Approximation theorems in weighted Lorentz spaces
Author(s): Yunus E. Yildirir and Daniyal M. Israfilov
Source: *Carpathian Journal of Mathematics*, Vol. 26, No. 1 (2010), pp. 108-119
Published by: Sinus Association
Stable URL: https://www.jstor.org/stable/43999439

CARPATHIAN J. MATH. 26 (2010), No. 1, 108 - 119
Online version available at http://carpathian.ubm.ro
Print Edition: ISSN 1584 - 2851 Online Edition: ISSN 1843 - 4401

Approximation theorems in weighted Lorentz spaces

YUNUS E. YILDIRIR and DANIYAL M. ISRAFILOV

ABSTRACT. In this paper we deal with the converse and simultaneous approximation problems of functions possessing derivatives of positive orders by trigonometric polynomials in the weighted Lorentz spaces with weights satisfying the so called Muckenhoupt's A_p condition.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbf{T} = [-\pi, \pi]$ and $w : \mathbf{T} \to [0, \infty]$ be an almost everywhere positive, integrable function. Let $f_w^*(t)$ be a decreasing rearrangement of $f : \mathbf{T} \to [0, \infty]$ with respect to the Borel measure

$$w(e) = \int_e w(x)dx,$$

i.e.,

$$f_w^*(t) = \inf \{\tau \geq 0 : w(x \in \mathbf{T} : |f(x)| > \tau) \leq t\}.$$

Let $1 < p, s < \infty$ and let $L_w^{ps}(\mathbf{T})$ be a weighted Lorentz space, i.e., the set of all measurable functions for which

$$\|f\|_{L_w^{ps}} = \left(\int_{\mathbf{T}} (f^{**}(t))^s \, t^{\frac{s}{p}} \frac{dt}{t} \right)^{1/s} < \infty,$$

where

$$f^{**}(t) = \frac{1}{t} \int_0^t f_w^*(u)du.$$

If $p = s$, $L_w^{ps}(\mathbf{T})$ is the weighted Lebesgue space $L_w^p(\mathbf{T})$ [5, p.20].

The weights w used in the paper are those which belong to the Muckenhoupt class $A_p(\mathbf{T})$, i.e., they satisfy the condition

$$\sup \frac{1}{|I|} \int_I w(x)dx \left(\frac{1}{|I|} \int_I w^{1-p'}(x)dx \right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1}$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

Received: 16.01.2009; In revised form: 25.07.2009; Accepted: 08.02.2010
2000 *Mathematics Subject Classification.* 41A10, 42A10.
Key words and phrases. *Best approximation, weighted Lorentz space, mean modulus of smoothness, fractional derivative and phrases.*

Let

$$(1.1) \qquad f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

be the Fourier and the conjugate Fourier series of $f \in L^1(\mathbf{T})$, respectively. In addition, we put

$$S_n(x,f) := \sum_{k=-n}^{n} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 1, 2, \ldots.$$

By $L_0^1(\mathbf{T})$ we denote the class of $L^1(\mathbf{T})$ functions f for which the constant term c_0 in (1.1) equals zero. If $\alpha > 0$, then $\alpha - th$ *integral* of $f \in L_0^1(\mathbf{T})$ is defined as

$$I_\alpha(x,f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where $(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k}$ and $\mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$.

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f),$$

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f)$$

if the right hand sides exist, where $r \in \mathbb{Z}^+ := \{1, 2, 3, \ldots\}$ [15, p. 347].

By c, $c(\alpha, \ldots)$ we denote the constants, which can be different in different place, such that they are absolute or depend only on the parameters given in their brackets.

Let $x, t \in \mathbb{R} := (-\infty, \infty)$, $r \in \mathbb{R}^+ := (0, \infty)$ and let

$$(1.2) \qquad \triangle_t^r f(x) := \sum_{k=0}^{\infty} (-1)^k [C_k^r] f(x + (r-k)t), \quad f \in L^1(\mathbf{T}),$$

where $[C_k^r] := \dfrac{r(r-1)\ldots(r-k+1)}{k!}$ for $k > 1$, $[C_k^r] := r$ for $k = 1$ and $[C_k^r] := 1$ for $k = 0$.

Since [15, p. 14]

$$|[C_k^r]| = \left| \frac{r(r-1)\ldots(r-k+1)}{k!} \right| \leq \frac{c(r)}{k^{r+1}}, \quad k \in \mathbb{Z}^+$$

we have

$$\sum_{k=0}^{\infty} |[C_k^r]| < \infty,$$

and therefore $\triangle_t^r f(x)$ is defined a.e. on $\mathbb{R}$. Furthermore, the series in (1.2) converges absolutely a.e. and $\triangle_t^r f(x)$ is measurable [17].

If $r \in \mathbb{Z}^+$, then the fractional difference $\triangle_t^r f(x)$ coincides with usual forward difference.

