Also the quotient space $\mathcal{U}/\Gamma$ where $\mathcal{U}$ is the upper half plane, is a sphere with one puncture and two elliptic fixed points of order 2 and 3. Hence the surface $\mathcal{U}/\Gamma$ is a Riemann surface.

The extended modular group Π has been defined by adding the reflection $r(z) = 1/\bar{z}$ to the generators of the modular group Γ. The extended modular group Π has a presentation, see [5],

$$\Pi = < t, s, r | t^2 = s^3 = r^2 = I, \quad rt = tr, \quad rs = s^{-1}r >,$$

or, equivalently,

$$\Pi = < t, s, r | t^2 = s^3 = r^2 = (rt)^2 = (rs)^2 = I > \cong D_2 *_{\mathbb{Z}_2} D_3.$$

Here t, s and r have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively (in this work, we identify each matrix A in $GL(2,\mathbb{Z})$ with $-A$, so that they each represent the same element of $PGL(2,\mathbb{Z})$) . Thus the modular group $\Gamma = PSL(2,\mathbb{Z})$ is a subgroup of index 2 in the extended modular group Π.

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The signature of the extended modular group Π is $(0;+;[-];\{2,3,\infty\})$. Since the extended modular group Π contain a reflection, it is a non-Euclidean crystallographic (NEC) group, which is a discrete subgroup Π of the group $PGL(2,\mathbb{R})$ of isometries of $\mathcal{U}$ such that the quotient $\mathcal{U}/\Pi$ is a Klein surface. Also $\mathcal{U}/\Gamma$ is the canonical double cover of $\mathcal{U}/\Pi$.

The extended modular group and its normal subgroups has been studied for many aspects in the literature, for example, automorphism groups of compact Klein surfaces, number theory, Belyi's theory, graph theory, regular maps.see [1–3,6,7]. Now we give the followings from [5]. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent a typical element of Π. For each integer $n \geqslant 1$, we define

$$\Pi(n) = \{A \in \Pi | a \equiv d \equiv \pm 1 \text{ and } b \equiv c \equiv 0(n)\},$$
$$\Gamma(n) = \Pi(n) \cap \Gamma.$$

These are normal subgroups of finite index in Π, and called the *principal congruence subgroups*. If $n > 2$ then $\Pi(n) = \Gamma(n)$ and if $n = 2$ then $\Pi(2) \geqslant \Gamma(2) \geqslant \Pi(4) = \Gamma(4)$.

Also the function

$$\alpha : t \to rt, \quad s \to s, \quad r \to r$$

is an automorphism of Π which is not inner. The effect of α on congruence subgroups of small index is shown in [5, p. 32] and [4, p. 135].

In this paper, we give the group structures and the signatures of some normal subgroups of the extended modular group Π containing $\Gamma(12)$, given in [5] and [4]. We achieve this by applying standart techniques of combinatorial group theory (the Reidemeister–Schreier method and the permutation method).

2. Supergroups of $\Gamma(12)$

Theorem 2.1

(i) *There are exactly 3 normal subgroups of index 2 in Π containing the principal congruence subgroup $\Gamma(12)$. Explicitly these are*

$$\Gamma = \langle t, s | t^2 = s^3 = I \rangle \cong C_2 * C_3,$$

$$\Pi_0 = \langle r, s, tst | r^2 = s^3 = (tst)^3 = (rs)^2 = (rtst)^2 = I \rangle \cong D_3 *_{\mathbb{Z}_2} D_3,$$

$$\Gamma\alpha = \langle tr, s | (tr)^2 = s^3 = I \rangle \cong C_2 * C_3,$$

where $tr = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $tst = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) *There is only a normal subgroup of index 4 in Π containing the principal congruence subgroup $\Gamma(12)$. Explicitly this is*

$$\Pi' = \langle s, tst | s^3 = (tst)^3 = I \rangle \cong C_3 * C_3.$$

(iii) *There are exactly 2 normal subgroups of index 6 in Π containing the principal congruence subgroup $\Gamma(12)$. Explicitly these are*

$$\Pi(2)\alpha = \langle t, sts^2, s^2ts | t^2 = (sts^2)^2 = (s^2ts)^2 = I \rangle \cong C_2 * C_2 * C_2,$$

$$\Pi(2) = \langle tr, rsts, rs^2ts^2 | (tr)^2 = (rsts)^2 = (rs^2ts^2)^2 = I \rangle \cong C_2 * C_2 * C_2,$$

where $sts^2 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, $s^2ts = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$, $rsts = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ *and* $rs^2ts^2 = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$.

