Kuwait J. Sci. Eng. **37** (2A) pp. 17-30, 2010

On submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection

CIHAN ÖZGÜR

Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, 10145, Cagis, Balikesir, Turkey, e-mail:cozgur@balikesir.edu.tr

ABSTRACT

Under investigation were the submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. We have proved that the induced connection is also a semi-symmetric non-metric connection. The totaly geodesicness and the minimality of a submanifold of a Riemannian manifold with a semi-symmetric non-metric connection were also considered. We have obtained the Gauss, Codazzi and Ricci equations with respect to a semi-symmetric, non-metric connection. The relation between the sectional curvatures of the Levi-Civita connection and the semi-symmetric non-metric connection is also obtained.

Keywords: Semi-symmetric non-metric connection, submanifold.

INTRODUCTION

Hayden (1932) introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Yano (1970) studied some properties of a Riemannian manifold endowed with a semi-symmetric metric connection. Imai (1972_a & 1972_b) found some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection. Nakao (1976) studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection. Agashe and Chafle (1992 & 1994) introduced the notion of a semi-symmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with semi-symmetric non-metric connections. Sengupta, De & Binh (2000) defined a new type of semi-symmetric non-metric connection.

In the present paper, we have studied submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection as defined in Sengupta, De & Binh (2000). The paper is organized as follows: in Section 2, we have given some properties of the semi-symmetric non-metric connection; in Section 3, some necessary information about a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection has been given and we have proved that the induced connection is also a semi-symmetric non-metric connection. We have also considered the totaly geodesicness and the minimality of a submanifold of a Riemannian manifold with the semi-symmetric non-metric connection. In Section 4, we have obtained the Gauss, Codazzi and Ricci equations with respect to the semi-symmetric non-metric connection. The relation between the sectional curvatures of the Levi-Civita connection and the semi-symmetric non-metric connection has been also found.

PRELIMINARIES

Let $\tilde{M}$ be an $(n+d)-$dimensional Riemannian manifold with a Riemannian metric g, and let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. Sengupta, De & Binh (2000) defined a linear connection on $\tilde{M}$ by

$$\overset{*}{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y} + \omega(\tilde{Y})\tilde{X} - g(\tilde{X}, \tilde{Y})\tilde{U} + g(\tilde{X}, \tilde{Y})\tilde{E}, \tag{1}$$

where $\tilde{U}$ is a vector field associated with the 1-form ω defined by

$$\omega(\tilde{X}) = g(\tilde{X}, \tilde{U}) \tag{2}$$

and $\tilde{E}$ is a vector field associated with the 1-form η by

$$\eta(\tilde{X}) = g(\tilde{X}, \tilde{E}). \tag{3}$$

Using (1), the torsion tensor T of $\tilde{M}$ with respect to the connection $\overset{*}{\tilde{\nabla}}$ is given by

$$T(\tilde{X}, \tilde{Y}) = \overset{*}{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} - \overset{*}{\tilde{\nabla}}_{\tilde{Y}}\tilde{X} - \left[\tilde{X}, \tilde{Y}\right] = \omega(\tilde{Y})\tilde{X} - \omega(\tilde{X})\tilde{Y}. \tag{4}$$

A linear connection $\overset{*}{\tilde{\nabla}}$ satisfying (4) is called a semi-symmetric connection. If $\overset{*}{\tilde{\nabla}}g \neq 0$ then $\overset{*}{\tilde{\nabla}}$ is called a *non-metric connection*. Using (1), we have

$$\left(\overset{*}{\tilde{\nabla}}_{\tilde{X}}g\right)(\tilde{Y}, \tilde{Z}) = -\eta(\tilde{Y})g(\tilde{X}, \tilde{Z}) - \eta(\tilde{Z})g(\tilde{X}, \tilde{Y}). \tag{5}$$

Hence the connection $\overset{*}{\tilde{\nabla}}$ is not a metric connection. Because of this reason, this connection is called a semi-symmetric non-metric connection (for more details see Sengupta, De & Binh, 2000).

