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ELLIPSES AND HARMONIC MÖBIUS TRANSFORMATIONS

Nihal Yilmaz Özgür

Abstract

Harmonic Möbius transformations are the generalization of Möbius transformations to harmonic mappings. Their basic geometric property is that they take circles to ellipses. In this paper, we determine the images of ellipses under the harmonic Möbius transformations.

1 Introduction

In [2], Chuaqui, Duren and Osgood introduced the harmonic Möbius transformations as a generalization of Möbius transformations to harmonic mappings. A harmonic Möbius transformation is a harmonic mapping of the form

$$f = h + \alpha\overline{h}, \tag{1}$$

where h is a Möbius transformation and α is a complex constant with $|\alpha| < 1$. We know that their basic geometric property is that they take circles to ellipses. In [3], it was also shown that a harmonic mapping taking circles to ellipses is a harmonic Möbius transformation (see [6] for more details about harmonic mappings).

In [5] and [4], Coffman and Frantz considered the images of non-circular ellipses under the Möbius transformations. They proved that the only Möbius transformations which take ellipses to ellipses are the similarity transformations. Then, it seems natural to consider the image $f(E)$ of a non-circular ellipse $E \subset \mathbb{C}$ for any harmonic Möbius transformation f. In this paper we investigate the images of non-circular ellipses under the harmonic Möbius transformations.

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2 Harmonic Möbius Transformations and Ellipses

We begin with a brief review of the basic properties of Möbius transformations. Möbius transformations are the automorphisms of the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, that is, the meromorphic bijections $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$. A Möbius transformation T has the form

$$T(z) = \frac{az+b}{cz+d}; a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0. \tag{2}$$

The set of all Möbius transformations is a group under composition. The Möbius transformations with $c = 0$ form the subgroup of *similarities*. Such transformations have the form

$$S(z) = Az + B; A, B \in \mathbb{C}, A \neq 0. \tag{3}$$

The transformation $J(z) = \frac{1}{z}$ is called an *inversion*. Every Möbius transformation T of the form (2) is a composition of finitely many similarities and inversions. It is well-known that Möbius transformations take circles to circles. This is their most basic geometric property (see [1] and [10] for more details about Möbius transformations). For some other geometric properties of Möbius transformations one can see the references [7]-[9].

As noted in [2], a harmonic Möbius transformation $f = h + \alpha\overline{h}$ is the composition of the Möbius transformation h with the linear map $z \to z + \alpha\overline{z}$. We use this fact to determine the images of ellipses under the harmonic Möbius transformations.

In general, the image of any ellipse E under a Möbius transformation h is a real biquadratic curve. We recall the definition of a real biquadratic curve, (see [5]).

Definition 2.1. *A "real biquadratic curve" is a plane curve that satisfies an implicit equation of the form*

$$c_{22}z^2\overline{z}^2 + c_{21}z^2\overline{z} + c_{12}z\overline{z}^2 + c_{20}z^2 + c_{11}z\overline{z} + c_{02}\overline{z}^2 + c_{10}z + c_{01}\overline{z} + c_{00} = 0, \tag{4}$$

where the complex coefficients satisfy $c_{jk} = \overline{c_{kj}}$.

The real conics are the real biquadratic curves with $c_{22} = c_{21} = c_{12} = 0$.

Now we can prove the following theorem.

Theorem 2.1. *The only harmonic Möbius transformations which take ellipses to ellipses are the harmonic similarity transformations of the form* $f = h + \alpha\overline{h}$ *where* h *is a similarity transformation.*

Proof. It is easy to see that the linear map $z \to z + \alpha\overline{z}$ take ellipses to ellipses. Indeed, without loss of generality let us consider the ellipse E with the following equation:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{5}$$

If $z \in E$, we can write

$$z = A\cos\theta + iB\sin\theta.$$

Applying the linear map $w = z + \alpha\overline{z}$ to the ellipse E we find the curve

$$w = u + iv = (1+\alpha)A\cos\theta + i(1-\alpha)B\sin\theta,$$

which coordinates satisfy the equation

$$\frac{u^2}{A^2(1+\alpha)^2} + \frac{v^2}{B^2(1-\alpha)^2} = 1. \tag{6}$$

Clearly Equation (6) is an equation of an ellipse. This curve can be a circle if $A^2(1+\alpha)^2 = B^2(1-\alpha)^2$.

Also we know that the only Möbius transformations which take ellipses to ellipses are the similarity transformations as noted in the introduction. So, any harmonic Möbius transformation of the form $f = h + \alpha\overline{h}$, where h is a similarity transformation, take ellipses to ellipses.

If h is not a similarity transformation, then the image of any ellipse under h is a real biquadric curve. Then it remains to show that the image of a real biquadric curve under the linear map $z \to z + \alpha\overline{z}$ can not be an ellipse.

