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On the approximation by trigonometric polynomials in weighted Lorentz spaces

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Abstract. We obtain estimates of structural characteristics of 2π -periodic functions by the best trigonometric approximations in weighted Lorentz spaces, and show that the order of generalized modulus of smoothness depends not only on the rate of the best approximation, but also on the metric of the spaces. In weighted Lorentz spaces L^{ps}, this influence is expressed not only in terms of the parameter p, but also in terms of the second parameter s.

1. Introduction

The well known Weiersterass theorem on approximation of continuous functions by trigonometric polynomials and its quantitative refinement represented by Jackson's inequality (see, e.g., [21, Section 5.1.2])

$$E_n(f) \leq C\omega\left(f, \frac{1}{n+1}\right) \tag{1.1}$$

are the basics of the approximation theory.

In inequality (1.1), $E_n(f)$ denotes the best approximation of a 2π-periodic continuous function f by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f) = \inf \max_{x\in[0,2\pi]} |f(x) - T_k(x)|\,,$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$, and

$$\omega(f,\delta) = \sup_{|h|\leq\delta} \max_{x\in[0,2\pi]} |f(x+h) - f(x)|$$

denotes the modulus of continuity of f. The analog of Jackson's inequality is valid also for the integral metrics and moduli of continuity of higher orders (see, e.g., [21, Section 5.3.1]).

Yet by the year 1912, S. Bernstein obtained the estimate inverse to Jackson's inequality in the space of continuous functions for some special cases [3]. Later, Quade [17], brothers A. and M. Timan [22], S. B. Stechkin [19], M. Timan [20], etc. proved such inverse estimates, including the case of of the spaces L^p, $1 < p < \infty$. Inequalities of this type played an important role in the investigation of properties of the conjugate functions [1], in the study of absolutely convergent Fourier series [18], and in related problems. In the case of Lebesgue spaces, the inverse inequalities for classical moduli of smoothness and the best approximation theorems were obtained in papers [20], [4]. In [10], this result was extended to reflexive Orlicz spaces. For the study of the approximation problems in weighted Lebesgue and Orlicz spaces we refer to [7], [13], [9], [12], [23].

The order of the modulus of smoothness, as it has been shown in [20] and [4], depends not only on the rate of the best approximation but also on the metric of the spaces. In the present paper we reveal that the similar influence in weighted Lorentz spaces L^{ps} is expressed not only in terms of the "leading" parameter p, but also in terms of the second parameter s. In the role of structural characteristic we consider the general modulus of continuity defined by the Steklov means. It is caused by the failure of the shift operator continuity in the weighted Lebesgue spaces. The generalized shift operator suits well for the spaces mentioned above.

Let $\mathbf{T} = [-\pi,\pi)$ and $w : \mathbf{T} \to \mathbb{R}^1$ be an almost everywhere positive, integrable function. Let $f_w^*(t)$ be a nondecreasing rearrangement of $f : \mathbf{T} \to \mathbb{R}^1$ with respect to the Borel measure $w(e) = \int_e w(x)dx$, i.e.,

$$f_w^*(t) = \inf\{\tau \geq 0 : w\,(x \in \mathbf{T} : |f(x)| > \tau) \leq t\}\,.$$

Let $1 < p, s < \infty$ and let $L_w^{ps}(\mathbf{T})$ be a weighted Lorentz space, i.e., the set of all measurable functions for which

$$\|f\|_{L_w^{ps}} = \left(\int_{\mathbf{T}} \left(f^{**}(t)\right)^s t^{\frac{s}{p}} \frac{dt}{t} \right)^{1/s} < \infty,$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f_w^*(u)du$.

By $E_n(f)_{L_w^{ps}}$ we denote the best approximation of $f \in L_w^{ps}(\mathbf{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L_w^{ps}} = \inf \|f - T_k\|_{L_w^{ps}} ,$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$.

The generalized modulus of smoothness of a function $f \in L_w^{ps}(\mathbf{T})$ is defined as

$$\Omega_l\left(f, \delta\right)_{L_w^{ps}} = \sup_{0<h_i<\delta} \left\| \prod_{i=1}^{l} \left(I - A_{h_i}\right) f \right\|_{L_w^{ps}}, \qquad \delta > 0,$$

where I is the identity operator and

$$\left(A_{h_i} f\right)(x) := \frac{1}{2h_i} \int_{x-h_i}^{x+h_i} f(u)du.$$

The weights w used in the paper are those which belong to the Muckenhoupt class $A_p(\mathbf{T})$, i.e., they satisfy the condition

$$\sup \frac{1}{|I|} \int_I w(x)dx \left(\frac{1}{|I|} \int_I w^{1-p'}(x)dx \right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1}$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

Whenever $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$, the Hardy-Littlewood maximal function of every $f \in L_w^{ps}(\mathbf{T})$, and therefore the average $A_{h_i} f$ belong to $L_w^{ps}(\mathbf{T})$ ([5, Theorem 3]). Thus $\Omega_l(f, \delta)_{L_w^{ps}}$ makes sense for every $w \in A_p(\mathbf{T})$.

We use the convention that c denotes a generic constant, i.e. a constant whose values can change even between different occurrences in a chain of inequalities.

2. Main results

In the present paper we prove the following results.

Theorem 1. *Let* $1 < p < \infty$ *and* $1 < s \leq 2$ *or* $p > 2$ *and* $s \geq 2$. *Let* $w \in A_p(\mathbf{T})$. *Then there exists a positive constant* c *such that*

$$\Omega_l \left(f, \frac{1}{n} \right)_{L_w^{ps}} \leq \frac{c}{n^{2l}} \left(\sum_{k=1}^{n} k^{2l\gamma-1} E_{k-1}^{\gamma}(f)_{L_w^{ps}} \right)^{1/\gamma} \tag{2.1}$$

for arbitrary $f \in L_w^{ps}(\mathbf{T})$ *and natural* n, *where* $\gamma = \min(s, 2)$.