Now let

$$\sigma_\delta^r f(x) := \frac{1}{\delta}\int_0^\delta |\Delta_t^r f(x)|\, dt,$$

for $f \in L_w^{ps}(\mathbf{T})$, $1 < p, s < \infty$, $w \in A_p(\mathbf{T})$. Since the series in (1.2) converges absolutely a.e., we have $\sigma_\delta^r f(x) < \infty$ a.e. and using boundedness of the Hardy-Littlewood Maximal function [3, Th. 3] in $L_w^{ps}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, we get

$$\|\sigma_\delta^r f(x)\|_{L_w^{ps}} \leq c\, \|f\|_{L_w^{ps}} < \infty. \tag{1.3}$$

Hence, if $r \in \mathbb{R}^+$ and $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$, we can define the *r-th mean modulus of smoothness* of a function $f \in L_w^{ps}(\mathbf{T})$ as

$$\Omega_r(f,h)_{L_w^{ps}} := \sup_{|\delta| \leq h} \|\sigma_\delta^r f(x)\|_{L_w^{ps}}. \tag{1.4}$$

Remark 1.1. Let f, f_1, $f_2 \in L_w^{ps}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$. The r-th mean modulus of smoothness $\Omega_r(f,h)_{L_w^{ps}}$, $r \in \mathbb{R}^+$, has the following properties:

(*i*) $\Omega_r(f,h)_{L_w^{ps}}$ is non-negative and non-decreasing function of $h \geq 0$.

(*ii*) $\Omega_r(f_1+f_2,\cdot)_{L_w^{ps}} \leq \Omega_r(f_1,\cdot)_{L_w^{ps}} + \Omega_r(f_2,\cdot)_{L_w^{ps}}$.

(*iii*) $\lim_{h\to 0} \Omega_r(f,h)_{L_w^{ps}} = 0$.

The best approximation of $f \in L_w^{ps}(\mathbf{T})$ in the class Π_n of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{L_w^{ps}} = \inf\left\{\|f - T_n\|_{L_w^{ps}} : T_n \in \Pi_n\right\}.$$

A polynomial $T_n(x,f) := T_n(x)$ of degree n is said to be a *near best approximant* of f if

$$\|f - T_n\|_{L_w^{ps}} \leq cE_n(f)_{L_w^{ps}},\ n = 0,1,2,\ldots.$$

Let $W_{ps,w}^{\alpha}(\mathbf{T})$, $\alpha > 0$, be the class of functions $f \in L_w^{ps}(\mathbf{T})$ such that $f^{(\alpha)} \in L_w^{ps}(\mathbf{T})$. $W_{ps,w}^{\alpha}(\mathbf{T})$, $1 < p, s < \infty$, $\alpha > 0$, becomes a Banach space with the norm

$$\|f\|_{W_{ps,w}^{\alpha}(\mathbf{T})} := \|f\|_{L_w^{ps}} + \left\|f^{(\alpha)}\right\|_{L_w^{ps}}.$$

In this paper we deal with the converse and simultaneous approximation problems of functions possessing derivatives of positive orders by trigonometric polynomials in the weighted Lorentz spaces $L_w^{ps}(\mathbf{T})$ with weights satisfying so called Muckenhoupt's A_p condition.

Our new results are the following

Theorem 1.1. *Let $1 < p < \infty$ and $1 < s \leq 2$ or $p > 2$ and $s \geq 2$. Then for a given $f \in L_w^{ps}(\mathbf{T})$, $w \in A_p(\mathbf{T})$, and $r \in \mathbb{R}^+$ we have*

$$\Omega_r(f,\pi/(n+1))_{L_w^{ps}} \leq \frac{c}{(n+1)^r}\left(\sum_{k=0}^{n}(k+1)^{r\gamma-1}E_k^{\gamma}(f)_{L_w^{ps}}\right)^{1/\gamma},\quad n = 0,1,2,\ldots$$

with a positive constant c independent of n, where $\gamma = \min(s,2)$.

In case of $r \in \mathbb{Z}^+$ this result was proved in [10]. In the space $L^p(\mathbf{T})$, $1 \leq p \leq \infty$, using the usual modulus of smoothness, it was obtained in [17] without γ. In case of $r \in \mathbb{Z}^+$ in the spaces $L_w^p(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, this theorem was proved in [12] without γ. In case of $r \in \mathbb{Z}^+$,Theorem 1 without γ in term of Butzer-Wehrens's type modulus of smoothness in the spaces $L^p_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, and in the weighted Orlicz spaces was obtained in [6] and [8], respectively. Note that the above defined modulus of smoothness is more general than Butzer-Wehrens's type modulus of smoothness and in special case, when $r \in \mathbb{Z}^+$ is even, it coincides with Butzer-Wehrens's type modulus of smoothness.

Theorem 1.2. *Let $1 < p < \infty$ and $1 < s \le 2$ or $p > 2$ and $s \ge 2$. Let $w \in A_p(\mathbf{T})$ and $f \in L^{ps}_w(\mathbf{T})$. Assume that*

$$\sum_{k=1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L^{ps}_w} < \infty$$

for some $\alpha \in (0,\infty)$ and $\gamma = \min(s,2)$. Then $f \in W^{\alpha}_{ps,w}(\mathbf{T})$ and for $n = 0,1,2,...$ the estimate

$$E_n(f^{(\alpha)})_{L^{ps}_w} \le c\left\{ n^{\alpha} E_n(f)_{L^{ps}_w} + \left(\sum_{k=n+1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L^{ps}_w} \right)^{1/\gamma} \right\} \tag{1.5}$$

holds with a constant c independent of n and f.