We denote by $\overset{*}{\tilde{R}}$ the curvature tensor of $\tilde{M}$ with respect to the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$. So we have

$$\begin{aligned}\overset{*}{\tilde{R}}(\tilde{X},\tilde{Y})\tilde{Z} &= \overset{*}{\tilde{\nabla}}_{\tilde{X}}\overset{*}{\tilde{\nabla}}_{\tilde{Y}}\tilde{Z} - \overset{*}{\tilde{\nabla}}_{\tilde{Y}}\overset{*}{\tilde{\nabla}}_{\tilde{X}}\tilde{Z} - \overset{*}{\tilde{\nabla}}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \\ &= \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} - s(\tilde{Y},\tilde{Z})\tilde{X} + s(\tilde{X},\tilde{Z})\tilde{Y}+ \\ &+g(\tilde{Y},\tilde{Z})\{\lambda(\tilde{X},\tilde{E}) - \lambda(\tilde{X},\tilde{U})\} - g(\tilde{X},\tilde{Z})\{\lambda(\tilde{Y},\tilde{E}) - \lambda(\tilde{Y},\tilde{U})\},\end{aligned} \tag{6}$$

where

$$\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}$$

is the curvature tensor of the manifold with respect to the Levi-Civita connection $\tilde{\nabla}$ and s is a $(0,2)-$tensor field defined by

$$s(\tilde{X},\tilde{Y})Z = \left(\tilde{\nabla}_{\tilde{X}}\omega\right)\tilde{Y} - \omega(\tilde{X})\omega(\tilde{Y}) \tag{7}$$

and

$$\lambda(\tilde{X},\tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} + \omega(\tilde{Y})\tilde{X} - g(\tilde{X},\tilde{Y})\tilde{U} + g(\tilde{X},\tilde{Y})\tilde{E}, \tag{8}$$

(see Sengupta, De & Binh, 2000). The Riemannian Christoffel tensors of the connections $\overset{*}{\tilde{\nabla}}$ and $\tilde{\nabla}$are defined by

$$\overset{*}{\tilde{R}}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = g(\overset{*}{\tilde{R}}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})$$

and

$$\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}) = g(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}),$$

respectively.

SUBMANIFOLDS

Let M be an $n-$dimensional submanifold of an $(n+d)-$dimensional Riemannian manifold $\tilde{M}$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$. Decomposing the vector fields $\tilde{U}$and $\tilde{E}$ on M uniquely into their tangent and normal components $U^T, U^\perp$ and $E^T, E^\perp$,respectively, we have

$$\tilde{U} = U^T + U^\perp, \tag{9}$$

$$\tilde{E} = E^T + E^\perp. \tag{10}$$

The Gauss formula for a submanifold M of a Riemannian manifold $\tilde{M}$ with respect to the Riemannian connection $\tilde{\nabla}$ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{11}$$

where $X, Y$are vector fields tangent to M, and $h$is the second fundamental form of M in $\tilde{M}$. If $h = 0$,then M is called *totally geodesic*. $H = \frac{1}{n}$ *traceh* is called the *mean curvature vector* of the submanifold. If $H = 0$ then M is called *minimal*. For the second fundamental form h, the covariant derivative of h is defined by

$$\left(\bar{\nabla}_X h\right)(Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to M. Then $\bar{\nabla}h$ is a normal bundle valued tensor field of type $(0, 3)$and is called the *third fundamental form* of M. $\bar{\nabla}$ is called the van der Waerden-Bortolotti connection of M,i. e., $\bar{\nabla}$is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$ Chen (1973).