Theorem 2. *Let* $1 < p < 2 < s < \infty$ *and let* $w \in A_p(\mathbf{T})$. *Then for arbitrary* p_0, $1 < p_0 < p$, *there exists a positive constant* c *such that*

$$\Omega_l \left(f, \frac{1}{n} \right)_{L_w^{ps}} \leq \frac{c}{n^{2l}} \left(\sum_{k=1}^{n} k^{2lp_0-1} E_{k-1}^{p_0}(f)_{L_w^{ps}} \right)^{1/p_0}$$

for arbitrary $f \in L_w^{ps}(\mathbf{T})$ *and natural* n.

Theorem 3. *Let* $1 < p < \infty$ *and* $1 < s \leq 2$ *or* $p > 2$ *and* $s \geq 2$. *Let* $w \in A_p(\mathbf{T})$ *and* $f \in L_w^{ps}(\mathbf{T})$. *Assume that*

$$\sum_{k=1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^{ps}} < \infty \tag{2.2}$$

for some natural number r *and* $\gamma = \min(s, 2)$. *Then there exists the absolutely continuous* $(r-1)$*th order derivative* $f^{(r-1)}(x)$ *such that* $f^{(r)} \in L_w^{ps}(\mathbf{T})$ *and*

$$E_n(f^{(r)})_{L_w^{ps}} \leq c \left\{ n^r E_n(f)_{L_w^{ps}} + \left(\sum_{k=n+1}^{\infty} k^{r\gamma-1} E_k^{\gamma}(f)_{L_w^{ps}} \right)^{1/\gamma} \right\} \tag{2.3}$$

for arbitrary natural n, *where* $\gamma = \min(s, 2)$ *and the constant* c *does not depend on* f *and* n.

Theorem 4. *Let* $1 < p < \infty$ *and* $1 < s \leq 2$ *or* $p > 2$ *and* $s \geq 2$. *Assume that* (2.2) *is fulfilled for some natural number* r *and* $\gamma = \min(s, 2)$. *Then* *there exists a positive constant* c *such that*

$$\Omega_l\left(f^{(r)},\frac{1}{n}\right)_{L_w^{ps}} \le \frac{c}{n^{2l}}\left(\sum_{k=1}^{n} k^{(r+2l)\gamma-1}E_{k-1}^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma} + c\left(\sum_{k=n+1}^{\infty} k^{r\gamma-1}E_k^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma} \tag{2.4}$$

for arbitrary $f\in L_w^{ps}(\mathbf{T})$ *and natural* n, *where* $\gamma=\min(s,2)$.

Corollary. *Let* $1<p<\infty$ *and* $1<s\le 2$ *or* $p>2$ *and* $s\ge 2$. *Assume that* $E_n(f)_{L_w^{ps}}=O\left(\frac{1}{n^{r+2l}}\right)$ *for some integer* $r\ge 1$ *and* $l\ge 1$. *Then*

$$\Omega_l\left(f^{(r)},\frac{1}{n}\right)_{L_w^{ps}} = O\left(\frac{(\ln n)^{1/\gamma}}{n^{2l}}\right) \tag{2.5}$$

where $\gamma=\min(s,2)$.

Let $\{\alpha_n\}$ be a monotonic sequence of positive numbers convergent to zero. Let $\Phi_w^{ps}(\alpha_n)$ be the set of functions $f\in L_w^{ps}$ for which $c_1\alpha_n\le E_n(f)_{L_w^{ps}}\le c_2\alpha_n$ for some constants c_1 and c_2 independent of f.

When $s,p>2$ the sharpness of (2.1) is shown by the following theorem.

Theorem 5. *For each* $\alpha_n\downarrow 0$ *there exists* $f_0\in\Phi_w^{ps}(\alpha_n)$ *satisfying the inequality*

$$\Omega_1\left(f_0,\frac{1}{n}\right)_{L_w^{ps}} \ge \frac{c}{n^2}\left(\sum_{k=1}^{n} k^3\alpha_{k-1}^2\right)^{1/2} \tag{2.6}$$

with a constant $c>0$ *independent of* n.

3. Auxiliary results

In this section we present some known results in weighted Lorentz spaces.

Proposition 3.1. *Let* $1<p,s<\infty$. *Then there exists a positive constant* c *such that for arbitrary* $f\in L_w^{ps}$

$$c^{-1}\|f\|_{L_w^{ps}} \le \sup\left|\int_{\mathbf{T}} f(x)g(x)w(x)dx\right| \le c\|f\|_{L_w^{ps}}, \tag{3.1}$$

where the supremum is taken with respect to all those functions g for which $\|g\|_{L_w^{p's'}} \le 1$ (see [5], also [11, Proposition 5.1.2]). Here $p' = p/(p-1)$.

