

CHEN INEQUALITIES FOR SUBMANIFOLDS OF REAL SPACE FORMS WITH A SEMI-SYMMETRIC METRIC CONNECTION

Adela Mihai and Cihan Özgür

Abstract. In this paper we prove Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric metric connection, i.e., relations between the mean curvature associated with the semi-symmetric metric connection, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

1. INTRODUCTION

In [9], H.A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. K. Yano studied in [16] some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. In [10] and [11], T. Imai found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Z. Nakao [14] studied submanifolds of a Riemannian manifold with semi-symmetric connections.

On the other hand, one of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. B. Y. Chen [4, 5, 8] established inequalities in this respect, well-known as *Chen inequalities*.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [1-3, 12, 13], and [15].

2. PRELIMINARIES

Let N^{n+p} be an $(n + p)$ -dimensional Riemannian manifold and $\tilde{\nabla}$ a linear connection on N^{n+p} . If the torsion tensor \tilde{T} of $\tilde{\nabla}$, defined by

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$$\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}\tilde{X} - [\tilde{X}, \tilde{Y}],$$

for any vector fields \tilde{X} and \tilde{Y} on N^{n+p} , satisfies

$$\tilde{T}(\tilde{X}, \tilde{Y}) = \phi(\tilde{Y})\tilde{X} - \phi(\tilde{X})\tilde{Y}$$

for a 1-form ϕ , then the connection $\tilde{\nabla}$ is called a *semi-symmetric connection*.

Let g be a Riemannian metric on N^{n+p} . If $\tilde{\nabla}g = 0$, then $\tilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p} .

Following [16], a semisymmetric metric connection $\tilde{\nabla}$ on N^{n+p} is given by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \overset{\circ}{\nabla}_{\tilde{X}}\tilde{Y} + \phi(\tilde{Y})\tilde{X} - g(\tilde{X}, \tilde{Y})P,$$

for any vector fields \tilde{X} and \tilde{Y} on N^{n+p} , where $\overset{\circ}{\nabla}$ denotes the Levi-Civita connection with respect to the Riemannian metric g and P is a vector field defined by $g(P, \tilde{X}) = \phi(\tilde{X})$, for any vector field \tilde{X} .

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let M^n be an n -dimensional submanifold of an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . On the submanifold M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\overset{\circ}{\nabla}$.

Let \tilde{R} be the curvature tensor of N^{n+p} with respect to $\tilde{\nabla}$ and $\overset{\circ}{R}$ the curvature tensor of N^{n+p} with respect to $\overset{\circ}{\nabla}$. We also denote by R and $\overset{\circ}{R}$ the curvature tensors of ∇ and $\overset{\circ}{\nabla}$, respectively, on M^n .

The Gauss formulas with respect to ∇ , respectively $\overset{\circ}{\nabla}$, can be written as:

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \quad X, Y \in \chi(M^n), \\ \overset{\circ}{\nabla}_X Y &= \overset{\circ}{\nabla}_X Y + \overset{\circ}{h}(X, Y), \quad X, Y \in \chi(M^n),\end{aligned}$$

where $\overset{\circ}{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0, 2)$ -tensor on M^n . According to the formula (7) from [14] h is also symmetric.

One denotes by $\overset{\circ}{H}$ the mean curvature vector of M^n in N^{n+p} .

Let $N^{n+p}(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$.

The curvature tensor $\overset{\circ}{R}$ with respect to the Levi-Civita connection $\overset{\circ}{\nabla}$ on $N^{n+p}(c)$ is expressed by

$$(2.1) \quad \overset{\circ}{R}(X, Y, Z, W) = c \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}.$$

Then the curvature tensor \tilde{R} with respect to the semi-symmetric metric connection $\tilde{\nabla}$ on $N^{n+p}(c)$ can be written as [11]

$$\begin{aligned}
 \tilde{R}(X, Y, Z, W) &= \overset{\circ}{R}(X, Y, Z, W) - \alpha(Y, Z)g(X, W) \\
 &+ \alpha(X, Z)g(Y, W) - \alpha(X, W)g(Y, Z) \\
 &+ \alpha(Y, W)g(X, Z),
 \end{aligned}
 \tag{2.2}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where α is a $(0, 2)$ -tensor field defined by

$$\alpha(X, Y) = \left(\overset{\circ}{\nabla}_X \phi \right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X, Y), \quad \forall X, Y \in \chi(M^n).$$