Let $\overset{*}{\nabla}$ be the induced connection from the semisymmetric non-metric connection. We define

$$\overset{*}{\tilde{\nabla}}_X Y = \overset{*}{\nabla}_X Y + \overset{*}{h}(X, Y). \tag{12}$$

Equation (12) is the Gauss equation with respect to the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$. Hence using (1), (11) and (12) we have

$$\begin{aligned} \overset{*}{\nabla}_X Y + \overset{*}{h}(X, Y) &= \nabla_X Y + h(X, Y) + \omega(Y)X \\ &-g(X, Y)U^T - g(X, Y)U^{\perp} + g(X, Y)E^T + g(X, Y)E^{\perp}. \end{aligned} \tag{13}$$

So comparing the tangential and normal parts of equation (13), we obtain

$$\overset{*}{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U^T + g(X, Y)E^T \tag{14}$$

and

$$\overset{*}{h}(X, Y) = h(X, Y) - g(X, Y)U^{\perp} + g(X, Y)E^{\perp}. \tag{15}$$

If $\overset{*}{h} = 0$,then M is called *totally geodesic with respect to the semi-symmetric non-metric connection* (see Agashe & Chafle, 1994).

From equation (12), we have

$$\overset{*}{T}(X, Y) = \overset{*}{\nabla}_X Y - \overset{*}{\nabla}_Y X - [X, Y] = \omega(Y)X - \omega(X)Y, \tag{16}$$

where $\overset{*}{T}$ is the torsion tensor of $M$with respect to $\overset{*}{\nabla}$and X, Y are vector fields tangent to M. Moreover using equation (14), we have

$$\begin{aligned}\left(\overset{*}{\nabla}_X g\right)(Y, Z) &= \overset{*}{\nabla}_X g(Y, Z) - g(\overset{*}{\nabla}_X Y, Z) - g(Y, \overset{*}{\nabla}_X Z) \\ &= -\eta(Y)g(X, Z) - \eta(Z)g(X, Y),\end{aligned} \tag{17}$$

for all vector fields X, Y, Z tangent to M.In view of equations (1), (14), (16) and (17), we can state the following theorem:

Theorem 1. The induced connection $\overset{*}{\nabla}$ on a submanifold of a Riemannian manifold admitting the semi-symmetric non-metric connection in the sense of Sengupta, De & Binh (2000) is also a semi-symmetric non-metric connection.

Let $\{E_1, E_2, ..., E_n\}$be an orthonormal basis of the tangent space of M.We define the mean curvature vector $\overset{*}{H}$ of M with respect to the semi-symmetric non-metric connection $\overset{*}{\nabla}$ by

$$\overset{*}{H} = \frac{1}{n}\sum_{i=1}^{n} \overset{*}{h}(E_i, E_i),$$

(see Agashe & Chafle, 1994). So from equation (15) we find

$$\overset{*}{H} = H - U^{\perp} + E^{\perp}.$$

If $\overset{*}{H} = 0$ then M is called *minimal with respect to the semi-symmetric metric connection* (see Agashe & Chafle, 1994).

So we have the following result:

Theorem 2. Let M be an $n-$dimensional submanifold of an $(n+d)-$dimensional Riemannian manifold $\tilde{M}$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ in the sense of Sengupta, De & Binh (2000). Then

i) M is totally geodesic with respect to the Levi-Civita connection and with respect to the semi-symmetric non-metric connection if and only if the vector fields $\tilde{U}$ and $\tilde{E}$ are tangent to M or $U^{\perp} = E^{\perp}$.

ii) The mean curvature normal of M and that of M with respect to the semi-symmetric non-metric connection coincide if and only if the vector fields $\tilde{U}$ and $\tilde{E}$ are tangent to M or $U^{\perp} = E^{\perp}$. Hence M is minimal with respect to the Levi-Civita connection and with respect to the semi-symmetric non-metric connection if and only if the vector fields $\tilde{U}$ and $\tilde{E}$ are tangent to M or $U^{\perp} = E^{\perp}$.

