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Generalized Fibonacci sequences related to the extended hecke groups and an application to the extended modular group

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Abstract

The extended Hecke groups $\overline{H}(\lambda_q)$ are generated by $T(z) = -1/z$, $S(z) = -1/(z+\lambda_q)$ and $R(z) = 1/\overline{z}$ with $\lambda_q = 2\cos(\pi/q)$ for $q \geq 3$ integer. In this paper, we obtain a sequence which is a generalized version of the Fibonacci sequence given in [6] for the extended modular group $\overline{\Gamma}$, in the extended Hecke groups $\overline{H}(\lambda_q)$. Then we apply our results to $\overline{\Gamma}$ to find all elements of the extended modular group $\overline{\Gamma}$.

Key Words: Extended Hecke groups, extended modular group, Fibonacci numbers

1. Introduction

In [4], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

E. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, or $\lambda \geq 2$. These groups have come to be known as the *Hecke Groups*, and we will denote them $H(\lambda_q)$, $H(\lambda)$, for $q \geq 3$, $\lambda \geq 2$, respectively. Hecke group $H(\lambda_q)$ is the Fuchsian group of the first kind when $\lambda = \lambda_q$ or $\lambda = 2$, and $H(\lambda)$ is the Fuchsian group of the second kind when $\lambda > 2$. In this study, we will focus the case $\lambda = \lambda_q$, $q \geq 3$. Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q and it has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q. \tag{1}$$

In the literature, the Hecke groups $H(\lambda_q)$ and their normal subgroups have been extensively studied in

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many aspects (see [1], [2] and [5]).

The extended Hecke group, denoted by $\overline{H}(\lambda_q)$, has been defined in [11] and [12] by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the Hecke group $H(\lambda_q)$. In [11], [12] and [14], some normal subgroups of the extended Hecke groups $\overline{H}(\lambda_q)$ (commutator subgroups, even subgroups, principal congruence subgroups, Fuchsian subgroups) and some relations between them were studied. The extended Hecke group $\overline{H}(\lambda_q)$ has the presentation

$$< T, S, R \mid T^2 = S^q = R^2 = I, RT = TR, RS = S^{q-1}R > \cong D_2 *_{\mathbb{Z}_2} D_q. \tag{2}$$

The Hecke group $H(\lambda_q)$ is a subgroup of index 2 in $\overline{H}(\lambda_q)$. It is clear that $\overline{H}(\lambda_q) \subset PGL(2, \mathbb{Z}[\lambda_q])$ when $q > 3$ and $\overline{H}(\lambda_q) = PGL(2, \mathbb{Z}[\lambda_q])$ when $q = 3$.

Throughout this paper, we identify each matrix A in $GL(2, \mathbb{Z}[\lambda_q])$ with $-A$, so that they each represent the same element of $\overline{H}(\lambda_q)$. Thus we can represent the generators of the extended Hecke group $\overline{H}(\lambda_q)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix} \ \text{ and } \ R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $q = 3$, then the extended Hecke group $\overline{H}(\lambda_3)$ is the extended modular group $\overline{\Gamma} = PGL(2, \mathbb{Z})$. The extended modular group $\overline{\Gamma}$ has been intensively studied. For examples of these studies see [6], [15]. In [13], they have investigated the power and free subgroups of the extended modular group $\overline{\Gamma}$.

In [6], Jones and Thornton found that there is a relationship between Fibonacci numbers and the entries of a matrice representation of an element of the extended modular group $\overline{\Gamma}$. If

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \overline{\Gamma},$$

then the k^{th} power of f is

$$f^k = \begin{pmatrix} f_{k-1} & f_k \\ f_k & f_{k+1} \end{pmatrix},$$

where f_k is the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$ and $f_k = f_{k-1} + f_{k-2}$.

Also, there are some papers related with relationships between Pell-numbers, Fibonacci and Lucas numbers and modular group in [8], [9] and [10].

In this paper, we obtain a sequence which is a generalized version of the Fibonacci sequence given in [6] for the extended modular group $\overline{\Gamma}$, in the extended Hecke groups $\overline{H}(\lambda_q)$. Then we apply our results to $\overline{\Gamma}$ to find all elements of the extended modular group $\overline{\Gamma}$. In fact, in [16], Özgür found two sequences which are generalization of Fibonacci sequence and Lucas sequence in the Hecke groups $H(\sqrt{q}\ )$, $q \geq 5$ prime. The Hecke groups $H(\sqrt{q}\ )$, $q \geq 5$ prime, are Fuchsian groups of the second kind and they do not contain any anti-automorphism. Since our studied groups contain reflections, they are NEC groups. To obtain the results given in Section 2 we use same method used in [16].

2. Generalized Fibonacci sequences in the extended Hecke groups $\overline{H}(\lambda_q)$

Firstly, let

$$h = TSR = \begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix} \text{ and } f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & \lambda_q \end{pmatrix}$$

from $\overline{H}(\lambda_q)$.