In case of $\alpha \in \mathbb{Z}^+$ this result was obtained in [10]. In the space $L^p(\mathbf{T})$, $1 \le p \le \infty$, this inequality for $\alpha \in \mathbb{Z}^+$ was proved without γ in [16]. In case of $\alpha \in \mathbb{Z}^+$, in $L^p_w(\mathbf{T})$, $w \in A_p(\mathbf{T})$, $1 < p < \infty$, an inequality of type (1.5) was proved in [9].

Corollary 1.1. *Let $1 < p < \infty$ and $1 < s \le 2$ or $p > 2$ and $s \ge 2$. Let $w \in A_p(\mathbf{T})$ and $f \in L^{ps}_w(\mathbf{T})$. If*

$$\sum_{k=1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L^{ps}_w} < \infty$$

for $\alpha \in (0,\infty)$ and $\gamma = \min(s,2)$, then $f \in W^{\alpha}_{ps,w}$ and

$$\Omega_r(f^{(\alpha)}, \pi/(n+1))_{L^{ps}_w}$$

$$\le \frac{c}{(n+1)^r}\left\{ \left(\sum_{k=1}^{n} k^{(\alpha+r)\gamma-1} E_{k-1}^{\gamma}(f)_{L^{ps}_w} \right)^{1/\gamma} + \left(\sum_{k=n+1}^{\infty} k^{\alpha\gamma-1} E_k^{\gamma}(f)_{L^{ps}_w} \right)^{1/\gamma} \right\}$$

with a constant c independent of $n = 1,2,...$ and f.

In cases of $\alpha, r \in \mathbb{Z}^+$ and $\alpha, r \in \mathbb{R}^+$, this corollary in the spaces $L^p(\mathbf{T})$, $1 \le p \le \infty$, was proved without γ in [19] (See also [16]) and in [18], respectively. In the weighted Lebesgue spaces $L^p_w(\mathbf{T})$, $1 < p < \infty$, when $w \in A_p(\mathbf{T})$, and $\alpha, r \in \mathbb{Z}^+$, similar type estimation was obtained for the Butzer-Wehrens's type modulus of smoothness of $f^{(\alpha)}$ in [9].

The simultaneous approximation theorem in the weighted Lorentz space $L^{ps}_w(\mathbf{T})$ can be formulated as following.

Theorem 1.3. *Let $f \in W^{\alpha}_{ps,w}(\mathbf{T})$, $\alpha \in \mathbb{R}^+_0 := [0,\infty]$, $1 < p, s < \infty$, and $w \in A_p(\mathbf{T})$. If $T_n \in \Pi_n$ is a near best approximant of f, then*

$$\left\| f^{(\alpha)} - T_n^{(\alpha)} \right\|_{L^{ps}_w} \le cE_n(f^{(\alpha)})_{L^{ps}_w}, \quad n = 0,1,2,....$$

with a constant $c > 0$ independent of n and f.

In case of $\alpha \in \mathbb{Z}^+$, this theorem in the Lebesgue spaces $L^p(\mathbf{T})$, $1 \le p \le \infty$, was proved in [4].

We prove also the following inequality of Jackson type in the weighted Lorentz space $L_w^{ps}(\mathbf{T})$.

Theorem 1.4. *If* $f \in W_{ps,w}^r(\mathbf{T})$, $r \in \mathbb{R}^+$, $1 < p, s < \infty$, *and* $w \in A_p(\mathbf{T})$, *then*

$$\Omega_r(f,h)_{L_w^{ps}} \le ch^r \left\| f^{(r)} \right\|_{L_w^{ps}}, \quad 0 < h \le \pi$$

with a constant c independent of h and f.

This Theorem in case of $r \in \mathbb{R}^+$ in the Lebesgue spaces $L^p(\mathbf{T})$, $1 \le p \le \infty$, was obtained in [2] (See also [17]), and in case of $r \in \mathbb{Z}^+$, in the weighted Lebesgue spaces $L_w^p(\mathbf{T})$ with $w \in A_p(\mathbf{T})$ and $1 < p < \infty$, was proved in [12].

2. Auxiliary Results

Lemma 2.1. *Let* $w \in A_p(\mathbf{T})$ *and* $r \in \mathbb{R}^+$, $1 < p, s < \infty$. *If* $T_n \in \Pi_n, n \ge 1$, *then there exists a constant* $c > 0$ *depending only on r, p and s such that*

$$\Omega_r(T_n,h)_{L_w^{ps}} \le ch^r \left\| T_n^{(r)} \right\|_{L_w^{ps}}, \quad 0 < h \le \pi/n.$$