Let ξ be a normal vector field on M. From (1), we have

$$\overset{*}{\tilde{\nabla}}_X \xi = \tilde{\nabla}_X \xi + \omega(\xi) X. \tag{18}$$

It is well-known that

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \tag{19}$$

which is the Weingarten formula for a submanifold of a Riemannian manifold, where A_ξ is the shape operator of M in the direction of ξ. So from (19), equation (18) can be written as

$$\overset{*}{\tilde{\nabla}}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi + \omega(\xi) X. \tag{20}$$

Now we define a $(1,1)-$tensor field Aon M by

$$\overset{*}{A}_\xi = \big(A_\xi - \omega(\xi)\big) I. \tag{21}$$

Then equation (20) turns into

$$\overset{*}{\tilde{\nabla}}_X \xi = -\overset{*}{A}_\xi X + \nabla^{\perp}_X \xi. \tag{22}$$

Equation (22) is the Weingarten's formula with respect to the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$. Since A_ξ is symmetric, it is easy to see that

$$g\Big(\overset{*}{A}_\xi X, Y\Big) = g\Big(X, \overset{*}{A}_\xi Y\Big)$$

and

$$g\Big(\Big[\overset{*}{A}_\xi, \overset{*}{A}_v\Big] X, Y\Big) = g\big([A_\xi, A_v] X, Y\big), \tag{23}$$

where $\left[\overset{*}{A}_{\xi}, \overset{*}{A}_{\upsilon}\right] = \overset{*}{A}_{\xi}\overset{*}{A}_{\upsilon} - \overset{*}{A}_{\upsilon}\overset{*}{A}_{\xi}$ and $\left[A_{\xi}, A_{\upsilon}\right] = A_{\xi}A_{\upsilon} - A_{\upsilon}A_{\xi}$ and ξ, υ are unit normal vector fields on M.

So we can state the following theorem:

Theorem 3. Let M be an $n-$dimensional submanifold of an $(n+d)-$dimensional Riemannian manifold $\tilde{M}$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ in the sense of Sengupta, De & Binh (2000). Then the shape operators with respect to the semi-symmetric non-metric connection are simultaneously diagonalizable if and only if the shape operators with respect to the Levi-Civita connection are simultaneously diagonalizable.

By a similar proof of Theorem 3.3 in Agashe & Chafle (1994), we have the following theorem:

Theorem 4. Principal directions of the unit normal vector field ξ with respect to the Levi-Civita connection and the semi-symmetric non-metric connection in the sense of Sengupta, De & Binh (2000) coincide, and the principal curvatures are equal if and only if ξ is orthogonal to $U^{\perp}$.

GAUSS, CODAZZI AND RICCI EQUATIONS WITH RESPECT TO SEMI-SYMMETRIC NON-METRIC CONNECTION

We denote the curvature tensor of a submanifold M of a Riemannian manifold $\tilde{M}$ with respect to the induced semi-symmetric non-metric connection $\overset{*}{\nabla}$ and the induced Riemannian connection ∇ by

$$\overset{*}{R}(X, Y)Z = \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z - \overset{*}{\nabla}_Y \overset{*}{\nabla}_X Z - \overset{*}{\nabla}_{[X,Y]} Z \tag{24}$$

and

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively, where X, Y, Z are tangent vector fields on M.

From equations (12) and (20) we get

$$\begin{aligned} \overset{*}{\tilde{\nabla}}_X \overset{*}{\tilde{\nabla}}_Y Z &= \overset{*}{\nabla}_X \overset{*}{\nabla}_Y Z + \overset{*}{h}(X, \overset{*}{\nabla}_Y Z) - A_{\overset{*}{h}(Y,Z)} X \\ &+ \nabla^{\perp}_X \overset{*}{h}(Y, Z) + \omega(^{*}(Y, Z))X, \end{aligned} \tag{25}$$

$$\tilde{\overset{*}{\nabla}}_Y \tilde{\overset{*}{\nabla}}_X Z = \overset{*}{\nabla}_Y \underset{X}{\overset{*}{\nabla}} Z + \overset{*}{h}(Y, \overset{*}{\nabla}_X Z) - A_{\overset{*}{h}(X,Z)} Y + \nabla_Y^{\perp} \overset{*}{h}(X,Z) + \omega({}^{*}(X,Z))\, Y \tag{26}$$

and

$$\tilde{\overset{*}{\nabla}}_{[X,Y]} Z = \overset{*}{\nabla}_{[X,Y]} Z + \overset{*}{h}([X, Y], Z). \tag{27}$$