Proposition 3.2. *Let $1 < p, s < \infty$ and let φ be a measurable function of two variables. Then*

$$\left\| \int_{\mathbf{T}} \varphi(x,\cdot)dx \right\|_{L_w^{ps}} \le c \int_{\mathbf{T}} \|\varphi(x,\cdot)\|_{L_w^{ps}}\, dx.$$

Proof. By proposition 3.1, Fubini's theorem and the Hölder's inequality we obtain

$$\begin{aligned}
\left\| \int_{\mathbf{T}} \varphi(x,\cdot)dx \right\|_{L_w^{ps}} &\le c \sup_{\|g\|_{L_w^{p's'}} \le 1} \int_{\mathbf{T}} \left(\int_{\mathbf{T}} |\varphi(x,y)|\, dx \right) |g(y)|\, w(y)dy \\
&= c \sup_{\|g\|_{L_w^{p's'}} \le 1} \int_{\mathbf{T}} \left(\int_{\mathbf{T}} |\varphi(x,y)|\, |g(y)|\, w(y)dy \right) dx \\
&\le c \sup_{\|g\|_{L_w^{p's'}} \le 1} \int_{\mathbf{T}} \|\varphi(x,\cdot)\|_{L_w^{ps}} \|g\|_{L_w^{p's'}}\, dx \\
&\le c \int_{\mathbf{T}} \|\varphi(x,\cdot)\|_{L_w^{ps}}\, dx.
\end{aligned}$$

□

Proposition 3.3. *Let $1 < p$, $s < \infty$ and let $w \in A_p(\mathbf{T})$. The trigonometric Fourier series of any $f \in L_w^{ps}(\mathbf{T})$ converges in the norm and almost everywhere to $f(x)$.*

Proof. The norm convergence follows in the standard way from the boundedness of conjugate functions in L_w^{ps} with $1 < p, s < \infty$ and $w \in A_p(\mathbf{T})$ (see [11, Theorem 6.6.2]).

When $f \in L_w^{ps}$ with $w \in A_p(\mathbf{T})$ $(1 < p, s < \infty)$, then $f \in L^{p_0}$ for some $p_0 > 1$. Indeed, from the inclusion $L_w^{p_1} \subset L_w^{ps}$, $1 < p_1 < p$ and the openness of A_p it follows that there exist p_0 and p_1, $1 < p_0 < p_1 < p$ such that $f \in L_w^{p_1}(\mathbf{T})$ and $w \in A_{p_1/p_0}$. Thus $w^{1-\left(\frac{p_1}{p_0}\right)'} \in A_{\left(\frac{p_1}{p_0}\right)'}$ and $w^{1-\left(\frac{p_1}{p_0}\right)'} \in L^1$. By the Hölder inequality we have

$$\int_{\mathbf{T}} |f(x)|^{p_0}\, dx \le \left(\int_{\mathbf{T}} |f(x)|^{p_1}\, w(x)dx \right)^{p_0/p_1} \left(\int_{\mathbf{T}} w^{-\frac{p_0}{p_1}\left(\frac{p_1}{p_0}\right)'}(x)dx \right)^{\frac{p_1-p_0}{p_1}}.$$

But $-\frac{p_0}{p_1}\left(\frac{p_1}{p_0}\right)' = -\frac{p_0}{p_1-p_0} = 1-\left(\frac{p_1}{p_0}\right)'$. Therefore the right-hand side of the last inequality is finite and $f \in L^{p_0}(\mathbf{T})$. Using the Hunt almost everywhere convergence theorem for the trigonometric Fourier series of $f \in L^{p_0},\ 1 < p_0 < \infty$, (see [8, Theorem 1]) we obtain the desired result. □

Proposition 3.4. *Let $1 < p, s < \infty$ and let $w \in A_p(\mathbf{T})$. Then there exists a positive constant c such that $\|f - S_n(f)\|_{L^{ps}_w} \le cE_n(f)_{L^{ps}_w}$ for each $f \in L^{ps}_w$ and $n \ge 1$, where $S_n(f)$ stands for the n-th partial sum of trigonometric Fourier series of f.*

Proof. The last inequality is obtained in the standard way as a consequence of the boundedness of the conjugate function in $L^{ps}_w(\mathbf{T})$ with $w \in A_p(\mathbf{T})$ which, (see [24, Chapter VI]) implies that $\|S_n(f)\|_{L^{ps}_w} \le c\|f\|_{L^{ps}_w}$. Indeed, let T_n be a trigonometric polynomial of the best approximation. Then we have

$$\begin{aligned}
\|f - S_n(f)\|_{L^{ps}_w} &\le \|f - T_n\|_{L^{ps}_w} + \|T_n - S_n(f)\|_{L^{ps}_w} \\
&= \|f - T_n\|_{L^{ps}_w} + \|S_n(T_n - f)\|_{L^{ps}_w} \\
&\le cE_n(f)_{L^{ps}_w}.
\end{aligned}$$

□

The following theorem is a weighted version of the Littlewood-Paley decomposition for trigonometric Fourier series (see [16], [24, Chapter XV, Theorem 4.24]).

Theorem A. *Let $1 < p, s < \infty$ and let $w \in A_p(\mathbf{T})$. Suppose that*

$$f(x) \sim \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Then there exist positive constants c_1 and c_2 independent of f such that

$$c_1 \|f\|_{L^{ps}_w} \le \left\| \left(\sum_{m=0}^{\infty} \Gamma_m^2(x) \right)^{1/2} \right\|_{L^{ps}_w} \le c_2 \|f\|_{L^{ps}_w},$$

where

$$\Gamma_m(x) = \sum_{n=2^{m-1}}^{2^m-1} (a_n \cos nx + b_n \sin nx), \qquad \Gamma_0 = a_0.$$

One can derive this result by means of interpolation arguments for Lorentz spaces from its L^p_w-version (see [14],[15]) and openness of A_p. Indeed, let

$w \in A_p(\mathbf{T})$. It is well known that there exists p_1, $1 < p_1 < p < \infty$, such that $w \in A_{p_1}(\mathbf{T})$ and $w \in A_{p_2}(\mathbf{T})$ for arbitrary $p_2 > p$. According to Theorem 4.1 in [14], we have

$$c_j \left\| f \right\|_{L_w^{p_j}} \leq \left\| \left(\sum_{m=0}^{\infty} \Gamma_m^2(x) \right)^{1/2} \right\|_{L_w^{p_j}} \leq c_j \left\| f \right\|_{L_w^{p_j}}, \quad j = 1, 2.$$

Applying the interpolation theorem for Lorentz spaces (see [2, Theorem 5.5]), we get the desired result.