Proof. Since

$$\triangle_t^r T_n \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i \sin \frac{t}{2} \nu \right)^r c_\nu e^{i\nu x},$$

$$\triangle_t^{[r]} T_n^{(r-[r])} \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \left(2i \sin \frac{t}{2} \nu \right)^{[r]} (i\nu)^{r-[r]} c_\nu e^{i\nu x}$$

with $\mathbb{Z}_n^* := \{\pm 1, \pm 2, \pm 3, ...\}$, $[r] \equiv$ integer part of r, putting

$$\varphi(z) := \left(2i \sin \frac{t}{2} z \right)^{[r]} (iz)^{r-[r]}, \; g(z) := \left(\frac{2}{z} \sin \frac{t}{2} z \right)^{r-[r]}, -n \le z \le n, g(0) := t^{r-[r]},$$

we get

$$\triangle_t^{[r]} T_n^{(r-[r])} \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) c_\nu e^{i\nu x}, \triangle_t^r T_n \left(x - \frac{[r]}{2} t \right) = \sum_{\nu \in \mathbb{Z}_n^*} \varphi(\nu) g(\nu) c_\nu e^{i\nu x}.$$

Taking into account the fact that [17]

$$g(z) = \sum_{k=-\infty}^{\infty} d_k e^{ik\pi z/n}$$

uniformly in $[-n, n]$, with $d_0 > 0$, $(-1)^{k+1} d_k \ge 0$, $d_{-k} = d_k$ $(k = 1, 2, ...)$, we have

$$\triangle_t^r T_n(\cdot) = \sum_{k=-\infty}^{\infty} d_k \triangle_t^{[r]} T_n^{(r-[r])} \left(\cdot + \frac{k\pi}{n} + \frac{r-[r]}{2} t \right).$$

Consequently we get

$$\begin{aligned}\left\|\frac{1}{\delta}\int_0^\delta |\triangle_t^r T_n(\cdot)|\,dt\right\|_{L_w^{ps}} &= \left\|\frac{1}{\delta}\int_0^\delta \left|\sum_{k=-\infty}^{\infty} d_k \triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^{ps}} \\ &\le \sum_{k=-\infty}^{\infty} |d_k| \left\|\frac{1}{\delta}\int_0^\delta \left|\triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^{ps}}\end{aligned}$$

and since [20, p. 103]

$$\triangle_t^{[r]} T_n^{(r-[r])}(\cdot) = \int_0^t \ldots \int_0^t T_n^{(r)}(\cdot + t_1 + \ldots t_{[r]})dt_1\ldots dt_{[r]}$$

we find

$$\Omega_r(T_n,h)_{L_w^{ps}} \le \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\|\frac{1}{\delta}\int_0^\delta \left|\triangle_t^{[r]} T_n^{(r-[r])}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^{ps}}$$

$$= \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\|\frac{1}{\delta}\int_0^\delta \left|\int_0^t \ldots \int_0^t T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+\ldots t_{[r]}\right) dt_1\ldots dt_{[r]}\right| dt\right\|_{L_w^{ps}}$$

$$\le\ h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k|$$

$$\times \left\|\frac{1}{\delta}\int_0^\delta \frac{1}{\delta^{[r]}}\int_0^\delta \ldots \int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+\ldots t_{[r]}\right)\right|\ dt_1\ldots dt_{[r]}dt\right\|_{L_w^{ps}}$$

$$\le\ h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k|$$

$$\times \left\|\frac{1}{\delta^{[r]}}\int_0^\delta \ldots \int_0^\delta \left\{\frac{1}{\delta}\int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t+t_1+\ldots t_{[r]}\right)\right| dt\right\} dt_1\ldots dt_{[r]}\right\|_{L_w^{ps}}$$

$$\le c h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\|\frac{1}{\delta}\int_0^\delta \left|T_n^{(r)}\left(\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}t\right)\right| dt\right\|_{L_w^{ps}}$$

$$\le c h^{[r]} \sup_{|\delta|\le h} \sum_{k=-\infty}^{\infty} |d_k| \left\|\frac{1}{\frac{r-[r]}{2}\delta}\int_{\cdot+\frac{k\pi}{n}}^{\cdot+\frac{k\pi}{n}+\frac{r-[r]}{2}\delta} \left|T_n^{(r)}(u)\right| du\right\|_{L_w^{ps}}$$

On the other hand [17]

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2t^{r-[r]},\ \ 0<t\le\pi/n$$

and for $0 < t < \delta < h \leq \pi/n$ we have

$$\sum_{k=-\infty}^{\infty} |d_k| < 2g(0) = 2h^{r-[r]}.$$

Therefore the boundedness of Hardy-Littlewood maximal function in $L_w^{ps}(\mathbf{T})$ implies that

$$\Omega_r(T_n, h)_{L_w^{ps}} \leq ch^r \left\| T_n^{(r)} \right\|_{L_w^{ps}}.$$

By similar way for $0 < -h \leq \pi/n$, the same inequality also holds and the proof of Lemma 1 is completed. □

Lemma 2.2. *Let $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$. If $T_n \in \Pi_n$ and $\alpha > 0$, then there exists a constant $c > 0$ depending only on α, p and ssuch that*

$$\left\| T_n^{(\alpha)} \right\|_{L_w^{ps}} \leq cn^{\alpha} \left\| T_n \right\|_{L_w^{ps}}.$$

Proof. Since $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$, we have [21, Chap. VI]

$$\begin{aligned} \|S_n(f)\|_{L_w^{ps}} &\leq c\|f\|_{L_w^{ps}}, \\ \left\|\tilde{f}\right\|_{L_w^{ps}} &\leq c\|f\|_{L_w^{ps}}. \end{aligned}$$