Hence in view of (24), from (25)-(27), we have

$$\begin{aligned}\tilde{\overset{*}{R}}(X, Y)Z &= \overset{*}{R}(X, Y)Z + \overset{*}{h}\left(X, \overset{*}{\nabla}_Y Z\right) - \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right) - \overset{*}{h}([X, Y], Z) \\ &-A_{\overset{*}{h}(Y,Z)} X + A_{\overset{*}{h}(X,Z)} Y + \nabla_X^{\perp} \overset{*}{h}(Y,Z) - \nabla_Y^{\perp} \overset{*}{h}(X,Z) \\ &+\omega\left(\overset{*}{h}(Y,Z)\right) X - \omega\left(\overset{*}{h}(X,Z)\right) Y.\end{aligned} \tag{28}$$

Since, $g(A_\xi X, Y) = g(h(X, Y), \xi)$, using (15) we find

$$\begin{aligned}\tilde{\overset{*}{R}}(X, Y, Z, W) &= \overset{*}{R}(X, Y, Z, W) - g(h(Y,Z), h(X, W)) + g(h(X,Z), h(Y, W)) \\ &+g(Y,Z)[\omega(h(X, W)) - \eta(h(X, W))] \\ &+g(X,Z)[\eta(h(Y, W)) - \omega(h(Y, W))] \\ &+\omega(h(Y,Z))g(X, W) - \omega(h(X,Z))g(Y, W) \\ &+\omega(E^{\perp})[g(Y,Z)g(X, W) - g(X,Z)g(Y, W)] \\ &+\omega(U^{\perp})[g(X,Z)g(Y, W) - g(Y,Z)g(X, W)],\end{aligned} \tag{29}$$

where W is a tangent vector field on M.

From (28), the normal component of $\tilde{\overset{*}{R}}(X, Y)Z$ is given by

$$\begin{aligned}&\overset{*}{h}\left(X, \overset{*}{\nabla}_Y Z\right) - \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right) - \overset{*}{h}([X, Y], Z) + \nabla_X^{\perp} \overset{*}{h}(Y,Z) - \nabla_Y^{\perp} \overset{*}{h}(X,Z) \\ &= \left(\bar{\overset{*}{\nabla}}_X \overset{*}{h}\right)(Y,Z) - \left(\bar{\overset{*}{\nabla}}_Y \overset{*}{h}\right)(X,Z) \\ &+\omega(Y)\, \overset{*}{h}(X,Z) - \omega(X)\, \overset{*}{h}(Y,Z) = \left(\tilde{\overset{*}{R}}(X, Y)Z\right)^{\perp},\end{aligned} \tag{30}$$

where

$$\left(\overset{*}{\tilde{\nabla}}_X \overset{*}{h}\right)(Y, Z) = \nabla^{\perp}_X \overset{*}{h}(Y, Z) - \overset{*}{h}\left(\overset{*}{\nabla}_X Y, Z\right) - \overset{*}{h}\left(Y, \overset{*}{\nabla}_X Z\right).$$

$\overset{*}{\tilde{\nabla}}$ is the connection in $TM \oplus T$ built with $\overset{*}{\nabla}$ and $\nabla^{\perp}$. It can be called the *van der Waerden-Bortolotti connection with respect to the semi-symmetric non-metric connection*. Equation (30) is the equation of Codazzi with respect to the semi-symmetric non-metric connection.

