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Determination of Genus of Normal Subgroups of Discrete Groups

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Abstract. In this work, subgroups of a special class of discrete subgroups of $PLS(2,\mathbf{R})$, namely the ones of the first kind with genus 0, have been studied. We establish a technique to compute the genus of these subgroups in terms of the genus of easier groups. The method established here can be used for triangle groups, surface groups and Hecke groups (including the well-known modular group).

Keywords: Hecke groups, permutation method, Fuchsian group, signature, Riemann surface.
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INTRODUCTION

In this work, we consider an important class of discrete groups, namely those of the first kind with genus 0. A discrete group Γ is of the first kind iff its limit set is $\mathbf{R}$. These might be classified into three classes:

(i) Those with elliptic elements but parabolics,
(ii) Those with parabolic elements but elliptics,
(iii) Those with both elliptics and parabolics.

Some examples of those are the triangle groups, surface groups and Hecke groups (including the well-known modular group), respectively.

If one defines

$$\mu(\Gamma) = 2g - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + t,$$

where g is the genus of the underlying Riemann surface, t is the parabolic class number and m_i are the periods of Γ, then $2\pi.\mu(\Gamma)$ is the hyperbolic area of a fundamental region of the group. Let Γ_1 be a subgroup of Γ of finite index. Then

$$[\Gamma : \Gamma_1] = \frac{\mu(\Gamma_1)}{\mu(\Gamma)}$$

is known as the Riemann-Hurwitz formula (RHF).

Now let Θ be an epimorphism between two such groups:

$$\Theta : \Gamma \longrightarrow \Delta.$$

Let Λ be a subgroup of Δ having genus g. We are interested in finding the genus of the inverse image group $\Theta^{-1}(\Lambda)$ of Λ in terms of g. The group Δ is usually "simpler" than Γ. Therefore by means of the RHF, it is easier to find the

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genus g of Λ rather than the genus, say g', of any subgroup of Γ. For this reason, we shall use the inverse image of Λ and hence g.

To use the RHF, one needs to know the periods of Λ and $\Theta^{-1}(\Lambda)$. One way of doing this is to make use of a result of D. Singerman, [4]. The original form of this theorem applies to all Fuchsian groups, but here, as we noted earlier, we restrict ourselves to the ones of the first kind. It is sometimes convenient to consider the parabolic elements as elliptic elements of infinite order. So we can assume that a group Γ_2 has signature $(g;m_1,\ldots,m_r,m_{r+1},\ldots,m_{r+t})$ where $m_{r+1}=\cdots=m_{r+t}=\infty$

CALCULATIONS

Let now Γ_1 be a subgroup of Γ_2 of finite index μ. Let v_i be the exponent of x_i modulo Γ_1, i.e. the least integer such that $x_i^{v_i}\in\Gamma_1$, (Here x_i denotes the generator of order m_i). It follows that $v_i<\infty$ and $v_i|m_i$ if $m_i<\infty$. Some of the x_i's in Γ_2 may have exponent m_i modulo Γ_1. Rearranging the periods so that $v_i=m_i$ only for $1\le i\le p$ and x_{i+p} has exponent $n_i<m_{i+p}$ otherwise, we find that the signature of Γ_2 can be rewritten as $(g;m_1,\ldots,m_p,n_1k_1,\ldots,n_qk_q)$ where $p+q=r+t$ and $1<k_i\le\infty$. Then Singerman's result can be deduced to the following form:

Theorem 1 *Let Γ_1 be a subgroup of Γ_2 of finite index μ. Then Γ_1 has signature*

$$(g_1;k_1^{(\mu/n_1)},\ldots,k_q^{(\mu/n_q)})$$

where $k_i^{(\mu/n_i)}$ means that the period k_i occurs μ/n_i times. Here g_1 can be found by the RHF.