Now, following the method given in [13] we obtain the request result. □

Definition 2.1. For $f \in L_w^{ps}(\mathbf{T})$, $1 < p, s < \infty$, $\delta > 0$ and $r = 1, 2, 3, ...$, the Peetre K-functional is defined as

$$K\left(\delta, f; L_w^{ps}(\mathbf{T}), W_{ps,w}^r(\mathbf{T})\right) := \inf_{g \in W_{ps,w}^r(\mathbf{T})} \left\{ \|f-g\|_{L_w^{ps}} + \delta \left\| g^{(r)} \right\|_{L_w^{ps}} \right\}. \tag{2.6}$$

Lemma 2.3. *Let $w \in A_p(\mathbf{T})$, $1 < p, s < \infty$. If $f \in L_w^{ps}(\mathbf{T})$ and $r = 1, 2, 3, ...$, then*
(i) the K-functional (2.6) and the modulus (1.4) are equivalent and
(ii) there exists a constant $c > 0$ depending only on r, p and s such that

$$E_n(f)_{L_w^{ps}} \leq c\Omega_r(f, 1/n)_{L_w^{ps}}.$$

Proof. (i) can be proved by the similar way to that of Theorem 1 in [12] and later (ii) is proved by standard way (see for example, [12], [8]). □

3. Proofs of the main results

Proof of Theorem 1. Let S_n be the $n-th$ partial sum of the Fourier series of $f \in L_w^p(\mathbf{T})$, $w \in A_p(\mathbf{T})$ and let $m \in \mathbb{Z}^+$. By Remark 1 (ii), (1.3) and [10, prop.3.4]

$$\begin{aligned} \Omega_r(f, \pi/(n+1))_{L_w^{ps}} &\leq \Omega_r(f - S_{2^m}, \pi/(n+1))_{L_w^{ps}} + \Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \\ &\leq cE_{2^m}(f)_{L_w^{ps}} + \Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \end{aligned}$$

and by Lemma 1,

$$\Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \leq c\left(\frac{\pi}{n+1}\right)^r \left\| S_{2^m}^{(r)} \right\|_{L_w^{ps}}, \quad n+1 \geq 2^m.$$

Since

$$S_{2^m}^{(r)}(x) = S_1^{(r)}(x) + \sum_{\nu=0}^{m-1} \left\{ S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x) \right\},$$

we have

$$\begin{aligned} &\Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \\ (3.7)\qquad &\leq c\left(\frac{\pi}{n+1}\right)^r \left\{ \left\|S_1^{(r)}\right\|_{L_w^{ps}} + \left\| \sum_{\nu=0}^{m-1} \left[S_{2^{\nu+1}}^{(r)} - S_{2^\nu}^{(r)}\right] \right\|_{L_w^{ps}} \right\}. \end{aligned}$$

Following the method used in [10, Proof of Theorem 1] , we obtain

$$\begin{aligned} &\left\| \sum_{\nu=0}^{m-1} \left[S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x)\right] \right\|_{L_w^{ps}} \\ \leq\ & c\left(\sum_{\nu=0}^{m-1} \left\|S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x)\right\|_{L_w^{ps}}^{\gamma} \right)^{\frac{1}{\gamma}}, \quad \gamma = \min(s,2). \end{aligned}$$

By Lemma 2, we get

$$\left\|S_{2^{\nu+1}}^{(r)}(x) - S_{2^\nu}^{(r)}(x)\right\|_{L_w^{ps}} \leq c2^{\nu r}\left\|S_{2^{\nu+1}}(x) - S_{2^\nu}(x)\right\|_{L_w^{ps}} \leq c(p,r)2^{\nu r+1}E_{2^\nu}(f)_{L_w^{ps}}$$

and

$$\left\|S_1^{(r)}\right\|_{L_w^{ps}} = \left\|S_1^{(r)} - S_0^{(r)}\right\|_{L_w^{ps}} \leq cE_0(f)_{L_w^{ps}}.$$

Then from (3.7) we have

$$\Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \leq c\left(\frac{\pi}{n+1}\right)^r \left\{ E_0(f)_{L_w^{ps}} + \left(\sum_{\nu=0}^{m-1} 2^{(\nu+1)r\gamma} E_{2^\nu}^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}} \right\}.$$

It is easily seen that

$$(3.8)\qquad 2^{(\nu+1)r\gamma}E_{2^\nu}^{\gamma}(f)_{L_w^{ps}} \leq c\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\gamma r-1}E_\mu^{\gamma}(f)_{L_w^{ps}}, \quad \nu = 1,2,3,\ldots.$$

Therefore,

$$\begin{aligned} &\Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \\ \leq\ & c\left(\frac{\pi}{n+1}\right)^r \left\{ E_0(f)_{L_w^{ps}} + 2^rE_1(f)_{L_w^{ps}} + c\left(\sum_{\nu=0}^{m-1} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{\gamma r-1}E_\mu^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}} \right\} \\ \leq\ & c\left(\frac{\pi}{n+1}\right)^r \left\{ E_0(f)_{L_w^{ps}} + \left(\sum_{\mu=1}^{2^m} \mu^{\gamma r-1}E_\mu^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}} \right\} \\ \leq\ & c\left(\frac{\pi}{n+1}\right)^r \left(\sum_{\nu=0}^{2^m-1} (\nu+1)^{\gamma r-1}E_\nu^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}}. \end{aligned}$$