From equations (22) and (12), we get

$$\overset{*}{\tilde{\nabla}}_X \overset{*}{\tilde{\nabla}}_Y \xi = -\overset{*}{\nabla}_X\left(\overset{*}{A}_\xi Y\right) - \overset{*}{h}\left(X, \overset{*}{A}_\xi Y\right) - \overset{*}{A}_{\nabla^{\perp}_Y \xi} X + \nabla^{\perp}_X \nabla^{\perp}_Y \xi, \tag{31}$$

$$\overset{*}{\tilde{\nabla}}_Y \overset{*}{\tilde{\nabla}}_X \xi = -\overset{*}{\nabla}_Y\left(\overset{*}{A}_\xi X\right) - \overset{*}{h}\left(Y, \overset{*}{A}_\xi X\right) - \overset{*}{A}_{\nabla^{\perp}_X \xi} Y + \nabla^{\perp}_Y \nabla^{\perp}_X \xi \tag{32}$$

and

$$\overset{*}{\tilde{\nabla}}_{[X, Y]} \xi = -\overset{*}{}_{\xi}[X, Y] + \nabla^{\perp}_{[X, Y]} \xi. \tag{33}$$

So using (31)-(33), we have

$$\overset{*}{\tilde{R}}(X, Y, \xi, \upsilon) = R^{\perp}(X, Y, \xi, \upsilon) - g\left(\overset{*}{h}\left(X, \overset{*}{A}_\xi Y\right), \upsilon\right) + g\left(\overset{*}{h}\left(Y, \overset{*}{A}_\xi X\right), \upsilon\right),$$

where ξ, υ are unit normal vector fields on M. Hence in view of (15) and (21) the last equation turns into

$$\overset{*}{\tilde{R}}(X, Y, \xi, \upsilon) = R^{\perp}(X, Y, \xi, \upsilon) - g\big(h\big(X, A_\xi Y\big), \upsilon\big) + g\big(h\big(Y, A_\xi X\big), \upsilon\big),$$

which is equivalent to

$$\begin{aligned} \overset{*}{\tilde{R}}(X, Y, \xi, \upsilon) &= R^{\perp}(X, Y, \xi, \upsilon) + g\big(\big(A_\upsilon A_\xi - A_\xi A_\upsilon\big)X, Y\big) \\ &= R^{\perp}(X, Y, \xi, \upsilon) + g\big(\big[A_\upsilon, A_\xi\big]X, Y\big). \end{aligned} \tag{34}$$

Equation (34) is the equation of Ricci with respect to the semi-symmetric non-metric connection.

Now assume that $\tilde{M}$ is a space of constant curvature c with the semi-symmetric non-metric connection $\tilde{\overset{*}{\nabla}}$. Then

$$\begin{aligned}\tilde{\overset{*}{R}}(X,Y)Z &= c(g(Y,Z)X - g(X,Z)Y) - s(Y,Z)X + s(X,Z)Y \\ &+g(Y,Z)\big(\lambda(X,\tilde{E}) - \lambda(X,\tilde{U})\big) \\ &-g(X,Z)\big(\lambda(Y,\tilde{E}) - \lambda(Y,\tilde{U})\big).\end{aligned} \tag{35}$$

Hence

$$\begin{aligned}\left(\tilde{\overset{*}{R}}(X,Y)Z\right)^{\perp} &= g(Y,Z)\left(\lambda(X,\tilde{E})^{\perp} - \lambda(X,\tilde{U})^{\perp}\right) \\ &-g(X,Z)\left(\lambda(Y,\tilde{E})^{\perp} - \lambda(Y,\tilde{U})^{\perp}\right),\end{aligned}$$

which gives us

$$\begin{aligned}\left(\tilde{\overset{*}{R}}(X,Y)Z\right)^{\perp} &= g(Y,Z)\{h(X,E^T) + \nabla^{\perp}_X E^{\perp} - h(X,U^T) - \nabla^{\perp}_X U^{\perp} \\ &+(\omega(X) - \eta(X))(U^{\perp} - E^{\perp})\} \\ &- g(X,Z)\{h(Y,E^T) + \nabla^{\perp}_Y E^{\perp} - h(Y,U^T) - \nabla^{\perp}_Y U^{\perp} \\ &+(\omega(Y) - \eta(Y))(U^{\perp} - E^{\perp})\}.\end{aligned}$$