4. Proofs of the main results

Theorem A is basic for our proofs. We need also some further auxiliary statements. In the sequel, we say that $f \in W_{ps,w}^{(k)}$ if the derivative $f^{(k-1)}(x)$ is absolutely continuous and $f^{(k)} \in L_w^{ps}(\mathbf{T})$.

Lemma 4.1. *Let* $f \in W_{ps,w}^{(2l)}$. *Then*

$$\Omega_l(f^{(2l)}, \delta)_{L_w^{ps}} \leq c\delta^{2l} \left\| f^{(l)} \right\|_{L_w^{ps}}, \tag{4.1}$$

where the positive constant c *is independent of* f *and* δ.

Proof. It is sufficient to prove that $\Omega_l(f, \delta) \leq c\delta^2 \Omega_{l-1}(f'', \delta)$. Let

$$\beta(x) = \prod_{i=2}^{l} (I - A_{h_i}) f(x).$$

Then

$$\begin{aligned} \prod_{i=1}^{l} (I - A_{h_i}) f(x) &= \beta(x) - \frac{1}{2h_1} \int_{-h_1}^{h_1} \beta(x+t) dt \\ &= \frac{1}{2h_1} \int_{-h_1}^{h_1} [\beta(x) - \beta(x+t)]\, dt \\ &= -\frac{1}{4h_1} \int_{-h_1}^{h_1} [\beta(x+t) - 2\beta(x) + \beta(x-t)]\, dt \\ &= -\frac{1}{8h_1} \int_0^{h_1} \int_0^{t} \int_{-y}^{y} \beta''(x+z) dz dy dt. \end{aligned}$$

Applying Proposition 3.2 we obtain

$$\left\| \prod_{i=1}^{l} \left(I - A_{h_i}\right) f \right\|_{L_w^{ps}} \leq \frac{1}{8h_1} \int_0^{h_1} \int_0^t \left\| \int_{-y}^{y} \beta''(\cdot + z) dz \right\|_{L_w^{ps}} dydt$$

$$= \frac{1}{8h_1} \int_0^{h_1} \int_0^t 2y \left\| \frac{1}{2y} \int_{-y}^{y} \beta''(\cdot + z) dz \right\|_{L_w^{ps}} dydt.$$

Using the uniform boundedness of $A_{h_i}(f)$ in L_w^{ps} with respect to h we get

$$\left\| \prod_{i=1}^{l} \left(I - A_{h_i}\right) f \right\|_{L_w^{ps}} \leq \frac{c}{8h_1} \int_0^{h_1} \int_0^t \|A_y \beta''\|_{L_w^{ps}} dydt \leq c h_1^2 \|\beta''\|_{L_w^{ps}}.$$

From the last inequality we conclude that

$$\Omega_l(f, \delta)_{L_w^{ps}} \leq c \sup_{\substack{0<h_i<\delta \\ 1\leq i\leq l}} h_1^2 \|\beta''\|_{L_w^{ps}}$$

$$= c\delta^2 \sup_{\substack{0<h_i<\delta \\ 2\leq i\leq l}} \left\| \prod_{i=2}^{l} \left(I - A_{h_i}\right) f'' \right\|_{L_w^{ps}} = c\delta^2 \Omega_{l-1}(f'', \delta).$$

□

Lemma 4.2. *Let $1 < p < \infty$ and $1 < s \leq 2$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_w^{ps}$ we have*

$$\left\| \left(\sum_{j=1}^{m} {\varphi_j}^2 \right)^{1/2} \right\|_{L_w^{ps}} \leq c \left(\sum_{j=1}^{m} \|\varphi_j\|_{L_w^{ps}}^s \right)^{1/s} \tag{4.2}$$

with a constant c independent of φ_j and m.

Proof. We shall use the following well-known relations $(f^*)^\alpha = (f^\alpha)^*$ and $(f+g)^{**} \leq (f^{**} + g^{**})$ [2, pp. 41 and 54]. By the Hardy inequality (see [2, pp. 129]), we get

$$I = \left\| \left(\sum_{j=1}^{m} {\varphi_j}^2 \right)^{1/2} \right\|_{L_w^{ps}}$$

$$\leq c\left(\int_{\mathbf{T}}\left(\left(\left(\sum_{j=1}^{m}{\varphi_j}^2\right)^{1/2}\right)^*\right)^s(t)t^{\frac{s}{p}-1}dt\right)^{1/s}$$

$$= c\left(\int_{\mathbf{T}}\left(\left(\sum_{j=1}^{m}{\varphi_j}^2\right)^{s/2}\right)^*(t)t^{\frac{s}{p}-1}dt\right)^{1/s}$$

$$\leq c\left(\int_{\mathbf{T}}\left(\sum_{j=1}^{m}{\varphi_j}^s\right)^*(t)t^{\frac{s}{p}-1}dt\right)^{1/s}.$$

Thus

$$\begin{aligned}
I &\leq c\left(\int_{\mathbf{T}}\left(\sum_{j=1}^{m}{\varphi_j}^s\right)^{**}(t)t^{\frac{s}{p}-1}dt\right)^{1/s} \\
&\leq c\left(\int_{\mathbf{T}}\left(\sum_{j=1}^{m}\left({\varphi_j}^s\right)^{**}(t)\right)t^{\frac{s}{p}-1}dt\right)^{1/s} \\
&= c\left(\sum_{j=1}^{m}\int_{\mathbf{T}}\left({\varphi_j}^s\right)^{**}(t)t^{\frac{s}{p}-1}dt\right)^{1/s}.
\end{aligned}$$