From (2.1) and (2.2) it follows that the curvature tensor \tilde{R} can be expressed as

$$\begin{aligned}
 \tilde{R}(X, Y, Z, W) &= c \{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\
 &- \alpha(Y, Z)g(X, W) + \alpha(X, Z)g(Y, W) \\
 &- \alpha(X, W)g(Y, Z) + \alpha(Y, W)g(X, Z).
 \end{aligned}
 \tag{2.3}$$

Denote by λ the trace of α .

The Gauss equation for the submanifold M^n into the real space form $N^{n+p}(c)$ is

$$\begin{aligned}
 \overset{\circ}{\tilde{R}}(X, Y, Z, W) &= \overset{\circ}{R}(X, Y, Z, W) + g(\overset{\circ}{h}(X, Z), \overset{\circ}{h}(Y, W)) \\
 &- g(\overset{\circ}{h}(X, W), \overset{\circ}{h}(Y, Z)).
 \end{aligned}
 \tag{2.4}$$

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇ . For any orthonormal basis $\{e_1, \dots, e_m\}$ of the tangent space $T_x M^n$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).$$

We recall the following algebraic Lemma:

Lemma 2.1. [4]. *Let a_1, a_2, \dots, a_n, b be $(n + 1)$ ($n \geq 2$) real numbers such that*

$$\left(\sum_{i=1}^n a_i \right)^2 = (n - 1) \left(\sum_{i=1}^n a_i^2 + b \right).$$

Then $2a_1 a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_n$.

Let M^n be an n -dimensional Riemannian manifold, L a k -plane section of $T_x M^n$, $x \in M^n$, and X a unit vector in L .

We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$.

One defines [6] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$Ric_L(X) = K_{12} + K_{13} + \dots + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j . For each integer k , $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L, X} Ric_L(X), \quad x \in M^n,$$

where L runs over all k -plane sections in $T_x M^n$ and X runs over all unit vectors in L .

3. CHEN FIRST INEQUALITY

Recall that the *Chen first invariant* is given by

$$\delta_M(x) = \tau(x) - \inf \{K(\pi) \mid \pi \subset T_x M^n, x \in M^n, \dim \pi = 2\},$$

(see for example [8]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_x M^n$, $x \in M^n$ and τ is the scalar curvature at x .

For submanifolds of real space forms endowed with a semi-symmetric metric connection we establish the following optimal inequality, which will call *Chen first inequality*:

Theorem 3.1. *Let $M^n, n \geq 3$, be an n -dimensional submanifold of an $(n+p)$ -dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c , endowed with a semi-symmetric metric connection $\tilde{\nabla}$. We have:*

$$(3.1) \quad \tau(x) - K(\pi) \leq (n-2) \left[\frac{n^2}{2(n-1)} \|H\|^2 + (n+1) \frac{c}{2} - \lambda \right] - \text{trace} \left(\alpha_{|\pi^\perp} \right),$$

where π is a 2-plane section of $T_x M^n$, $x \in M^n$.

Proof. From [14], the Gauss equation with respect to the semi-symmetric metric connection is

$$(3.2) \quad \begin{aligned} \tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ &\quad - g(h(Y, Z), h(X, W)). \end{aligned}$$

Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{n+p}\}$ be orthonormal basis of $T_x M^n$ and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j, i \neq j$, from the equation (2.3) it follows that:

$$(3.3) \quad \tilde{R}(e_i, e_j, e_j, e_i) = c - \alpha(e_i, e_i) - \alpha(e_j, e_j).$$

From (3.2) and (3.3) we get

$$c - \alpha(e_i, e_i) - \alpha(e_j, e_j) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