So the Ricci equation becomes

$$\begin{aligned}&\left(\overset{*}{\tilde{\nabla}}_X \overset{*}{h}\right)(Y,Z) - \left(\overset{*}{\tilde{\nabla}}_Y \overset{*}{h}\right)(X,Z) + \omega(Y)\overset{*}{h}(X,Z) - \omega(X)\overset{*}{h}(Y,Z) \\ &= g(Y,Z)\{h(X,E^T) + \nabla^{\perp}_X E^{\perp} - h(X,U^T) - \nabla^{\perp}_X U^{\perp} \\ &\quad +(\omega(X) - \eta(X))(U^{\perp} - E^{\perp})\} \\ &\quad -g(X,Z)\{h(Y,E^T) + \nabla^{\perp}_Y E^{\perp} - h(Y,U^T) - \nabla^{\perp}_Y U^{\perp} \\ &\quad +(\omega(Y) - \eta(Y))(U^{\perp} - E^{\perp})\}.\end{aligned}$$

Since $\tilde{M}$ is a space of constant curvature c with the semi-symmetric non-metric connection, from (35), we have $\tilde{\overset{*}{R}}(X,Y,\xi,\upsilon) = 0$. Therefore using (34) and (23) we obtain

$$R^{\perp}(X,Y,\xi,\upsilon) = g\big([A_\xi, A_\upsilon]X, Y\big) = g\left(\left[\overset{*}{A}_{\xi}, \overset{*}{A}_{\upsilon}\right]X, Y\right).$$

Hence using (23), we can state the following theorem:

Theorem 5. Let M be an $n-$dimensional submanifold of an $(n+d)-$dimensional space of constant curvature $\tilde{M}(c)$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ in the sense of Sengupta, De & Binh (2000). Then the normal connection $\nabla^{\perp}$ is flat if and only if all second fundamental tensors with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are simultaneously diagonalizable.

Now assume that X and Y are orthogonal unit tangent vector fields on M. Then in view of (29) we can write

$$\begin{aligned}\overset{*}{\tilde{R}}(X,Y,Y,X) &= \overset{*}{R}(X,Y,Y,X) - g(h(Y,Y),h(X,X))\\ &+g(h(X,Y),h(Y,X)) + [\omega(h(X,X)) - \eta(h(X,X))]\\ &+\omega(h(Y,Y)) + \omega(E^{\perp}) - \omega(U^{\perp}).\end{aligned}$$

So we get

$$\begin{aligned}\overset{*}{\tilde{K}}(\pi) &= \overset{*}{K}(\pi) - g(h(Y,Y),h(X,X))\\ &+g(h(X,Y),h(Y,X)) + [\omega(h(X,X)) - \eta(h(X,X))]\\ &+\omega(h(Y,Y)) + \omega(E^{\perp}) - \omega(U^{\perp}).\end{aligned} \tag{36}$$

Let M be an $n-$dimensional submanifold of an $(n+d)-$dimensional Riemannian manifold $\tilde{M}$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$in the sense of Sengupta, De & Binh (2000) and π be a subspace of the tangent space spanned by the orthonormal base $\{X, Y\}$. Denote by $\overset{*}{\tilde{K}}(\pi)$ and $\overset{*}{K}(\pi)$ the sectional curvatures of $\tilde{M}$ and M at a point $p \in \tilde{M}$, respectively with respect to the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ in the sense of Sengupta, De & Binh (2000). Let γ be a geodesic in $\tilde{M}$ which lies in M, and T be a unit tangent vector field of γ in M. Then from (15) we have

$$h(T,T) = 0,$$

$$\overset{*}{h}(T,T) = -U^{\perp} + E^{\perp}. \tag{37}$$

Let π be the subspace of the tangent space spanned by X, T, and $\tilde{U}$ and $\tilde{E}$ be vector fields tangent to M. Then from (37), we have $\overset{*}{h}(T, T) = 0$. Hence using (36), we obtain

$$\overset{*}{\tilde{K}}(\pi) = \overset{*}{K}(\pi) + g(h(X, T), h(X, T)).$$

Let X be a unit tangent vector field on M which is parallel along γ in $M. So \nabla_T X = 0$.