If we choose $2^m \leq n+1 \leq 2^{m+1}$, then

$$\Omega_r(S_{2^m}, \pi/(n+1))_{L_w^{ps}} \leq \frac{c}{(n+1)^r}\left(\sum_{\nu=0}^{n} (\nu+1)^{\gamma r-1}E_\nu^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}}.$$

Taking also the relation

$$E_{2^m}(f)_{L_w^{ps}} \le E_{2^{m-1}}(f)_{L_w^{ps}} \le \frac{c}{(n+1)^r}\left(\sum_{\nu=0}^{n}(\nu+1)^{\gamma r-1}E_\nu^\gamma(f)_{L_w^{ps}}\right)^{\frac{1}{\gamma}}$$

into account we obtain the required inequality of Theorem 1. □

Proof of Theorem 2. If T_n is the best approximating polynomial of f, then by Lemma 2

$$\left\|T_{2^{m+1}}^{(\alpha)} - T_{2^m}^{(\alpha)}\right\|_{L_w^{ps}} \le c2^{(m+1)\alpha}E_{2^m}(f)_{L_w^{ps}}$$

and hence by this inequality, (3.8) and hypothesis of Theorem 2 we have

$$\begin{aligned}\sum_{m=1}^{\infty}\|T_{2^{m+1}} - T_{2^m}\|_{W_{ps,w}^\alpha(\mathbf{T})} &= \sum_{m=1}^{\infty}\|T_{2^{m+1}} - T_{2^m}\|_{L_w^{ps}} \\ +\sum_{m=1}^{\infty}\left\|T_{2^{m+1}}^{(\alpha)} - T_{2^m}^{(\alpha)}\right\|_{L_w^{ps}} &\le c\sum_{m=1}^{\infty}2^{(m+1)\alpha}E_{2^m}(f)_{L_w^{ps}} \\ \le c\sum_{m=1}^{\infty}\sum_{j=2^{m-1}+1}^{2^m} j^{\alpha-1}E_j(f)_{L_w^{ps}} &\le c\sum_{j=2}^{\infty} j^{\alpha-1}E_j(f)_{L_w^{ps}} < \infty.\end{aligned}$$

Therefore

$$\sum_{m=1}^{\infty}\|T_{2^{m+1}} - T_{2^m}\|_{W_{ps,w}^\alpha(\mathbf{T})} < \infty,$$

which implies that $\{T_{2^m}\}$ is a Cauchy sequence in $W_{ps,w}^\alpha(\mathbf{T})$. Since $T_{2^m} \to f$ in the Banach space $L_w^{ps}(\mathbf{T})$, we have $f \in W_{ps,w}^\alpha(\mathbf{T})$. It is clear that

$$E_n(f^{(\alpha)})_{L_w^{ps}} \le \left\|f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{L_w^{ps}}$$

$$\le \left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{L_w^{ps}} + \left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right]\right\|_{L_w^{ps}}.$$

By Lemma 2

$$\begin{aligned}\left\|S_{2^{m+2}}f^{(\alpha)} - S_n f^{(\alpha)}\right\|_{L_w^{ps}} &\le c2^{(m+2)\alpha}E_n(f)_{L_w^{ps}} \\ &\le c(n+1)^\alpha E_n(f)_{L_w^{ps}}\end{aligned}$$

for $2^m < n < 2^{m+1}$.

On the other hand, following the method used in [10, Proof of Theorem 1], we get

$$\left\|\sum_{k=m+2}^{\infty}\left[S_{2^{k+1}}f^{(\alpha)} - S_{2^k}f^{(\alpha)}\right]\right\|_{L_w^{ps}} \le c\left(\sum_{k=m+2}^{\infty}\left\|S_{2^{k+1}}^{(\alpha)}(x) - S_{2^k}^{(\alpha)}(x)\right\|_{L_w^{ps}}^\gamma\right)^{\frac{1}{\gamma}},$$

where $\gamma = \min(s,2)$. Since by Lemma 2

$$\left\|S_{2^{k+1}}^{(\alpha)}(x) - S_{2^k}^{(\alpha)}(x)\right\|_{L_w^{ps}} \le c2^{k\alpha}\|S_{2^{k+1}}(x) - S_{2^k}(x)\|_{L_w^{ps}} \le c2^{k\alpha+1}E_{2^k}(f)_{L_w^{ps}},$$

we get

$$\left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L_w^{ps}} \le c \left(\sum_{k=m+2}^{\infty} 2^{\gamma k \alpha + 1} E_{2^k}^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}}$$

Therefore, we have

$$\left\| \sum_{k=m+2}^{\infty} \left[S_{2^{k+1}} f^{(\alpha)} - S_{2^k} f^{(\alpha)} \right] \right\|_{L_w^{ps}} \le c \left(\sum_{k=n+1}^{\infty} k^{\gamma\alpha - 1} E_{k}^{\gamma}(f)_{L_w^{ps}} \right)^{\frac{1}{\gamma}}$$

for $2^m < n < 2^{m+1}$. This completes the proof. □

Proof of Theorem 3. We set

$$W_n(f) := W_n(x,f) := \frac{1}{n+1} \sum_{\nu=n}^{2n} S_\nu(x,f),\ n = 0,1,2,\dots .$$