By means of Theorem 1, we can find the periods of both Λ and $\Theta^{-1}(\Lambda)$. Then it is easy to find g' in terms of g. In fact we obtain the following main result of this work:

Theorem 2 *Let Θ be the homomorphism between Γ and Δ, two discrete groups of the first kind, defined as above. Then the genus g of a subgroup Λ of Δ is equal to the genus g' of $\Theta^{-1}(\Lambda)$; i.e. Θ^{-1} preserves the genus.*

Proof We prove this result in three cases. All other cases can be reduced to one of those.

Case 1. Let

$$\Theta:\Gamma=(0;m_1,\ldots,m_r)\longrightarrow\Delta=(0;n_1,\ldots,n_r)$$

be a homomorphism for $m_i\ge 2$, $n_j\ge 1$, so that $n_i|m_i$ for every i. Let Λ be a subgroup of Δ of genus g. Then $\Theta^{-1}(\Lambda)$ is a subgroup of Γ with genus g, as well, i.e. Θ^{-1} preserves the genus.Let $\Theta^{-1}(\Lambda)$ have genus g', and let $y_1,\ldots,y_r$ be the generators of Γ. Then

$$\Theta(y_i)=(v_{i1})(v_{i2})\ldots(v_{i\alpha_i})$$

such that

$$\sum_{j=1}^{\alpha_i}v_{ij}=k$$

for $1\le i\le r$, where $k=[\Delta:\Lambda]$ and (v_{ij}) denotes a cycle of length v_{ij} in the permutation representation of $\Theta(y_i)$. Since Θ is an epimorphism, we also have $k=\left[\Gamma:\Theta^{-1}(\Lambda)\right]$. The periods of Λ are

$$\frac{n_1}{v_{11}},\ldots,\frac{n_1}{v_{1\alpha_1}},\ldots,\frac{n_r}{v_{r1}},\ldots,\frac{n_r}{v_{r\alpha_r}}$$

and the periods of $\Theta^{-1}(\Lambda)$ are

$$\frac{m_1}{v_{11}},\ldots,\frac{m_1}{v_{1\alpha_1}},\ldots,\frac{m_r}{v_{r1}},\ldots,\frac{m_r}{v_{r\alpha_r}}.$$

Hence by the Riemann-Hurwitz Formula,

$$2g-2+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{n_i}\right)=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right)\right)$$

and

$$2g'-2+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{m_i}\right)=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right).$$

Now $g=g'$ iff

$$k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right)\right)-\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{n_i}\right)=k\left(-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right)-\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\left(1-\frac{v_{ij}}{m_i}\right)$$

iff

$$-k\sum_{i=1}^{r}\frac{1}{n_i}+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{v_{ij}}{n_i}=-k\sum_{i=1}^{r}\frac{1}{m_i}+\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{v_{ij}}{m_i}$$

iff

$$-k\sum_{i=1}^{r}\frac{1}{n_i}+k\sum_{i=1}^{r}\frac{1}{n_i}=-k\sum_{i=1}^{r}\frac{1}{m_i}+k\sum_{i=1}^{r}\frac{1}{m_i}$$

since $\sum_{j=1}^{\alpha_i}v_{ij}=k$. Therefore for every k, $g=g'$.

Case 2. For $m_i\geq 2$, $n_j\geq 1$, so that for every i, $n_j|m_i$, let

$$\Theta:\Gamma=(0;m_1,\ldots,m_r,\infty^{(t)})\longrightarrow\Delta=(0;n_1,\ldots,n_r,n_{r+1},\ldots,n_{r+s},\infty^{(t-s)})$$

be a homomorphism. Let Λ be a subgroup of Δ with genus g. Then $\Theta^{-1}(\Lambda)$ is a subgroup of Γ with genus g as well.