Hence we have the following theorem:

Theorem 6. Let M be an $n-$dimensional submanifold of an $(n + d)-$dimensional Riemannian manifold $\tilde{M}$ with the semi-symmetric non-metric connection $\overset{*}{\tilde{\nabla}}$ in the sense of Sengupta, De & Binh (2000), and γ be a geodesic in $\tilde{M}$ which lies in M, and T be a unit tangent vector field of γ in M, π be the subspace of the tangent space spanned by X, T. If the vector fields $\tilde{U}$ and $\tilde{E}$are tangent to M, then

i) $\overset{*}{\tilde{K}}(\pi) \geq \overset{*}{K}(\pi)$ along γ.

ii) If $X$is a unit tangent vector field on M which is parallel along γ in M, then the equality case of (i) holds if and only if $X$is parallel along γ in $\tilde{M}$.

REFERENCES

Agashe, N. S. & Chafle, M. R. 1994. On submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection. Tensor (New Series) **55:** 120-130.

Agashe, N. S. & Chafle, M. R. 1992. A semi-symmetric nonmetric connection on a Riemannian manifold. Indian Journal of Pure and Applied Mathematics 23: 399-409.

Chen, B. Y. 1973. Geometry of submanifolds. Pure and Applied Mathematics No. 22. Marcel Dekker, Inc., New York.

Friedmann, A. & Schouten, J. A. 1924. Über die Geometrie der halbsymmetrischen Übertragungen. Mathematische Zeitschrift **21:** 211-223.

Hayden, H. A. 1932. Subspaces of a space with torsion. Proceedings of the London Mathematical Society **34:** 27-50.

Imai, T. 1972$_a$. Notes on semi-symmetric metric connections. Tensor (New Series) **24:** 293- 296.

Imai, T. 1972$_b$. Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection. Tensor (New Series), **23:** 300-306.

Nakao, Z. 1976. Submanifolds of a Riemannian manifold with semisymmetric metric connections. Proceedings American Mathematical Society **54:** 261-266.

Sengupta, J., De, U. C. & Binh, T. Q. 2000. On a type of semi-symmetric non-metric connection on a Riemannian manifold. Indian Journal of Pure and Applied Mathematics **31:** 1659-1670.

Yano, K. 1970. On semi-symmetric metric connection. Revue Roumaine de Mathématiques Pures et Appliquées **15:** 1579-1586.

Submitted : *19/11/2009*
Revised : *13/5/2010*
Accepted : *31/5/2010*

الاتصال غير المتري شبه المتماثل مع عديد من متعددات ريماني

سيهان أوزقور

قسم الرياضيات – كلية الآداب والعلوم – جامعة باليكسير
تركيا

خلاصة

لقد تم دراسة عديد من متعددات الريماني مع الاتصال غير المتري شبه المتماثل. ولقد تم إثبات أن الاتصال المستحث هو أيضاً غير متري شبه متماثل. وكذلك تم الأخذ في الاعتبار كل geodesicness وأقل عدد من متعددات الريماني مع الاتصال غير المتري شبه المتماثل. ولقد تم الحصول على معادلات Gauss وCodazzi وRicci فيما يتعلق في اتصال غير المتري شبه المتماثل. وأيضاً تم الحصول على العلاقة بين الانحناءات لاتصال levi-civita والاتصال غير المتري شبه المتماثل.