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On the Efficiency of Semi-Direct Products of Finite Cyclic Monoids by One-Relator Monoids

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Abstract. In this paper we give necessary and sufficient conditions for the efficiency of a standard presentation for the semi-direct product of finite cyclic monoids by one-relator monoids.

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INTRODUCTION

Let $\mathscr{P} = [\mathbf{x};\mathbf{r}]$ be a finite presentation for a monoid M. Then the *Euler characteristic* of $\mathscr{P}$ is defined by $\chi(\mathscr{P}) = 1 - |\mathbf{x}| + |\mathbf{r}|$ and an upper bound of M is defined by $\delta(M) = 1 - rk_{\mathbf{Z}}(H_1(M)) + d(H_2(M))$. In an unpublished work, S.J. Pride has shown that $\chi(\mathscr{P}) \geq \delta(M)$. With this background, one can define a monoid presentation $\mathscr{P}$ to be *efficient* if $\chi(\mathscr{P}) = \delta(M)$, and then M is called *efficient* if it has an efficient presentation.

It is well known that one of the effective way to show efficiency for the monoid M is to use spherical monoid pictures over $\mathscr{P}$. These geometric configurations are the representative elements of the Squier complex denoted by $\mathscr{D}(\mathscr{P})$ (see, for example [4], [5], [7]). Suppose $\mathbf{Y}$ is a collection of spherical monoid pictures over $\mathscr{P}$. Two monoid pictures $\mathbf{P}$ and $\mathbf{P}'$ are *equivalent relative to* $\mathbf{Y}$ if there is a finite sequence of monoid pictures $\mathbf{P} = \mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_m = \mathbf{P}'$ where, for $1 \leq i \leq m$, the monoid picture $\mathbf{P}_i$ is obtained from the picture $\mathbf{P}_{i-1}$ either by the insertion, deletion and replacement operations. By definition, a set $\mathbf{Y}$ of spherical monoid pictures over $\mathscr{P}$ is a *trivializer of* $\mathscr{D}(\mathscr{P})$ if every spherical monoid picture is equivalent to an empty picture relative to $\mathbf{Y}$. The trivializer is also called a set of generating pictures.

For any monoid picture $\mathbf{P}$ over $\mathscr{P}$ and for any $R \in \mathbf{r}$, $\exp_R(\mathbf{P})$ denotes the *exponent sum* of R in $\mathbf{P}$ which is the number of positive discs labelled by R_+, minus the number of negative discs labelled by R_-. For a non-negative integer n, $\mathscr{P}$ is said to be *n-Cockcroft* if $\exp_R(\mathbf{P}) \equiv 0 \,(\mathrm{mod}\, n)$, (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbf{P}$ over $\mathscr{P}$. Then a monoid M is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation. In fact to verify the n-Cockcroft property, it is enough to check for pictures $\mathbf{P} \in \mathbf{Y}$, where $\mathbf{Y}$ is a trivializer (see [4], [5]). The 0-Cockcroft property is usually just called Cockcroft.

The following result is also an unpublished result by S.J. Pride.

Theorem 1 *Let $\mathscr{P}$ be a monoid presentation. Then $\mathscr{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

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Let K be a finite cyclic monoid with the presentation $\mathscr{P}_K = [y\,;\, y^k = y^l\ (l < k)]$, and let A be one-relator monoid with presentation $\mathscr{P}_A = [\mathbf{x}\,;\, R_+ = R_-]$. Also let ψ be an endomorphism of K. Then we have a mapping

$$x \longrightarrow End(K), \quad x \longmapsto \psi_x,$$

where $x \in \mathbf{x}$.

In fact this induces a homomorphism $\theta : A \longrightarrow End(K)$, $x \longmapsto \psi_x$ if and only if $\psi_{R_+} = \psi_{R_-}$ and $(y^k)\psi_x = (y^l)\psi_x$ where $x \in \mathbf{x}$. For $R_+ = a_1 a_2 \cdots a_r$ and $R_- = z_1 z_2 \cdots z_s$, let us define $(y)\psi_{a_i}$ by y^{m_i} and $(y)\psi_{z_j}$ by y^{n_j}, where $a_i, z_j \in \mathbf{x}$ and $m_i, n_j \in \mathbf{Z}^+$. Therefore in order to define semi-direct product, we must have

$$[y^{m_1 m_2 \cdots m_r}] = [y^{n_1 n_2 \cdots n_s}], \tag{1}$$

and then we get the corresponding semi-direct product $M = K \times_\theta A$ with the presentation

$$\mathscr{P}_M = [y, \mathbf{x}\,;\, y^k = y^l,\ R_+ = R_-,\ T_{yx_i}\,], \tag{2}$$

where $T_{yx_i} : yx_i = x_i y^{t_i}$ $(x_i \in \mathbf{x}, t_i \in \mathbf{Z}^+)$ such that we have the relation (1) for y^{t_i}.