Let now $\Theta^{-1}(\Lambda)$ have genus g'. Then with the notation of Case 1,

$$\begin{array}{rl}\Theta(y_i)=(v_{i1})\ldots(v_{i\alpha}), & for\ 1\leq i\leq r\\ \Theta(y_{r+i})=(v_{r+i,1})\ldots(v_{r+i,\alpha}), & for\ 1\leq i\leq s\\ \Theta(y_{r+i})=(v_{r+i,1})\ldots(v_{r+i,\alpha}), & for\ s+1\leq i\leq t\end{array}.$$

Then the periods of Λ are

$$\frac{n_1}{v_{11}},\ldots,\frac{n_r}{v_{r\alpha_r}},\frac{n_{r+1}}{v_{r+1,1}},\ldots,\frac{n_{r+1}}{v_{r+1,\alpha_{r+1}}},\ldots,\frac{n_{r+s}}{v_{r+s,1}},\ldots,\frac{n_{r+s}}{v_{r+s,\alpha_{r+s}}},\ldots,\infty^{(\alpha_{r+s+1})},\ldots,\infty^{(\alpha_{r+t})}$$

and the periods of $\Theta^{-1}(\Lambda)$ are

$$\frac{m_1}{v_{11}},\ldots,\frac{m_1}{v_{1\alpha_1}},\ldots,\frac{m_r}{v_{r1}},\ldots,\frac{m_r}{v_{r\alpha_r}},\infty^{(\alpha_{r+1})},\ldots,\infty^{(\alpha_{r+t})}.$$

Hence as in the proof of Case 1, we have $g=g'$ for every index k.

Case 3. Let $m_i\geq 2$, $n_j\geq 1$ and $n_i|m_i$ for $1\leq i\leq r$. Let

$$\Theta:\Gamma=(0;m_1,\ldots,m_r,\infty^{(t)})\longrightarrow\Delta=(0;n_1,\ldots,n_{r+t})$$

be a homomorphism. Proceeding similar to the first two cases, we obtain the required result.

Therefore we have completed the discussion of all three cases. All other cases can be reduced to one of these, e.g. if

$$\Theta:(0;m_1,\ldots,m_r,\infty^{(t)})\longrightarrow(0;n_1,\ldots,n_s,\infty^{(t)}),$$

this can be considered as a special case of Case 2 with $s = 0$. If

$$\Theta : (0; \infty^{(t)}) \longrightarrow (0; n_1, \ldots, n_s, \infty^{(t-s)}),\ 0 \leq s \leq t$$

this also can be considered as a special case of Case 2 with $r = 0$. This completes the proof of Theorem 2.

Some restricted applications of Theorem 2 has been done in the special case of Hecke groups $H(\lambda_q)$ in [1]. These are the discrete subgroups of $PSL(2, \mathbf{R})$ of the first kind having signature $(0; 2, q, \infty)$ for $q \in \mathbf{Z}$, $q \geq 3$. Therefore they fall into the class (iii) in our classification. In [1] and [3], a classification of some normal subgroups of $H(\lambda_q)$ has been done and this technique was often used to establish information about them.

As an example let us consider the homomorphism from $H(\lambda_q)$ to $(2, q, 2q)$, for odd q, taking the generator **R** of order 2 of $H(\lambda_q)$ to the generator r of order 2, and the generator S of order q to the generator s of order q. This is an infinite image of $H(\lambda_q)$, for odd q. If we take the subgroup Λ of $(2, q, 2q)$ having the relations $r^2 = s^q = rsrs^{-1} = 1$, then Λ has the signature $\left(\frac{q-1}{2}; \infty\right)$ by the RHF. Hence $\Theta^{-1}(\Lambda)$ has genus $\frac{q-1}{2}$ and also the same signature. Note that this gives us the commutator subgroup $\Theta^{-1}(\Lambda) = H'(\lambda_q)$ of $H(\lambda_q)$. When q is even, by mapping $H(\lambda_q)$ to $(2, q, q)$ and applying the same method we can obtain the commutator subgroup with signature $(\frac{q}{2} - 1; \infty, \infty)$. Note that $H'(\lambda_q)$ is therefore isomorphic to a free group of rank $q - 1$, [2].

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