Lemma 1 *For the element $h = TSR$ in $\overline{H}(\lambda_q)$, the k^{th} power of h is as follows,*

$$h^k = \begin{pmatrix} a_k & a_{k-1} \\ a_{k-1} & a_{k-2} \end{pmatrix}$$

where $a_0 = 1$, $a_1 = \lambda_q$ and $a_k = \lambda_q a_{k-1} + a_{k-2}$, for $k \geq 2$.

Proof. In order to prove, first of all, let us show

$$h^k = \begin{pmatrix} \lambda_q a_{k-1} + b_{k-1} & a_{k-1} \\ a_{k-1} & b_{k-1} \end{pmatrix}.$$

For this we use induction method. Let

$$h = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{ and } h^k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}.$$

If we continue using $h = \begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix}$, we find h^2 as

$$h^2 = \begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+\lambda_q^2 & \lambda_q \\ \lambda_q & 1 \end{pmatrix} = \begin{pmatrix} \lambda_q a_1 + b_1 & a_1 \\ a_1 & b_1 \end{pmatrix}.$$

Thus the correct result for $k = 2$ is obtained. Now, let us assume that

$$h^{k-1} = \begin{pmatrix} \lambda_q a_{k-2} + b_{k-2} & a_{k-2} \\ a_{k-2} & b_{k-2} \end{pmatrix}.$$

Finally h^k is found as

$$\begin{aligned} h^k &= \begin{pmatrix} \lambda_q a_{k-2} + b_{k-2} & a_{k-2} \\ a_{k-2} & b_{k-2} \end{pmatrix}\begin{pmatrix} \lambda_q & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_{k-2} + \lambda_q(\lambda_q a_{k-2} + b_{k-2}) & b_{k-2} + \lambda_q a_{k-2} \\ b_{k-2} + \lambda_q a_{k-2} & a_{k-2} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_q a_{k-1} + b_{k-1} & a_{k-1} \\ a_{k-1} & b_{k-1} \end{pmatrix}. \end{aligned}$$

Notice that $b_2 = a_1$, $b_{k-1} = a_{k-2}$ and $b_k = a_{k-1}$. Together with these, due to the boundary condition of $a_0 = 1$, we get $b_1 = a_0$ and

$$h^k = \begin{pmatrix} a_k & a_{k-1} \\ a_{k-1} & a_{k-2} \end{pmatrix}.$$

Therefore, we get a real number sequence a_k. The definition and boundary conditions of this sequence are

$$\begin{aligned} a_k &= \lambda_q a_{k-1} + a_{k-2}, \text{ for } k \geq 2, \\ a_0 &= 1, \ a_1 = \lambda_q. \end{aligned} \tag{3}$$

□

Similar to the previous theorem we can give the following corollary.

Corollary 2 *The* k^{th} *power of* f *is*

$$f^k = \begin{pmatrix} a_{k-1} & a_k \\ a_k & a_{k+1} \end{pmatrix}$$

where $a_0 = 1$, $a_1 = \lambda_q$ *and* $a_k = \lambda_q a_{k-1} + a_{k-2}$, *for* $k \geq 2$.

Notice that this result coincides with the ones given by Jones and Thornton in [6, p. 28].

We mentioned a sequence a_k in the Lemma 1. Now, let us give the general formula of this sequence a_k. We will get a generalized Fibonacci sequence by this formula.

Proposition 3 *For all* $k \geq 2$,

$$a_k = \frac{1}{\sqrt{\lambda_q^2 + 4}} \left[\left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2} \right)^{k+1} - \left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2} \right)^{k+1} \right]. \tag{4}$$

Proof. To solve the equation (3), let a_k to be a characteristic polynomial r^k. Then we get the equation

$$r^k = \lambda_q r^{k-1} + r^{k-2} \Rightarrow r^2 - \lambda_q r - 1 = 0.$$

The roots of this equation are

$$r_{1,2} = \frac{\lambda_q \pm \sqrt{\lambda_q^2 + 4}}{2}.$$

Benefiting from these roots $r_{1,2}$, we will reach a general formula of a_k. If we write a_k as combinations of the roots $r_{1,2}$, we get

$$a_k = A \left(\frac{\lambda_q + \sqrt{\lambda_q^2 + 4}}{2} \right)^k + B \left(\frac{\lambda_q - \sqrt{\lambda_q^2 + 4}}{2} \right)^k.$$

Notice that $a_0 = 1$ and $a_1 = \lambda_q$, we can compute constants A and B.

$$\begin{aligned} a_0 &= 1 = A + B, \\ a_1 &= \lambda_q = A\left(\frac{\lambda_q + \sqrt{\lambda_q^2+4}}{2}\right) + B\left(\frac{\lambda_q - \sqrt{\lambda_q^2+4}}{2}\right) \end{aligned}$$

and so

$$2\lambda_q = A(\lambda_q + \sqrt{\lambda_q^2+4}) + (1-A)(\lambda_q - \sqrt{\lambda_q^2+4}).$$

Hence constants A and B are

$$A = \frac{\lambda_q + \sqrt{\lambda_q^2+4}}{2\sqrt{\lambda_q^2+4}} \text{ and } B = \frac{\sqrt{\lambda_q^2+4} - \lambda_q}{2\sqrt{\lambda_q^2+4}}.$$