If $U = x_1 x_2 \cdots x_n$ is a positive word on $\mathbf{x}$, then we denote the word $(y)\psi_U$ by $(\cdots(((y)\psi_{x_1})\psi_{x_2})\psi_{x_3}\cdots)\psi_{x_n})$ and this can be represented by a monoid picture, say $\mathbf{A}_{U,y}$. For the relation $R_+ = R_-$, we have two important special cases $\mathbf{A}_{R_+,y}$ and $\mathbf{A}_{R_-,y}$ of this consideration. We should note that these non-spherical pictures consist of only T_{yx_i}-discs ($x_i \in \mathbf{x}$). Also, since $[(y^k)\psi_x]_{\mathscr{P}_K} = [(y^l)\psi_x]_{\mathscr{P}_K}$, there exists a non-spherical picture, say $\mathbf{B}_{S,x}$, over $\mathscr{P}_K$ where $S : y^k = y^l$. On the other hand, since we have $R_+ = R_-$, we get $[(y)\psi_{R_+}]_{\mathscr{P}_K} = [(y)\psi_{R_-}]_{\mathscr{P}_K}$. Hence there is a non-spherical picture over $\mathscr{P}_K$ which is denoted by $\mathbf{C}_{y,\psi_R}$ with $\iota(\mathbf{C}_{y,\psi_R}) = (y)\psi_{R_+}$ and $\tau(\mathbf{C}_{y,\psi_R}) = (y)\psi_{R_-}$, where $R : R_+ = R_-$.

Now, by considering the presentation $\mathscr{P}_M$ in (2), we prove the following theorem as a main result in the present paper.

Theorem 2 *Let p be a prime or* 0. *Then the presentation $\mathscr{P}_M$ is p-Cockcroft if and only if the following conditions hold.*

(i) $k - l \equiv 0\,(mod\ p)$,
(ii) *For each i,* $t_i \equiv 1\,(mod\ p)$,
(iii) $\frac{m_1 m_2 \cdots m_r - n_1 n_2 \cdots n_s}{k-l} \equiv 0\ (mod\ p)$,
(iv) $\exp_{T_{yx_i}}(\mathbf{A}_{R_+,y}) \equiv \exp_{T_{yx_i}}(\mathbf{A}_{R_-,y})\ (mod\ p)$.

We may refer [1, 2, 3, 4, 5, 6, 7] to the reader for most of the fundamental material (for instance, *semidirect products of monoids, Squier complex, a trivializer set* of the Squier complex, *spherical* and *non-spherical monoid pictures*) which will be needed here.

TRIVIALIZER SET $\mathscr{D}(\mathscr{P}_M)$

In [8], Wang constructed a finite trivializer set for the standard presentation for the semi-direct product. By using this paper, we can construct spherical monoid pictures, say $\mathbf{P}_{S,x}$ and $\mathbf{P}_{R,y}$, by the non-spherical pictures $\mathbf{B}_{S,x}$, $\mathbf{A}_{R_+,y}$, $\mathbf{A}_{R_-,y}$ and $\mathbf{C}_{y,\psi_R}$.

Let $\mathbf{X_A}$ and $\mathbf{X_K}$ be trivializer sets of $\mathscr{D}(\mathscr{P}_A)$ and $\mathscr{D}(\mathscr{P}_K)$, respectively. Also, let $\mathbf{C_1}$ consists of the pictures $\mathbf{P}_{S,x}$ and $\mathbf{C_2}$ consists of the pictures $\mathbf{P}_{R,y}$ where $S : y^k = y^l$ and $R : R_+ = R_-$.

Let us consider the presentation $\mathscr{P}_M$, as in(2). Then, by [8], a trivializer set of $\mathscr{D}(\mathscr{P}_M)$ is

$$\mathbf{X_A} \cup \mathbf{X_K} \cup \mathbf{C_1} \cup \mathbf{C_2}.$$

In here, the subpictures $\mathbf{C}_{y,\psi_R}$ and $\mathbf{B}^{-1}_{S,x_i}$ contain just $S : y^k = y^l$ $(l < k)$ discs. Moreover, each of the subpictures $\mathbf{A}_{R_+,y}$ and $\mathbf{A}^{-1}_{R_-,y}$ consists of just the T_{yx_i} discs. The reason for us keeping work on the above monoid pictures is their usage in the important connection between *efficiency* and *p-Cockcroft property*. Therefore, in the present paper, we will use this connection to get the efficiency. To do that we will count the exponent sums of the discs in these above pictures to obtain p-Cockcroft property for the presentation $\mathscr{P}_M$ given in (2).

PROOF OF THE MAIN RESULT AND ITS APPLICATION

Let us consider the pictures given in above. To prove Theorem 2, we will count the exponent sums of the discs in these pictures.