Applying again the Hardy inequality we have

$$\begin{aligned}
I &\leq c\left(\sum_{j=1}^{m}\int_{\mathbf{T}}\left({\varphi_j}^s\right)^*(t)t^{\frac{s}{p}-1}dt\right)^{1/s} \leq c\left(\sum_{j=1}^{m}\int_{\mathbf{T}}\left({\varphi_j}^*\right)^s(t)t^{\frac{s}{p}-1}dt\right)^{1/s} \\
&\leq c\left(\sum_{j=1}^{m}\int_{\mathbf{T}}\left({\varphi_j}^{**}\right)^s(t)t^{\frac{s}{p}-1}dt\right)^{1/s} = c\left(\sum_{j=1}^{m}\|\varphi_j\|^s_{L_w^{ps}}\right)^{1/s}.
\end{aligned}$$

□

Lemma 4.3. *Let* $2 < p < \infty$ *and* $s \geq 2$. *For an arbitrary system* $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_w^{ps}$, *we have*

$$\left\| \left(\sum_{j=1}^m {\varphi_j}^2 \right)^{1/2} \right\|_{L_w^{ps}} \leq c \left(\sum_{j=1}^m \|\varphi_j\|_{L_w^{ps}}^2 \right)^{1/2} \tag{4.3}$$

with a constant c *independent of* φ_j *and* m.

Proof. By the definition

$$I := \left\| \left(\sum_{j=1}^m {\varphi_j}^2 \right)^{1/2} \right\|_{L_w^{ps}} = \left(\int_{\mathbf{T}} \left(\left(\left(\sum_{j=1}^m {\varphi_j}^2 \right)^{1/2} \right)^{**} (t) \right)^s t^{\frac{s}{p}-1} dt \right)^{1/s}.$$

According to the Hardy inequality and taking into account that $L_w^{\frac{p}{2}\frac{s}{2}}$ is a normed space in the current situation, we have

$$\begin{aligned} I &\leq c \left(\int_{\mathbf{T}} \left(\left(\left(\sum_{j=1}^m {\varphi_j}^2 \right)^{1/2} \right)^{*} (t) \right)^s t^{\frac{s}{p}-1} dt \right)^{1/s} \\ &= c \left(\int_{\mathbf{T}} \left(\left(\sum_{j=1}^m {\varphi_j}^2 \right)^{*} (t) \right)^{s/2} t^{\frac{s}{p}-1} dt \right)^{1/s} \\ &\leq c \left(\sum_{j=1}^m \int_{\mathbf{T}} \left(\left(\varphi_j^2\right)^{**} (t) \right)^{s/2} t^{\frac{s}{p}-1} dt \right)^{\frac{2}{s}\frac{1}{2}}. \end{aligned}$$

If we use the Hardy inequality once more inside the sum, we get

$$\begin{aligned} I &\leq c \left(\sum_{j=1}^m \int_{\mathbf{T}} \left(\left(\varphi_j^2\right)^{*} (t) \right)^{s/2} t^{\frac{s}{p}-1} dt \right)^{\frac{2}{s}\frac{1}{2}} \\ &\leq c \left(\sum_{j=1}^m \int_{\mathbf{T}} \left(\varphi_j^*(t)\right)^s t^{\frac{s}{p}-1} dt \right)^{\frac{2}{s}\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le c\left(\sum_{j=1}^{m}\int_{\mathbf{T}}\left(\varphi_j^{**}(t)\right)^s t^{\frac{s}{p}-1}dt\right)^{\frac{2}{s}\frac{1}{2}} \\
&\le c\left(\sum_{j=1}^{m}\left(\int_{T}\left(\varphi_j^{**}(t)\right)^s t^{\frac{s}{p}-1}dt\right)^{\frac{2}{s}}\right)^{\frac{1}{2}} \\
&= c\left(\sum_{j=1}^{m}\|\varphi_j\|_{L_w^{ps}}^2\right)^{1/2}.
\end{aligned}
$$

□

Lemma 4.4. *Let $1 < p_0 < p < 2 < s < \infty$. Then for an arbitrary system of functions $\{\varphi_j(x)\}_{j=1}^m$, $\varphi_j \in L_w^{ps}$, we have*

$$
\left\|\left(\sum_{j=1}^{m}{\varphi_j}^2\right)^{1/2}\right\|_{L_w^{ps}} \le c\left(\sum_{j=1}^{m}\|\varphi_j\|_{L_w^{ps}}^{p_0}\right)^{\frac{1}{p_0}}
$$

with a constant c independent of φ_j and m.

Proof. Using the arguments of the proof of Lemmas 4.2 and 4.3, we claim that

$$
\begin{aligned}
\left\|\left(\sum_{j=1}^{m}{\varphi_j}^2\right)^{\frac{1}{2}}\right\|_{L_w^{ps}} &= \left\|\left(\sum_{j=1}^{m}{\varphi_j}^2\right)^{\frac{p_0}{2}\frac{1}{p_0}}\right\|_{L_w^{ps}} \\
&\le \left\|\left(\sum_{j=1}^{m}|\varphi_j|^{p_0}\right)^{\frac{1}{p_0}}\right\|_{L_w^{ps}} \\
&\le c\left\|\sum_{j=1}^{m}|\varphi_j|^{p_0}\right\|_{L_w^{\frac{p}{p_0}\frac{s}{p_0}}}^{\frac{1}{p_0}} \\
&\le c\left(\sum_{j=1}^{m}\left\||\varphi_j|^{p_0}\right\|_{L_w^{\frac{p}{p_0}\frac{s}{p_0}}}\right)^{\frac{1}{p_0}}
\end{aligned}
$$

$$\leq c\left(\sum_{j=1}^{m}\|\varphi_j\|_{L_w^{ps}}^{p_0}\right)^{\frac{1}{p_0}}.$$

□

Lemma 4.5. *Let $1 < p, s < \infty$, $f \in L_w^{ps}(T)$ and $w \in A_p(T)$. If $B_{k,\mu}(x) = a_k \cos\left(k + \mu\frac{\pi}{2}\right)x + b_k \sin\left(k + \mu\frac{\pi}{2}\right)x$, where a_k, b_k are Fourier coefficients of f, then*

$$\left\|\sum_{k=2^i+1}^{2^{i+1}} k^{\mu} B_{k,\mu}\right\|_{L_w^{ps}} \leq c2^{i\mu} E_{2^i}(f)_{L_w^{ps}}, \tag{4.4}$$

where the constant c is independent of f and i.