Equation 5 can be written in terms of complex coordinates, $z = x + iy$, $\overline{z} = x - iy$:

$$M\left(z^2 + \overline{z}^2\right) + Nz\overline{z} - 1 = 0, \tag{7}$$

where $M = \frac{1}{4}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ and $N = \frac{1}{2}\left(\frac{1}{A^2} + \frac{1}{B^2}\right)$.

Applying a Möbius transformation $w = h(z)$ of the form (2) with $c \neq 0$ to Equation (7) gives a real biquadratic curve. This curve satisfies the following implicit equation:

$$c_{22}w^2\overline{w}^2 + c_{21}w^2\overline{w} + c_{12}w\overline{w}^2 + c_{20}w^2 + c_{11}w\overline{w} + c_{02}\overline{w}^2 + c_{10}w + c_{01}\overline{w} + c_{00} = 0. \tag{8}$$

The coefficients c_{22} and c_{21} of this curve are the followings:

$$\begin{aligned} c_{22} &= Md^2\overline{c}^2 + M\overline{d}^2c^2 + N|c|^2|d|^2 - |c|^4, \\ c_{21} &= -2M\overline{ac}d^2 - 2M\overline{bd}c^2 - N\overline{a}c|d|^2 - N\overline{b}d|c|^2 + 2\overline{ac}c^2. \end{aligned} \tag{9}$$

We need these coefficients later.

If we apply the map $W = z + \alpha\overline{z}$ to equation (8), after some computations, we find a curve with the equation

$$\begin{gathered} c'_{40}W^4 + c'_{04}\overline{W}^4 + c'_{31}w^3\overline{W} + c'_{13}W\overline{W}^3 \\ +c'_{30}W^3 + c'_{03}\overline{W}^3 + c'_{22}W^2\overline{W}^2 + c'_{21}W^2\overline{W} \\ +c'_{12}W\overline{W}^2 + c'_{20}W^2 + c'_{11}W\overline{W} + c'_{02}\overline{W}^2 \\ +c'_{10}W + c'_{01}\overline{W} + c'_{00} = 0, \end{gathered} \tag{10}$$

where the complex coefficients satisfy $c'_{jk} = \overline{c'_{kj}}$. We obtain the coefficients as

$$\begin{aligned}
c'_{40} &= \left(1-|\alpha|^2\right)^{-4}\overline{\alpha}^2 c_{22,} \\
c'_{04} &= \left(1-|\alpha|^2\right)^{-4}\alpha^2 c_{22,} \\
c'_{31} &= -2\overline{\alpha}\left(1-|\alpha|^2\right)^{-4}\left(1+|\alpha|^2\right)c_{22,} \\
c'_{13} &= -2\alpha\left(1-|\alpha|^2\right)^{-4}\left(1+|\alpha|^2\right)c_{22,} \\
c'_{30} &= \overline{\alpha}\left(1-|\alpha|^2\right)^{-3}\left(\overline{\alpha}c_{12}-c_{21}\right), \\
c'_{03} &= \alpha\left(1-|\alpha|^2\right)^{-3}\left(\alpha c_{21}-c_{12}\right), \\
c'_{22} &= \left(1-|\alpha|^2\right)^{-4}\left(1+4|\alpha|^2+|\alpha|^4\right)c_{22}, \\
c'_{21} &= \left(1-|\alpha|^2\right)^{-3}\left[\left(1+2|\alpha|^2\right)c_{21}-\overline{\alpha}\left(2+|\alpha|^2\right)c_{12}\right], \\
c'_{12} &= \left(1-|\alpha|^2\right)^{-3}\left[\left(1+2|\alpha|^2\right)c_{12}-\alpha\left(2+|\alpha|^2\right)c_{21}\right], \\
c'_{20} &= \left(1-|\alpha|^2\right)^{-2}\left(c_{20}+\overline{\alpha}^2c_{02}-\overline{\alpha}c_{11}\right), \\
c'_{02} &= \left(1-|\alpha|^2\right)^{-2}\left(\alpha^2c_{20}+c_{02}-\alpha c_{11}\right), \\
c'_{11} &= \left(1-|\alpha|^2\right)^{-2}\left(\left(1+|\alpha|^2\right)c_{11}-2\alpha c_{20}-2\overline{\alpha}c_{02}\right), \\
c'_{10} &= \left(1-|\alpha|^2\right)^{-1}\left(c_{10}-\overline{\alpha}c_{01}\right), \\
c'_{01} &= \left(1-|\alpha|^2\right)^{-1}\left(c_{01}-\alpha c_{10}\right), \\
c'_{00} &= c_{00}.
\end{aligned}$$

Now we have two cases:

Case 1. If $c_{22} \neq 0$, this image curve cannot be an ellipse since we have

$$c'_{40} \neq 0, c'_{04} \neq 0, c'_{31} \neq 0, c'_{13} \neq 0, c'_{22} \neq 0.$$