(iv) *There are exactly 2 normal subgroups of index 12 in Π containing the principal congruence subgroup $\Gamma(12)$. Explicitly these are*

$$\Gamma' = \langle tsts^2, ts^2ts \rangle,$$
$$\Gamma(2) = \langle tsts, ts^2ts^2 \rangle,$$

where $tsts^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $ts^2ts = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $tsts = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *and* $ts^2ts^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Proof. For (i) and (iii) see [10], for (ii) see [5], and for (iv) see [11]. □

Using the permutation method and Riemann Hurwitz formula, we obtain the signatures of Π_0, $\Gamma\alpha$, Π', $\Pi(2)\alpha$, $\Pi(2)$, Γ' and $\Gamma(2)$ as $(0;+;[-];\{(2,2,3,3)\})$, $(0;+;[3];\{(2,2)\})$, $(0;+;[3,3,\infty];\{(-)\})$, $(0;+;[2,2,2,\infty];\{(-)\})$, $(0;+;[-];\{(2,2,2)\})$, $(1;+;[\infty];\{(-)\})$ and $(0;+;[\infty,\infty,\infty];\{(-)\})$, respectively.

Notice that $\Pi(2)\alpha = \Gamma^3$ where Γ^3 is the power subgroup of Γ(see [9]).

Theorem 2.2

(i) $|\Gamma:\Gamma(3)| = 12$

(ii) *The group $\Gamma(3)$ is a free group of rank 3 with basis tststs, $ts^2ts^2ts^2$ and $tsts^2ts^2tst$.*

Proof

(i) The quotient group $\Gamma/\Gamma(3)$ has the presentation

$$\Gamma/\Gamma(3) \cong \langle t,s | t^2 = s^3 = (ts)^3 = I\rangle \cong A_4.$$

Then, we obtain $|\Gamma:\Gamma(3)| = 12$.

(ii) Now we choose $\Sigma = \{I, t, s, s^2, ts, ts^2, tst, ts^2t, tsts, ts^2ts, tsts^2, ts^2ts^2\}$ as a Schreier transversal for $\Gamma(3)$. According to the Reidemeister–Schreier method, we can form all possible products:

$$\begin{array}{ll}
I.t.(t)^{-1} = I, & I.s.(s)^{-1} = I,\\
t.t.(I)^{-1} = I, & t.s.(ts)^{-1} = I,\\
s.t.(ts^2ts^2)^{-1} = ststst, & s.s.(s^2)^{-1} = I,\\
s^2.t.(tsts)^{-1} = s^2ts^2ts^2t, & s^2.s.(I)^{-1} = I,\\
ts.t.(tst)^{-1} = I, & ts.s.(ts^2)^{-1} = I,\\
ts^2.t.(ts^2t)^{-1} = I, & ts^2.s.(t)^{-1} = I,\\
tst.t.(ts)^{-1} = I, & tst.s.(tsts)^{-1} = I,\\
ts^2t.t.(ts^2)^{-1} = I, & ts^2t.s.(ts^2ts)^{-1} = I,\\
tsts.t.(s^2)^{-1} = tststs, & tsts.s.(tsts^2)^{-1} = I,\\
ts^2ts.t.(tsts^2)^{-1} = ts^2tststs^2t, & ts^2ts.s.(ts^2ts^2)^{-1} = I,\\
tsts^2.t.(ts^2ts)^{-1} = tsts^2ts^2tst, & tsts^2.s.(tst)^{-1} = I,\\
ts^2ts^2.t.(s)^{-1} = ts^2ts^2ts^2, & ts^2ts^2.s.(ts^2t)^{-1} = I.
\end{array}$$

Since $(ststst)^{-1} = ts^2ts^2ts^2$, $(s^2ts^2ts^2t)^{-1} = tststs$, and $(ts^2tststs^2t)^{-1} = tsts^2ts^2tst$, the generators of $\Gamma(3)$ are $\beta_1 = tststs$, $\beta_2 = ts^2ts^2ts^2$ and $\beta_3 = tsts^2ts^2tst$. Here $\beta_1 = \begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$, $\beta_2 = \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ and $\beta_3 = \begin{pmatrix}4 & -3\\ 3 & -2\end{pmatrix}$. Also, using the permutation method, we get also the signature of $\Gamma(3)$ as $(0;+;[\infty^{(4)}];\{(-)\})$. □