Proof. Let us introduce the notation $\tau_{j,\mu}(x) := \sum\limits_{k=1}^{j} B_{k,\mu}(x)$. By means of the Abel transformation we obtain

$$\begin{aligned}\sum_{n=2^i+1}^{2^{i+1}} k^{\mu} B_{k,\mu}(x) &= \sum_{n=2^i+1}^{2^{i+1}} \left[k^{\mu} - (k+1)^{\mu}\right]\left(\tau_{k,\mu}(x) - \tau_{2^i,\mu}(x)\right) \\ &\quad + 2^{i+1}\left(\tau_{2^{i+1},\mu}(x) - \tau_{2^i,\mu}(x)\right).\end{aligned}$$

But by Proposition 3.4 we deduce that

$$\begin{aligned}\left\|\sum_{k=2^i+1}^{2^{i+1}} k^{\mu} B_{k,\mu}\right\|_{L_w^{ps}} &\leq c\sum_{k=2^i+1}^{2^{i+1}}\left[(k+1)^{\mu} - k^{\mu}\right] E_{2^i}(f)_{L_w^{ps}} + \\ &\quad + c2^{i\mu} E_{2^i}(f)_{L_w^{ps}} \leq c2^{i\mu} E_{2^i}(f)_{L_w^{ps}}.\end{aligned}$$

□

Proof of Theorem 1. Let $2^m < n \leq 2^{m+1}$ and $\delta = \frac{1}{n}$. Let $S_n(f)$ be a partial sum of Fourier series of f. Then we have

$$\Omega_l(f,\delta)_{L_w^{ps}} \leq \Omega_l\left(\left(f - S_{2^{m+1}}(f)\right),\delta\right)_{L_w^{ps}} + \Omega_l(S_{2^{m+1}},\delta)_{L_w^{ps}}. \tag{4.5}$$

By the uniform boundedness of the averaging operator A_h in L_w^{ps} we obtain

$$\Omega_l\left(\left(f - S_{2^{m+1}}(f)\right),\delta\right)_{L_w^{ps}} \leq c\left\|f - S_{2^{m+1}}\right\|_{L_w^{ps}} \leq cE_n(f)_{L_w^{ps}}. \tag{4.6}$$

Then according to Lemma 4.1 we have

$$\Omega_l(S_{2^{m+1}},\delta)_{L_w^{ps}} \leq c\delta^{2l}\left\|S_{2^{m+1}}^{(2l)}\right\|_{L_w^{ps}}.$$

Then
(4.7)

$$\Omega_l(S_{2^{m+1}},\delta)_{L_w^{ps}} \le c\delta^{2l}\left\{\left\|S_1^{(2l)}-S_0^{(2l)}\right\|_{L_w^{ps}}+\left\|\sum_{i=0}^{m}\left[S_{2^{i+1}}^{(2l)}-S_{2^i}^{(2l)}\right]\right\|_{L_w^{ps}}\right\}.$$

For the first term on the right side of (4.7) we have

$$\left\|S_1^{(2l)}-S_0^{(2l)}\right\|_{L_w^{ps}} \le c\left(|a_1|+|b_1|\right) \le cE_0(f)_{L_w^{ps}}. \tag{4.8}$$

Applying Theorem A to the second term, we get

$$\begin{aligned}\left\|\sum_{i=0}^{m}\left[S_{2^{i+1}}^{(2l)}-S_{2^i}^{(2l)}\right]\right\|_{L_w^{ps}} &= \left\|\sum_{i=0}^{m}\sum_{k=2^i+1}^{2^{i+1}}k^{2l}B_{k,2l}(x)\right\|_{L_w^{ps}}\\ &\le c\left\|\left(\sum_{i=0}^{m}\left|\sum_{k=2^i+1}^{2^{i+1}}k^{2l}B_{k,2l}(x)\right|^2\right)^{1/2}\right\|_{L_w^{ps}}.\end{aligned}$$

Now with the aid of Lemmas 4.2 and 4.3 we conclude that

$$\left\|\sum_{i=0}^{m}\left[S_{2^{i+1}}^{(2l)}-S_{2^i}^{(2l)}\right]\right\|_{L_w^{ps}} \le c\left(\sum_{i=1}^{m}\left\|\sum_{k=2^i+1}^{2^{i+1}}k^{2l}B_{k,2l}(x)\right\|^{\gamma}\right)^{1/\gamma},$$

where $\gamma=\min(s,2)$. Then by Lemma 4.5 we have

$$\left\|\sum_{i=0}^{m}\left[S_{2^{i+1}}^{(2l)}-S_{2^i}^{(2l)}\right]\right\|_{L_w^{ps}} \le c\left(\sum_{i=1}^{m}2^{2\gamma li}E_{2^i}^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma}. \tag{4.9}$$

Thus from (4.5), (4.6), (4.8) and (4.9) we derive the estimate

$$\Omega_l(f,\delta)_{L_w^{ps}} \le c\delta^{2l}\left[E_0(f)_{L_w^{ps}}+E_n(f)_{L_w^{ps}}+\left(\sum_{i=1}^{m}2^{2\gamma li}E_{2^i}^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma}\right].$$