By summation over $1 \leq i, j \leq n$, it follows from the previous relation that

$$(3.4) \quad 2\tau + \|h\|^2 - n^2 \|H\|^2 = -2(n - 1)\lambda + (n^2 - n)c,$$

where we recall that λ is the trace of α and denote by

$$\begin{aligned} \|h\|^2 &= \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)), \\ H &= \frac{1}{n} \text{trace} h. \end{aligned}$$

One takes

$$(3.5) \quad \varepsilon = 2\tau - \frac{n^2(n - 2)}{n - 1} \|H\|^2 + 2(n - 1)\lambda - (n^2 - n)c.$$

Then, from (3.4) and (3.5) we get

$$(3.6) \quad n^2 \|H\|^2 = (n - 1) (\|h\|^2 + \varepsilon).$$

Let $x \in M^n, \pi \subset T_x M^n, \dim \pi = 2, \pi = sp \{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.6) we obtain:

$$\left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 = (n - 1) \left[\sum_{i,j=1}^n \sum_{r=n+1}^{n+p} (h_{ij}^r)^2 + \varepsilon \right],$$

or equivalently,

$$(3.7) \quad \begin{aligned} &\left(\sum_{i=1}^n h_{ii}^{n+1} \right)^2 \\ &= (n - 1) \left\{ \sum_{i=1}^n (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^n \sum_{r=n+2}^{n+p} (h_{ij}^r)^2 + \varepsilon \right\}. \end{aligned}$$

By using Lemma 2.1 we have from (3.7):

$$(3.8) \quad 2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^n \sum_{r=n+2}^{n+p} (h_{ij}^r)^2 + \varepsilon.$$

The Gauss equation for $X = W = e_1, Y = Z = e_2$ gives

$$\begin{aligned} K(\pi) &= R(e_1, e_2, e_2, e_1) = c - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+1}^p [h_{11}^r h_{22}^r - (h_{12}^r)^2] \\ &\geq c - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2} \left[\sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^n \sum_{r=n+2}^{n+p} (h_{ij}^r)^2 + \varepsilon \right] \\ &\quad + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p} (h_{12}^r)^2 = c - \alpha(e_1, e_1) - \alpha(e_2, e_2) \\ &\quad + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{i,j=1}^n \sum_{r=n+2}^{n+p} (h_{ij}^r)^2 + \frac{1}{2} \varepsilon + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p} (h_{12}^r)^2 \\ &= c - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{n+p} \sum_{i,j>2} (h_{ij}^r)^2 \\ &\quad + \frac{1}{2} \sum_{r=n+2}^{n+p} (h_{11}^r + h_{22}^r)^2 + \sum_{j>2} [(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2} \varepsilon \\ &\geq c - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{\varepsilon}{2}, \end{aligned}$$

which implies

$$K(\pi) \geq c - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{\varepsilon}{2}.$$

We remark that

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - \text{trace} \left(\alpha_{|\pi^\perp} \right).$$

Using (3.5) we get

$$K(\pi) \geq \tau + (n - 2) \left[-\frac{n^2}{2(n - 1)} \|H\|^2 - (n + 1) \frac{c}{2} + \lambda \right] + \text{trace} \left(\alpha_{|\pi^\perp} \right),$$

which represents the inequality to prove. ■

Recall the following important result (Proposition 1.2) from [10].

Proposition 3.2. *The mean curvature H of M^n with respect to the semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M^n with respect to the Levi-Civita connection if and only if the vector field P is tangent to M^n .*

Remark 3.3. According to the formula (7) from [14] it follows that $h = \overset{\circ}{h}$ if P is tangent to M^n .

In this case inequality (3.1) becomes

Corollary 3.4. *Under the same assumptions as in the Theorem 3.1, if the vector field P is tangent to M^n then we have*

$$(3.9) \quad \tau(x) - K(\pi) \leq (n - 2) \left[\frac{n^2}{2(n - 1)} \left\| \overset{\circ}{H} \right\|^2 + (n + 1) \frac{c}{2} - \lambda \right] - \text{trace} \left(\alpha_{|\pi^\perp} \right).$$

Theorem 3.5. *If the vector field P is tangent to M^n , then the equality case of inequality (3.1) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_x M^n$ and an orthonormal basis $\{e_{n+1}, \dots, e_{n+p}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in $N^{n+p}(c)$ at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^{n+i} & h_{12}^{n+i} & 0 & \cdots & 0 \\ h_{12}^{n+i} & -h_{11}^{n+i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq p,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n + 1 \leq r \leq n + p$.