Since

$$W_n(\cdot, f^{(\alpha)}) = W_n^{(\alpha)}(\cdot, f),$$

we have

$$\left\| f^{(\alpha)}(\cdot) - T_n^{(\alpha)}(\cdot, f) \right\|_{L_w^{ps}} \le \left\| f^{(\alpha)}(\cdot) - W_n(\cdot, f^{(\alpha)}) \right\|_{L_w^{ps}} +$$

$$+ \left\| T_n^{(\alpha)}(\cdot, W_n(f)) - T_n^{(\alpha)}(\cdot, f) \right\|_{L_w^{ps}} + \left\| W_n^{(\alpha)}(\cdot, f) - T_n^{(\alpha)}(\cdot, W_n(f)) \right\|_{L_w^{ps}} =: I_1 + I_2 + I_3.$$

Let $T_n(x,f)$ be the best approximating polynomial of degree at most n to f in $L_w^{ps}(\mathbf{T})$. From the boundedness of W_n in $L_w^{ps}(\mathbf{T})$ we have

$$I_1 \le \left\| f^{(\alpha)}(\cdot) - T_n(\cdot, f^{(\alpha)}) \right\|_{L_w^{ps}} + \left\| T_n(\cdot, f^{(\alpha)}) - W_n(\cdot, f^{(\alpha)}) \right\|_{L_w^{ps}}$$

$$\le cE_n(f^{(\alpha)})_{L_w^{ps}} + \left\| W_n(\cdot, T_n(f^{(\alpha)}) - f^{(\alpha)}) \right\|_{L_w^{ps}} \le cE_n(f^{(\alpha)})_{L_w^{ps}}$$

and by Lemma 2

$$I_2 \le cn^\alpha \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f) \right\|_{L_w^{ps}}$$

and

$$\begin{aligned} I_3 &\le c(2n)^\alpha \left\| W_n(\cdot, f) - T_n(\cdot, W_n(f)) \right\|_{L_w^{ps}} \\ &\le c(2n)^\alpha E_n(W_n(f))_{L_w^{ps}}. \end{aligned}$$

Taking into account that

$$\begin{aligned} & \left\| T_n(\cdot, W_n(f)) - T_n(\cdot, f) \right\|_{L_w^{ps}} \\ \le\ & \left\| T_n(\cdot, W_n(f)) - W_n(\cdot, f) \right\|_{L_w^{ps}} + \left\| W_n(\cdot, f) - f(\cdot) \right\|_{L_w^{ps}} + \left\| f(\cdot) - T_n(\cdot, f) \right\|_{L_w^{ps}} \\ \le\ & cE_n(W_n(f))_{L_w^{ps}} + cE_n(f)_{L_w^{ps}} + cE_n(f)_{L_w^{ps}} \end{aligned}$$

and

$$E_n(W_n(f))_{L_w^{ps}} \le cE_n(f)_{L_w^{ps}},$$

we get

$$\begin{aligned}&\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot,f)\right\|_{L_w^{ps}}\\ \leq\ & cE_n(f^{(\alpha)})_{L_w^{ps}}+cn^{\alpha}E_n(W_n(f))_{L_w^{ps}}+cn^{\alpha}E_n(f)_{L_w^{ps}}+c(2n)^{\alpha}E_n(W_n(f))_{L_w^{ps}}\\ \leq\ & cE_n(f^{(\alpha)})_{L_w^{ps}}+cn^{\alpha}E_n(f)_{L_w^{ps}}.\end{aligned}$$

Since [1]

$$E_n(f)_{L_w^{ps}}\leq\frac{c}{(n+1)^{\alpha}}E_n(f^{(\alpha)})_{L_w^{ps}},\tag{3.9}$$

we conclude that

$$\left\|f^{(\alpha)}(\cdot)-T_n^{(\alpha)}(\cdot)\right\|_{L_w^{ps}}\leq cE_n(f^{(\alpha)})_{L_w^{ps}}$$

and the proof is completed. □

Proof of Theorem 4. Let $T_n\in\Pi_n$ be the trigonometric polynomial of the best approximation of f in $L_w^{ps}(\mathbf{T})$ metric. By Remark 1 (*ii*), Lemma 1 and (1.3) we get

$$\Omega_r(f,h)_{L_w^{ps}}\leq\Omega_r(T_n,h)_{L_w^{ps}}+\Omega_r(f-T_n,h)_{L_w^{ps}}$$

$$\leq ch^r\left\|T_n^{(r)}\right\|_{L_w^{ps}}+cE_n(f)_{L_w^{ps}},\quad 0<h<\pi/n.$$