Case 2. If $c_{22} = 0$, then we find

$$c'_{40} = c'_{04} = c'_{31} = c'_{13} = c'_{22} = 0.$$

In this case, we see that it can not be $c'_{03} = 0$ and $c'_{12} = 0$ at the same time. Conversely, assume that $c'_{03} = 0$ and $c'_{12} = 0$. Then we have $c'_{30} = 0$ and $c'_{21} = 0$ since $c'_{03} = \overline{c'_{30}}$ and $c'_{12} = \overline{c'_{12}}$. From the equations $c'_{03} = 0$ and $c'_{12} = 0$, we find

$$\alpha c_{21} = c_{12} \text{ and } c_{12} = \frac{\alpha\left(2 + |\alpha|^2\right)}{1 + 2|\alpha|^2} c_{21}.$$

From last two equations, we find

$$(|\alpha|^2 - 1)c_{21} = 0.$$

If $c_{21} \neq 0$, then it would be $|\alpha| = 1$ which is a contradiction. If $c_{21} = 0$, then we have also $c_{12} = 0$ since $c_{12} = \overline{c_{21}}$. We see that it can not be $c_{22} = 0$ and $c_{21} = 0$ at the same time. Otherwise, from (9) we find $d = 0$ and this implies $c = 0$ which is a contradiction. Thus the image curve can not be an ellipse.

This completes the proof of the theorem. □

From the proof of Theorem 2.1, we have also obtained an alternative proof of the fact that the Möbius transformations of the form (2) with $c \neq 0$ can not map ellipses to ellipses. We can give the following corollary:

Corollary 2.1. *The images of ellipses under the harmonic Möbius transformation of the form $f = h + \alpha\overline{h}$, where h of the form (2) with $c \neq 0$, can not be a real conic.*

Thus in the proof of Theorem 2.1, we have seen that the images of ellipses under the harmonic Möbius transformations are the curves with equation (10). The family of curves with equation (10) contains real biquadric curves.

Example 2.1. *Let us consider the ellipse E with the equation $\frac{x^2}{4}+y^2 = 1$. The image of E under the harmonic Möbius transformation $f_1(z) = iz + \frac{1}{2}(\overline{iz}) = iz - \frac{i}{2}\overline{z}$ is another ellipse. But the image of E under the harmonic Möbius transformation $f_2(z) = \frac{-1}{z+1} + \frac{1}{2}\left(\frac{-1}{\overline{z}+1}\right)$ is not an ellipse (see Figure 1).*

Remark 2.1. *From the proof of Theorem 2.1, we observe an interesting situation. In [4], Coffman and Frantz proved that the image $h(E)$ is not contained in a circle for any Möbius transformation h. But we have seen that the image $h(E)$ can be a circle for the harmonic similarity transformations of the form $f = h + \alpha\overline{h}$ where h is a similarity transformation. More explicitly, in the proof of Theorem 2.1 we have seen that $h(E)$ is a circle if the equation $A^2(1+\alpha)^2 = B^2(1-\alpha)^2$ or equivalently $(1+\alpha^2)M = \alpha N$ holds. For example, the image $h(E)$ of the ellipse E with equation $\frac{x^2}{4} + \frac{y^2}{36} = 1$ is the unit circle under the harmonic Möbius transformation $f(z) = \frac{1}{3}z + \frac{1}{2}\left(\overline{\frac{1}{3}z}\right) = \frac{1}{3}z + \frac{1}{6}\overline{z}$.*

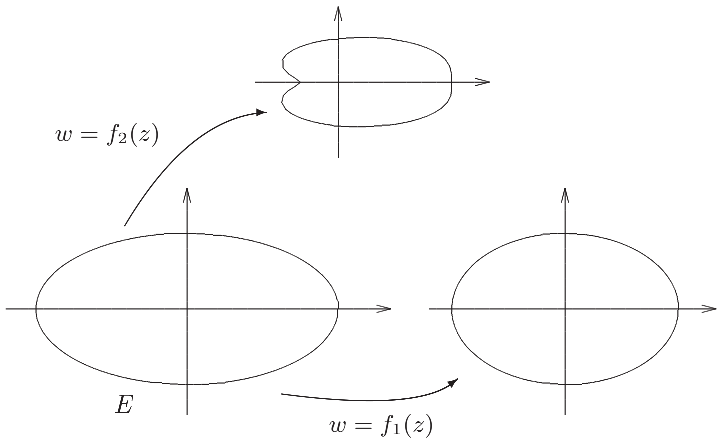

Figure 1: The images of the ellipse E under the harmonic Möbius transformations $f_1(z)$ and $f_2(z)$

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Balikesir University,
Department of Mathematics,
10145 Balikesir, TURKEY,
e-mail: nihal@balikesir.edu.tr