Since $E_k(f)_{L_w^{ps}}$ is monotonically decreasing, we conclude that

$$\Omega_l(f,\delta)_{L_w^{ps}} \le \frac{c}{n^{2l}}\left(\sum_{k=1}^{n}k^{2l\gamma-1}E_{k-1}^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma}.$$

□

Proof of Theorem 2. We can repeat the proof of Theorem 1 just using Lemma 4.4 instead of Lemma 4.3 for a system of functions

$$\varphi_j(x) = \sum_{k=2^i+1}^{2^{i+1}} k^{2l} B_{k,2l}(x).$$

□

Proof of Theorem 5. Let $\{\alpha_n\}$ be a decreasing sequence convergent to 0. Define the function f with lacunary Fourier expansion

$$f(x) = \sum_{n=0}^{\infty} \sqrt{\alpha_{2^n}^2 - \alpha_{2^{n+1}}^2} \sin 2^n x. \tag{4.10}$$

Then

$$(I - A_h) f(x) = \sum_{n=0}^{\infty} \sqrt{\alpha_{2^n}^2 - \alpha_{2^{n+1}}^2} \left(1 - \frac{\sin 2^n h}{2^n h}\right) \sin 2^n x.$$

As it was shown in the proof of Proposition 3.3, there exists $p_0 > 1$ such that

$$\|(I - A_h) f\|_{L^{p_s}} \geq c \|(I - A_h) f\|_{L^{p_0}}. \tag{4.11}$$

Since the series is lacunary (see [24, Vol. 2, pp. 132]), we get

$$\|(I - A_h) f\|_{L^{p_0}} \geq c \|(I - A_h) f\|_{L^2} \tag{4.12}$$

and then

$$\|(I - A_h) f\|_{L^2}^2 \geq c \sum_{k=0}^{\infty} \left(\alpha_{2^k}^2 - \alpha_{2^{k+1}}^2\right) \left(2^k h\right)^4.$$

Take $h = \frac{1}{n}$ and m such that $2^m \leq n < 2^{m+1}$. Then the right-hand side of (4.12) is not less than

$$\frac{c}{n^4} \sum_{k=0}^{m+1} \left(\alpha_{2^k}^2 - \alpha_{2^{k+1}}^2\right) 2^{4k} = \frac{c}{n^4} \left(\alpha_1^2 + \sum_{k=1}^{m+1} \frac{15}{16} 2^{4k} \alpha_{2^k}^2 - 2^{4(m+1)} \alpha_{2^{m+2}}\right). \tag{4.13}$$

Since $\frac{2^{4(m+1)}}{n^4} \alpha_{2^{m+2}} = \left(\frac{2^{m+1}}{2}\right)^4 \alpha_{4n} \leq 2^4 \alpha_{4n} \longrightarrow 0$ we can write that the last expression in (4.13) is more than

$$\frac{c}{n^4} \left(\alpha_1^2 + \sum_{k=1}^{m+1} \left(2^k\right)^3 2^k \alpha_{2^k}^2\right) \geq \frac{c}{n^4} \sum_{k=1}^{2^{m+1}} k^3 \alpha_k^2 \geq \frac{c}{n^4} \sum_{k=1}^{n} \left(k^3 \alpha_k^2\right). \tag{4.14}$$

Then inequality (2.6) follows from (4.11)-(4.14). □

Theorem 5 shows that estimation 2.1 cannot be improved when $2 < p$, $s < \infty$.

In order to prove Theorems 3 and 4 we need several Lemmas.

Lemma 4.6 *Let $\{f_n\}$ be a sequence of absolutely continuous functions and let $w \in A_p(\mathbf{T})$. If $\{f_n\}$ converges to a function f in $L_w^{ps}(\mathbf{T})$, $1 < p, s < \infty$, and the sequence of first derivatives $\{f_n'\}$ converges to a function g in $L_w^{ps}(\mathbf{T})$, then f is absolutely continuous and $f'(x) = g(x)$ almost everywhere.*

Proof. Since $\|f_n - f\|_{L_w^{ps}} \to 0$, there exists p_0, $1 < p_0 < p$, such that $\|f_n - f\|_{L^{p_0}} \to 0$. Thus there exists a subsequence $\{f_{n_k}\}$ of the sequence $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ almost everywhere. Let x_0 be a point of convergence. By Hölder's inequality for Lorentz spaces we get

$$\left| \int_{x_0}^{x} f_{n_k}'(t)dt - \int_{x_0}^{x} g(t)dt \right| \leq \left\| f_{n_k}' - g \right\|_{L_w^{ps}} \left\| w^{-1} \right\|_{L_w^{p's'}} .$$

Since $w \in A_p(\mathbf{T})$, we have $\left\| w^{-1} \right\|_{L_w^{p's'}} < \infty$ (see [5]). Thus we get

$$\lim_{k \longrightarrow \infty} \left| \int_{x_0}^{x} f_{n_k}'(t)dt - \int_{x_0}^{x} g(t)dt \right| = 0.$$

Therefore, we obtain

$$\int_{x_0}^{x} g(t)dt = \lim_{n \to \infty} \int_{x_0}^{x} f_{n_k}'(t)dt = \lim_{k \to \infty} \left(f_{n_k}(x) - f_{n_k}(x_0) \right) = f(x) - f(x_0).$$

This completes the proof. □

Proof of Theorem 3. Let $2^m < n < 2^{m+1}$. We have

$$\begin{aligned} \left\| f^{(r)} - S_n(f^{(r)}) \right\|_{L_w^{ps}} &\leq \left\| S_{2^{m+2}}(f^{(r)}) - S_n(f^{(r)}) \right\|_{L_w^{ps}} \\ &+ \sum_{k=m+2}^{\infty} \left\| S_{2^{k+1}}(f^{(r)}) - S_{2^k}(f^{(r)}) \right\|_{L_w^{ps}} . \end{aligned} \tag{4.15}$$