Using (3.9), Lemma 3 (*ii*) and the inequality

$$\Omega_l(f,h)_{L_w^{ps}}\leq ch^l\left\|f^{(l)}\right\|_{L_w^{ps}},\quad f\in W_{ps,w}^l(\mathbf{T}),\ l=1,2,3,...,$$

which can be showed using the judgements given in [12, Theorem 1], we have

$$\begin{aligned}E_n(f)_{L_w^{ps}}&\leq\frac{c}{(n+1)^{r-[r]}}E_n(f^{(r-[r])})_{L_w^{ps}}\leq\frac{c}{(n+1)^{r-[r]}}\Omega_{[r]}\left(f^{(r-[r])},\frac{2\pi}{n+1}\right)_{L_w^{ps}}\\&\leq\frac{c}{(n+1)^{r-[r]}}\left(\frac{2\pi}{n+1}\right)^{[r]}\left\|f^{(r)}\right\|_{L_w^{ps}}.\end{aligned}$$

On the other hand, by Theorem 3 we find

$$\begin{aligned}\left\|T_n^{(r)}\right\|_{L_w^{ps}}&\leq\left\|T_n^{(r)}-f^{(r)}\right\|_{L_w^{ps}}+\left\|f^{(r)}\right\|_{L_w^{ps}}\\&\leq cE_n(f^{(r)})_{L_w^{ps}}+\left\|f^{(r)}\right\|_{L_w^{ps}}\leq c\left\|f^{(r)}\right\|_{L_w^{ps}}.\end{aligned}$$

Choosing h with $\pi/(n+1)<h\leq\pi/n$, $(n=1,2,3,...)$, we obtain

$$\Omega_r(f,h)_{L_w^{ps}}\leq ch^r\left\|f^{(r)}\right\|_{L_w^{ps}}$$

and we are done. □

REFERENCES

[1] Akgun, R. and Israfilov, D. M., *Approximation in weighted Orlicz spaces*, submitted
[2] Butzer, P. L., Dyckhoff, H., Görlich, E. and Stens, R. L., *Best trigonometric approximation, fractional order derivatives and Lipschitz classes* Can. J. Math. **29** (1977), No. 3, 781-793
[3] Chang, H. M., Hunt R. A. and Kurtz, D. S., *The Hardy-Littlewood maximal functions on $L(p,q)$ spaces with weights*, Indiana Univ. Math. J. **31** (1982), 109-120
[4] Czipszer, J. and Freud, G., *Sur l'approximation d'une fonktion périodique et de ses dérivées succesives par un polynome trigonometrique et par ses dérivées succesives*, Acta Math. **99** (1958), 33-51
[5] Genebashvili, I., Gogatishvili, A., Kokilashvili, V. and Krbec, M., *Weight theory for integral transforms on spaces of homogenous type*, Pitman Monographs, 1998
[6] Haciyeva, E. A., *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces*, (in Russian), Author's summary of candidate dissertation, Tbilisi, 1986
[7] Hunt, R., Muckenhoupt, B. and Wheeden, R., *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227-251
[8] Israfilov, D. M. and Guven, A., *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Math. **174** (2006), No. 2, 147-168
[9] Kokilashvili, V. M. and Yildirir, Y. E., *On the approximation in weighted Lebesgue spaces*, Proceedings of A. Razmadze Math. Inst. **143** (2007), 103-113
[10] Kokilashvili, V. M. and Yildirir Y. E., *On the approximation by trigonometric polynomials in weighted Lorentz spaces*, Journal of Function Spaces and Applications (accepted for publication)
[11] Kurtz, D. S., *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. AMS, **259**(1980), No. 1, 235-254
[12] Ky, N. X., *Moduli of mean smoothness and approximation with A_p-weights*, Annales Univ. Sci., Budapest, **40** (1997), 37-48
[13] Ky, N. X., An Alexits's lemma and its applications in approximation theory, *Functions, Series, Operators*, Budapest, 2002 (L. Leindler, F. Schipp, J. Szabados, eds.), 287-296
[14] Muckenhoupt, B., *Weighted Norm Inequalities for the Hardy Maximal Function*, Transactions of the American Mathematical Society, **165** (1972), 207-226
[15] Samko, S. G., Kilbas, A. A. and Marichev, O. I., *Fractional integrals and derivatives*, Theory and applications, Gordon and Breach Science Publishers, 1993
[16] Steckin, S. B., *On the order of the best approximations of continuous functions*, Izv. Akad. Nauk SSSR, Ser. Mat. **15** (1951), 219-242
[17] Taberski, R., *Differences, moduli and derivatives of fractional orders*, Comment. Math. **19** (1977), 389-400
[18] Taberski, R., *Two indirect approximation theorems*, Demonstratio Math. **9** (1976), No. 2, 243-255
[19] Timan, A. F., *Investigation in the theory of approximation of functions*, Dissertation, Khar'kov, 1951
[20] Timan, A. F., *Theory of appoximation of functions of a real variable*, Pergamon Press and MacMillan, 1963; Russian original published by Fizmatzig, Moscow, 1960
[21] Zygmund, A., *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1968

BALIKESIR UNIVERSITY
DEPARTMENT OF MATHEMATICS
BALIKESIR, TURKEY
E-mail address: yildirir@balikesir.edu.tr

BALIKESIR UNIVERSITY
DEPARTMENT OF MATHEMATICS
BALIKESIR, TURKEY
E-mail address: mdaniyal@balikesir.edu.tr