Proof. The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$\begin{aligned} h_{ij}^r &= 0, \quad \forall i \neq j, i, j > 2, r = n+1, \dots, n+p, \\ h_{11}^r + h_{22}^r &= 0, \quad \forall r = n+2, \dots, n+p, \\ h_{1j}^{n+1} &= h_{2j}^{n+1} = 0, \quad \forall j > 2, \\ h_{11}^{n+1} + h_{22}^{n+1} &= h_{33}^{n+1} = \dots = h_{nn}^{n+1}. \end{aligned}$$

We may choose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = \dots = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. \blacksquare

4. RICCI CURVATURE IN THE DIRECTION OF A UNIT TANGENT VECTOR

In this section, we establish a sharp relation between the Ricci curvature in the direction of a unit tangent vector X and the mean curvature H with respect to the semi-symmetric metric connection $\tilde{\nabla}$.

Denote by

$$N(x) = \{X \in T_x M^n \mid h(X, Y) = 0, \quad \forall Y \in T_x M^n\}.$$

Theorem 4.1. *Let $M^n, n \geq 3$, be an n -dimensional submanifold of an $(n+p)$ -dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$. Then:*

(i) *For each unit vector X in $T_x M$ we have*

$$(4.1) \quad \|H\|^2 \geq \frac{4}{n^2} [\text{Ric}(X) - (n-1)c + (2n-3)\lambda - (n-2)\alpha(X, X)].$$

(ii) *If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of (4.1) if and only if $X \in N(x)$.*

Proof. (i) Let $X \in T_x M^n$ be a unit tangent vector at x . We choose an orthonormal basis $e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ such that e_1, e_2, \dots, e_n are tangent to M^n at x , with $e_1 = X$.

From (3.4) we obtain

$$(4.2) \quad \begin{aligned} n^2 \|H\|^2 &= 2\tau + \frac{1}{2} \sum_{r=n+1}^{n+p} \left[(h_{11}^r + \dots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \dots - h_{nn}^r)^2 \right] \\ &+ 2 \sum_{r=n+1}^{n+p} \sum_{1 \leq i < j \leq n} (h_{ij}^r)^2 - 2 \sum_{r=n+1}^{n+p} \sum_{1 \leq i < j \leq n} (h_{ii}^r h_{jj}^r) \\ &+ 2(n-1)\lambda - (n^2 - n)c. \end{aligned}$$

From Gauss equation (3.2) and the formula (3.3), for $X = W = e_i, Y = Z = e_j, i \neq j$, we get

$$K_{ij} = \tilde{R}(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) \\ = c - \alpha(e_i, e_i) - \alpha(e_j, e_j) + \sum_{r=n+1}^{n+p} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2].$$

By summation, one obtains

$$(4.3) \quad \sum_{2 \leq i < j \leq n} K_{ij} = \sum_{r=n+1}^{n+p} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ + \sum_{2 \leq i < j \leq n} (c - \alpha(e_i, e_i) - \alpha(e_j, e_j)) \\ = \sum_{r=n+1}^{n+p} \sum_{2 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] \\ + \frac{(n-2)(n-1)}{2} c - (n-2) [\lambda - \alpha(e_1, e_1)].$$

After substituting (4.3) into (4.2) we find

$$n^2 \|H\|^2 \geq \frac{1}{2} n^2 \|H\|^2 + 2 \left(\tau - \sum_{2 \leq i < j \leq n} K_{ij} \right) + 2 \sum_{r=n+1}^{n+p} \sum_{j=2}^n (h_{1j}^r)^2 \\ - 2(n-1)c + 2(2n-3)\lambda - 2(n-2)\alpha(e_1, e_1),$$

which gives us

$$\frac{1}{2} n^2 \|H\|^2 \geq 2 \text{Ric}(X) - 2(n-1)c + 2(2n-3)\lambda - 2(n-2)\alpha(X, X).$$

This proves the inequality (4.1).