As the last step, we get the formula of a_k as

$$\begin{aligned} a_k &= \left(\frac{\lambda_q + \sqrt{\lambda_q^2+4}}{2\sqrt{\lambda_q^2+4}}\right)\left(\frac{\lambda_q + \sqrt{\lambda_q^2+4}}{2}\right)^k + \left(\frac{\sqrt{\lambda_q^2+4} - \lambda_q}{2\sqrt{\lambda_q^2+4}}\right)\left(\frac{\lambda_q - \sqrt{\lambda_q^2+4}}{2}\right)^k \\ &= \frac{1}{\sqrt{\lambda_q^2+4}}\left[\left(\frac{\lambda_q + \sqrt{\lambda_q^2+4}}{2}\right)^{k+1} - \left(\frac{\lambda_q - \sqrt{\lambda_q^2+4}}{2}\right)^{k+1}\right]. \end{aligned}$$

□

This formula, as seen, is a generalized Fibonacci sequence. If $\lambda_q = 1$, we get the common Fibonacci sequence used in the literature. Here $a_k = h_{k+1}$ is the $(k+1)^{\text{th}}$ Fibonacci number. Also, the Fibonacci sequence is

$$a_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] = h_{k+1}.$$

So we get

$$h^k = \begin{pmatrix} h_{k+1} & h_k \\ h_k & h_{k-1} \end{pmatrix}$$

in the extended modular group $\overline{\Gamma}$.

This outcome is very important for us. Since, in the following section of this paper, we get all the elements of the extended modular group $\overline{\Gamma}$ by using the Fibonacci numbers. Thus the extended modular group $\overline{\Gamma}$ and related topics can be studied more thoroughly by the help of these results in future works.

3. An application to the extended modular group

Now we give an application of our findings given above to the extended modular group $\overline{\Gamma}$.

From [3] and [7], we know that the following matrices are called blocks in the modular group and the extended modular group:

$$TS = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } TS^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{5}$$

Let $W(T,S,R)$ be a reduced word in $\overline{\Gamma}$ such that the sum of exponents of R is even number; then this word is

$$S^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^j \tag{6}$$

and $W(T,S,R)$ is a reduced word in $\overline{\Gamma}$ such that the sum of exponents of R is odd number, then this word is

$$RS^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^j \tag{7}$$

for $i = 0, 1, 2$ and $j = 0, 1$. The exponents of blocks are positive integers, but m_0 and n_k may be zero. This representation is general and called a *block reduced form,* abbreviated as BRF in [7].

We can write any reduced word in BRF by these blocks. For examples, the word $TSTSTSTS^2TS^2TS$ in BRF is $(TS)^3(TS^2)^2(TS)$ and the word RTS^2RTS^2R in BRF is $R(TS^2)(TS)$.

By using these BRF's, in [3], Fine has studied trace classes in the modular group Γ. Then, in [7], Koruoğlu et al. have investigated trace classes in the extended modular group $\overline{\Gamma}$.

Now we need the following matrices to get the main results in the extended modular group $\overline{\Gamma}$.

$$f = RTS = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad h = RTS^2 = TSR = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

These matrices are important for our work and specific cases of f and h given in Section 2 for $\lambda_q = 1$. To obtain each element in the forms (6) or (7) in $\overline{\Gamma}$ by powers of h and f, we need the following definition.

Definition 4 *f and h are called new blocks. The word $W(T,S,R)$ in BRF is called a new block reduced form abbreviated as $NBRF$ if $W(T,S,R)$ is obtained by powers of h and f.*

Now we give the following corollary.

Corollary 5 *Each reduced word in the extended modular group $\overline{\Gamma}$ has a $NBRF$.*

Proof. Let $W(T,S,R)$ be a reduced word in $\overline{\Gamma}$. Then in BRF, $W(T,S,R)$ is either

$$S^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^j,$$

or

$$RS^i(TS)^{m_0}(TS^2)^{n_0}...(TS)^{m_k}(TS^2)^{n_k}T^j.$$

For the blocks TS and TS^2 in $W(T,S,R)$, we obtain the relations $TS = Rf = hR$ and $TS^2 = Rh = fR$. Therefore, if these relations are written instead of TS and TS^2 in $W(T,S,R)$, we get desired result. $\square$

By using the Corollary 5 we can find all elements of the extended modular group $\overline{\Gamma}$ by powers of h and f. Now, let us give an application by using results we found so far.

Example 6 *Let the word in BRF,*

$$W = (TS^2)(TS)^2(TS^2)(TS)^2,$$

be in the extended modular group $\overline{\Gamma}$. Owing to the relations $TS = Rf = hR$ and $TS^2 = Rh = fR$,

$$W = (Rh)(Rf)(Rf)(Rh)(Rf)(Rf).$$

Therefore, this word in $NBRF$ is obtained as

$$W = f^2h^3f = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix}\begin{pmatrix} h_4 & h_3 \\ h_3 & h_2 \end{pmatrix}\begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

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