Corollary 2.3. *There are exactly 2 normal subgroups of index 24 in Π containing the principal congruence subgroup Γ (12). Explicitly these are*

$$\Gamma(3) = \Pi(3) = \langle \beta_1, \beta_2, \beta_3\rangle,$$
$$\Pi(3)\alpha = \langle \beta_1\alpha, \beta_2\alpha, \beta_3\alpha\rangle,$$

where $\beta_1\alpha = ts^2tsts^2r = \begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$, $\beta_2\alpha = tsts^2tsr = \begin{pmatrix}3 & 2\\ 2 & 1\end{pmatrix}$ *and* $\beta_3\alpha = ts^2ts^2tststr = \begin{pmatrix}-1 & 2\\ -2 & 5\end{pmatrix}$. *Also using the permutation method, we get also the signature of Π (3) α as* $(0;+;[-];\{(\infty^{(4)})\})$.

Notice that the normal subgroups of Π different from Π, Π_0, $\Gamma\alpha$, $\Pi(2)$ and $\Pi(3)\alpha$, does not contain any reflection.

Theorem 2.4

(i) $|\Gamma^2 : (\Gamma^2)'| = 9$.

(ii) *The group $(\Gamma^2)'$ is a free group of rank 4 with basis $ststs^2ts^2t$, sts^2ts^2tst, s^2tstss^2t, s^2ts^2tstst and of index 3 in Γ'.*

(iii) $|\Gamma^3 : (\Gamma^3)'| = 8$.

(iv) *The group $(\Gamma^3)'$ is a free group of rank 5 with basis $tsts^2tsts^2$, ts^2tsts^2ts, $tststs ts^2ts^2ts^2$, $ststs^2tsts$, $tststs^2tstst$ and of index 4 in Γ'.*

Proof. (i) and (ii) Since $\Pi' = \Gamma^2$, we have $(\Gamma^2)' = \Pi''$. Then it is easy to see from [5, p. 28]. Also, using the permutation method, we get also the signature of $(\Gamma^2)'$ as $(1; +; [\infty, \infty, \infty]; \{(-)\}) = (1; +; [\infty^{(3)}]; \{(-)\})$.

(iii) It is well known that Γ^3 has the presentation

$$\langle t, sts^2, s^2ts | t^2 = (sts^2)^2 = (s^2ts)^2 = I\rangle \cong C_2 * C_2 * C_2.$$

The quotient group $\Gamma^3/(\Gamma^3)'$ is the group obtained by adding the abelianizing to the relations of Γ^3. Then

$$\Gamma^3/(\Gamma^3)' \cong C_2 \times C_2 \times C_2.$$

Therefore, we obtain $|\Gamma^3 : (\Gamma^3)'| = 8$.

(iv) Now we choose $\Sigma = \{I, t, sts^2, s^2ts, tsts^2, ts^2ts, ststs, tststs\}$ as a Schreier transversal for $(\Gamma^3)'$. According to the Reidemeister–Schreier method, we can form all possible products:

$$\begin{array}{ll}
I.t.(t)^{-1} = I, & tsts^2.sts^2.(t)^{-1} = I,\\
t.t.(I)^{-1} = I, & ts^2ts.sts^2.(tststs)^{-1} = ts^2ts^2tsts^2ts^2t,\\
sts^2.t.(tsts^2)^{-1} = sts^2tsts^2t, & ststs.sts^2.(s^2ts)^{-1} = ststs^2tsts,\\
s^2ts.t.(ts^2ts)^{-1} = s^2tsts^2tst, & tststs.sts^2.(ts^2ts)^{-1} = tststs^2tstst,\\
tsts^2.t.(sts^2)^{-1} = tsts^2tsts^2, & I.s^2ts.(s^2ts)^{-1} = I,\\
ts^2ts.t.(s^2ts)^{-1} = ts^2tsts^2ts, & t.s^2ts.(ts^2ts)^{-1} = I,\\
ststs.t.(tststs)^{-1} = ststststs^2ts^2ts^2t, & sts^2.s^2ts.(ststs)^{-1} = I,\\
tststs.t.(ststs)^{-1} = tstststs^2ts^2ts^2, & s^2ts.s^2ts.(I)^{-1} = I,\\
I.sts^2.(sts^2)^{-1} = I, & tsts^2.s^2ts.(tststs)^{-1} = I,\\
t.sts^2.(tsts^2)^{-1} = I, & ts^2ts.s^2ts.(t)^{-1} = I,\\
sts^2.sts^2.(I)^{-1} = I, & ststs.s^2ts.(sts^2)^{-1} = I,\\
s^2ts.sts^2.(ststs)^{-1} = s^2ts^2tsts^2ts^2, & tststs.s^2ts.(tsts^2)^{-1} = I.
\end{array}$$