At first, let us think the pictures $\mathbf{P}_{S,x_i}$. It is easy to see that

$$exp_S(\mathbf{P}_{S,x_i}) = 1 - exp_S(\mathbf{B}_{S,x_i}),$$

and to p-Cockcroft property be hold, we need to have

$$\begin{aligned} exp_S(\mathbf{P}_{S,x_i}) \equiv 0\,(mod\,p) \quad &\Leftrightarrow \quad exp_S(\mathbf{B}_{S,x_i}) \equiv 1\,(mod\,p) \\ &\Leftrightarrow \quad t_i \equiv 1\,(mod\,p). \end{aligned}$$

Moreover, in $\mathbf{P}_{S,x_i}$, we also have l times positive and k times negative $T_{yx_i} : yx_i = x_i y^{t_i}$ discs. That means

$$exp_{T_{yx_i}}(\mathbf{P}_{S,x_i}) = l - k,$$

and so we must have $k - l \equiv 0\,(mod\,p)$ to p-Cockcroft property be hold.

Secondly let us consider the pictures $\mathbf{P}_{R,y}$. In these pictures, we have only one positive and one negative S disc. Therefore $exp_S(\mathbf{P}_{R,y}) \equiv 0\ (mod\,p)$. Now let us check the exponent sums in the subpictures $\mathbf{A}_{R_{+},y}$ and $\mathbf{A}^{-1}_{R_{-},y}$. (We recall that these two subpictures are dual of each other). Then, the exponent sum of T_{yx_i} discs in pictures $\mathbf{P}_{R,y}$ are

$$exp_{T_{yx_i}}(\mathbf{P}_{R,y}) = exp_{T_{yx_i}}(\mathbf{A}_{R_{+},y}) - exp_{T_{yx_i}}(\mathbf{A}_{R_{-},y}), \tag{3}$$

So we have $\exp_{T_{yx_i}}(\mathbf{A}_{R_{+},y}) \equiv \exp_{T_{yx_i}}(\mathbf{A}_{R_{-},y}) \pmod p$ to get efficiency. Thus it remains to check the subpictures $\mathbf{C}_{y,\psi_R}$. By a simple calculations, one can show that the number of discs is $\frac{m_1 m_2 \cdots m_r - n_1 n_2 \cdots n_s}{k-l} = c$ in this picture. Thus we have $c \equiv 0\,(\text{mod}\ p)$. Hence the result.

We note that, by considering the trivializer sets $\mathbf{X_A}$ and $\mathbf{X_K}$ of the Squier complexes $\mathscr{D}(\mathscr{P}_A)$ and $\mathscr{D}(\mathscr{P}_K)$, respectively, it can be easily deduced that $\mathscr{P}_A$ and $\mathscr{P}_K$ are p-Cockcroft, in fact Cockcroft, presentations.

These all above procedure give us sufficient conditions to be the presentation $\mathscr{P}_M$ in (2) is p-Cockcroft for any prime p. In fact the converse part (necessary conditions) of the theorem is quite clear.

As an example, let us suppose that the monoid A is presented by $\mathscr{P}_A = [x_1, x_2\,;\, x_1^m x_2 x_1 = x_2^m x_1 x_2]$. Then in order to define semi-direct product with K, we must have $[y^{t_1^{m+1} t_2}] = [y^{t_2^{m+1} t_1}]$ by replacing $(y)\psi_{x_1}$ with y^{t_1} and $(y)\psi_{x_2}$ with y^{t_2}. Hence we get the corresponding semi-direct product M with the presentation

$$\begin{aligned} \mathscr{P}_M = [y, x_1, x_2\,;\, & y^k = y^l,\, x_1^m x_2 x_1 = x_2^m x_1 x_2, \\ & yx_1 = x_1 y^{t_1},\, yx_2 = x_2 y^{t_2}\,]. \end{aligned} \tag{4}$$

Let us consider presentation given in (4) .Then we can give the following corollary as a consequence of the main result.

Corollary 3 *Let p be a prime or 0. Then the presentation $\mathscr{P}_M$ is p-Cockcroft if and only if the following conditions hold.*

- (i) $k - l \equiv 0\,(mod\ p)$,
- (ii) *For each* $i \in \{1,2\}$, $t_i \equiv 1\,(mod\ p)$,
- (iii) $\frac{t_1^{m+1} t_2 - t_2^{m+1} t_1}{k-l} \equiv 0\ (mod\ p)$,
- (iv) $\frac{1-t_1^m}{1-t_1} + t_1^m t_2 - t_2^m \equiv 0\ (mod\ p)$ *and*
  $\frac{1-t_2^m}{1-t_2} + t_1 t_2^m - t_1^m \equiv 0\ (mod\ p)$.

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