By the weighted version of Bernstein's inequality (see [6, Theorem 4.1]), we have

$$\begin{aligned}\left\|S_{2^{m+2}}(f^{(r)})-S_n(f^{(r)})\right\|_{L^{ps}_w} &= \|S^r_{2^{m+2}}(f)-S^r_n(f)\|_{L^{ps}_w}\\ &\le c2^{(m+2)r}\left\|S_{2^{m+2}}(f)-S_n(f)\right\|\\ &\le c2^{(m+2)r}E_{2^{m+2}}(f)_{L^{ps}_w}\\ &\le cn^rE_n(f)_{L^{ps}_w}. \qquad (4.16)\end{aligned}$$

Applying Theorem A and Lemma 4.5, we obtain

$$\begin{aligned}\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}(f^{(r)})-S_{2^k}(f^{(r)})\right]\right\|_{L^{ps}_w} &\le c\left(\sum_{k=m+2}^{\infty}\left\|\sum_{\mu=2^k+1}^{2^{k+1}}\mu^rB_{\mu,r}(x)\right\|^{\gamma}_{L^{ps}_w}\right)^{\frac{1}{\gamma}}\\ &\le c\left(\sum_{k=m+2}^{\infty}2^{kr\gamma}E^{\gamma}_{2^k-1}(f)_{L^{ps}_w}\right)^{\frac{1}{\gamma}}\\ &\le c\left(\sum_{k=n+1}^{\infty}k^{r\gamma-1}E^{\gamma}_k(f)_{L^{ps}_w}\right)^{\frac{1}{\gamma}}. \qquad (4.17)\end{aligned}$$

Gathering formulas (4.15), (4.16) and (4.17), we deduce the desired inequality. □

Proof of Theorem 4. Let $2^m < n \le 2^{m+1}$ and $\delta = \frac{1}{n}$. Using Lemma 4.6, we conclude that under condition (2.3) there exists the absolutely continuous $(r-1)$th order derivative $f^{(r-1)}(x)$ and $f^{(r)} \in L^{ps}_w$. Then

$$\Omega_l(f^{(r)},\delta) \le \Omega_l\left(\left(f^{(r)}-S^{(r)}_{2^{m+1}}\right),\delta\right)+\Omega_l(S^{(r)}_{2^{m+1}},\delta). \qquad (4.18)$$

But

$$\left\|\Omega_l\left(\left(f^{(r)}-S^{(r)}_{2^{m+1}}\right),\delta\right)\right\| \le c\left\|f^{(r)}-S^{(r)}_{2^{m+1}}\right\|_{L^{ps}_w} \le cE_n(f^{(r)})_{L^{ps}_w}. \qquad (4.19)$$

On the other hand

$$\begin{aligned}\Omega_l(S^{(r)}_{2^{m+1}},\delta) &\le c\delta^{2l}\left\|S^{(r+2l)}_{2^{m+1}}\right\|_{L^{ps}_w}\\ &\le c\delta^{2l}\left\{\left\|S^{(r+2l)}_1-S^{(r+2l)}_0\right\|_{L^{ps}_w}+\left\|\sum_{i=0}^{m}\left[S^{(r+2l)}_{2^{i+1}}-S^{(r+2l)}_{2^i}\right]\right\|_{L^{ps}_w}\right\}.\end{aligned}$$

Consequently, applying Theorem A, we have

$$\Omega_l(S^{(r)}_{2^{m+1}},\delta) \le c\delta^{2l}\left\{E_0(f)_{L^{ps}_w} + \left\|\left(\sum_{i=0}^{m}\left|\sum_{k=2^i+1}^{2^{i+1}} k^{r+2l}B_{k,r+2l}(x)\right|^2\right)^{1/2}\right\|_{L^{ps}_w}\right\}.$$

Then by Lemmas 4.2 and 4.3
(4.20)

$$\Omega_l(S^{(r)}_{2^{m+1}},\delta) \le c\delta^{2l}\left\{E_0(f)_{L^{ps}_w} + \left(\sum_{i=0}^{m}\left\|\sum_{k=2^i+1}^{2^{i+1}} k^{r+2l}B_{k,r+2l}\right\|^{\gamma}_{L^{ps}_w}\right)^{1/\gamma}\right\}.$$

Using the arguments of the proof of Theorem 1, according to Lemma 4.5 and Theorem 3, we obtain the assertion from (4.18), (4.19) and (4.20). □

Note that using the standard method of proving inverse inequalities (see [21, Section 6.1], and [9, Theorem 4]), on the base of Proposition 3.2 and 3.4 and Lemma 4.1, we can establish the following statement.

Proposition 4.1 *Let* $1 < p,\ s < \infty$ *and let* $w \in A_p(\mathbf{T})$. *Then there exists a positive constant* c *such that*

$$\Omega_l\left(f, \frac{1}{n}\right) \le \frac{c}{n^{2l}}\sum_{k=1}^{n} k^{2l-1}E_{k-1}(f)_{L^{ps}_w} \tag{4.21}$$

for an arbitrary $f \in L^{ps}_w(\mathbf{T})$ *and every natural* n.

On the other hand, for arbitrary $\beta,\ 1 < \beta < \infty$, natural number μ and sequence $\alpha_n \downarrow 0$ the inequality

$$\left(\sum_{k=1}^{n} k^{\beta\mu-1}\alpha^{\beta}_{k-1}\right)^{1/\beta} \le c\left(\sum_{k=1}^{n} k^{\mu-1}\alpha_{k-1}\right)$$

holds, where the constant c does not depend on α_k and n (see, for example, [20]). Thus Theorems 1 and 2 improve the estimation (4.21).

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