Since $(sts^2tsts^2t)^{-1} = tsts^2tsts^2$, $(s^2tsts^2tst)^{-1} = ts^2tsts^2ts$, $tststs^2ts^2ts^2t)^{-1} = tstststs^2ts^2ts^2$, $(s^2ts^2tsts^2ts^2)^{-1} = ststs^2tsts$ and $(ts^2ts^2tsts^2ts^2t)^{-1} = tststs^2tstst$, the generators are $\delta_1 = tsts^2tsts^2$, $\delta_2 = ts^2tsts^2ts$, $\delta_3 = tstststs^2ts^2ts^2$, $\delta_4 = ststs^2tsts$ and $\delta_5 = tststs^2tstst$. Here $\delta_1 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, $\delta_2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\delta_3 = \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}$, $\delta_4 = \begin{pmatrix} -1 & -3 \\ 3 & 8 \end{pmatrix}$ and $\delta_5 = \begin{pmatrix} 8 & -3 \\ 3 & -1 \end{pmatrix}$. Also, since $|\Gamma : (\Gamma^3)'| = 24$ and $|\Gamma : \Gamma'| = 6$, we obtain $|\Gamma' : (\Gamma^3)'| = 4$. Also, using the permutation method, we obtain the signature of $(\Gamma^3)'$ as $(1; +; [\infty^{(4)}]; \{(-)\})$.

Notice that this theorem was proved by Newman and Smart in [8]. But they did not give the generators of the $(\Gamma^2)'$ and $(\Gamma^3)'$. Also this theorem generalized to the Hecke groups $H(\lambda_q)$, $q \geqslant 3$ prime, by Sahin and Koruoğlu in [12].

Corollary 2.5

(i) *$\Pi(4)\alpha = (\Gamma^3)'$ and $(\Pi(2))' = \Pi(4)$.*
(ii) *There are exactly 2 normal subgroups of index 48 in Π. Explicitly these are*

$$\Pi(4)\alpha = \langle \delta_1, \delta_2, \delta_3, \delta_4, \delta_5\rangle$$
$$\Pi(4) = \langle \delta_1\alpha, \delta_2\alpha, \delta_3\alpha, \delta_4\alpha, \delta_5\alpha\rangle$$

where $\delta_1\alpha = ts^2ts^2ts^2ts^2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $\delta_2\alpha = tstststs = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $\delta_3\alpha = ts^2tsts^2ts^2tsts^2 = \begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix}$, $\delta_4\alpha = sts^2ts^2ts^2ts = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$ *and* $\delta_5\alpha = ts^2tststs^2t = \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix}$. *Also the signature of $\Pi(4)$ is* $(0; +; [\infty^{(6)}]; \{(-)\})$.

Therefore we have only left the subgroups $\Gamma(6)$, $\Gamma(6)\alpha$ and $\Gamma(12)$ to consider. In these cases we can say only the following.

Theorem 2.6

(i) *The group $\Gamma(6)$ is a free group of rank 13 and of index 144 in Π. The signature of $\Gamma(6)$ is* $(1; +; [\infty^{(12)}]; \{(-)\})$.
(ii) *The group $\Gamma(12)$ is a free group of rank 25 and of index 576 in Π. The signature of $\Gamma(12)$ is* $(13; +; [\infty^{(24)}]; \{(-)\})$.

Corollary 2.7. *There are exactly 2 normal subgroups of index 144 in Π. Explicitly these are $\Gamma(6)$ and $\Gamma(6)\,\alpha$.There is exactly a normal subgroup of index 576 in Π. This is $\Gamma(12)$.*

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