(ii) Assume $H(x) = 0$. Equality holds in (4.1) if and only if

$$h_{12}^r = \dots = h_{1n}^r = 0,$$

$$h_{11}^r = h_{22}^r + \dots + h_{nn}^r, \quad r \in \{n+1, \dots, n+p\}.$$

Then $h_{1j}^r = 0, \forall j \in \{1, \dots, n\}, r \in \{n+1, \dots, n+p\}$, i.e. $X \in N(x)$. ■

Corollary 4.2. *If the vector field P is tangent to M^n , then the equality case of inequality (4.1) holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.*

Proof. The equality case of (4.1) holds for all unit tangent vectors at x if and only if

$$h_{ij}^r = 0, \quad i \neq j, \quad r \in \{n+1, \dots, n+p\},$$

$$h_{11}^r + \dots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, n+p\}.$$

We distinguish two cases:

- (a) $n \neq 2$, then x is a totally geodesic point;
- (b) $n = 2$, it follows that x is a totally umbilical point.

The converse is trivial. ■

5. k -RICCI CURVATURE

We first state a relationship between the sectional curvature of a submanifold M^n of a real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$ and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k -Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant).

In this section we suppose that the vector field P is tangent to M^n .

Theorem 5.1. *Let $M^n, n \geq 3$, be an n -dimensional submanifold of an $(n+p)$ -dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$ such that the vector field P is tangent to M^n . Then we have*

$$(5.1) \quad \|H\|^2 \geq \frac{2\tau}{n(n-1)} - c + \frac{2}{n}\lambda.$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and orthonormal basis of $T_x M^n$. The relation (3.4) is equivalent with

$$(5.2) \quad n^2 \|H\|^2 = 2\tau + \|h\|^2 + 2(n-1)\lambda - n(n-1)c.$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and e_1, \dots, e_n diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$(5.3) \quad A_{e_{n+1}} \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$$(5.4) \quad A_{e_r} = (h_{ij}^r), i, j = 1, \dots, n; r = n + 2, \dots, n + p, \text{trace } A_r = 0.$$

From (5.2), we get

$$(5.5) \quad n^2 \|H\|^2 = 2\tau + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^r)^2 + 2(n-1)\lambda - n(n-1)c.$$

On the other hand, since

$$0 \leq \sum_{i < j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i < j} a_i a_j,$$

we obtain

$$(5.6) \quad n^2 \|H\|^2 = \left(\sum_{i=1}^n a_i\right)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i < j} a_i a_j \leq n \sum_{i=1}^n a_i^2,$$

which implies

$$\sum_{i=1}^n a_i^2 \geq n \|H\|^2.$$

We have from (5.5)

$$(5.7) \quad n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 + 2(n-1)\lambda - n(n-1)c$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c + \frac{2}{n}\lambda. \quad \blacksquare$$

Using Theorem 5.1, we obtain the following

Theorem 5.2. *Let $M^n, n \geq 3$, be an n -dimensional submanifold of an $(n + p)$ -dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric metric connection $\tilde{\nabla}$, such that the vector field P is tangent to M^n . Then, for any integer $k, 2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$(5.8) \quad \|H\|^2(p) \geq \Theta_k(p) - c + \frac{2}{n}\lambda.$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x M^n$. Denote by $L_{i_1 \dots i_k}$ the k -plane section spanned by e_{i_1}, \dots, e_{i_k} . By the definitions, one has

$$\begin{aligned}\tau(L_{i_1 \dots i_k}) &= \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} Ric_{L_{i_1 \dots i_k}}(e_i), \\ \tau(x) &= \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}).\end{aligned}$$

From (5.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2} \Theta_k(p),$$

which implies (5.8). ■

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Adela Mihai
University of Bucharest,
Faculty of Mathematics,
Academiei 14,
010014 Bucharest,
Romania
E-mail: adela_mihai@fmi.unibuc.ro

Cihan Özgür
University of Balikesir,
Department of Mathematics,
10145, Cagis, Balikesir,
Turkey
E-mail: cozgur@balikesir.edu.tr