Digital Variable Sampling Integral Control of Infinite Dimensional Systems Subject to Input Nonlinearity

Necati Özdemir

***Abstract*—In this technical note we present a novel sampled-data low-gain I-control algorithm for infinite-dimensional systems in the presence of input nonlinearity. The system is assumed to be exponentially stable with invertible steady-state gain. We use an integral controller with fixed integrator gain, chosen on the basis of state gain information and a time varying sampling period determined by the growth bound of the system. We compare this new algorithm with two other algorithms one with fixed gain and sampling period, the other with time-varying gain.**

***Index Terms*—Infinite dimensional systems, input nonlinearities, integral control, robust tracking, steady-state gain matrix, variable sampling.**

I. INTRODUCTION

The design of low-gain integral (I) and proportional-plus-integral (PI) controllers for uncertain stable plants has been studied extensively during the last 30 years. More recently there has been considerable interest in low-gain integral control for infinite-dimensional systems.

The following principle of low-gain integral control is well known: Closing the loop around a stable, finite-dimensional, continuous-time, single-input, single-output plant, with transfer function $\mathbf{G}(s)$, pre-compensated by an integral controller $C(s) = k/s$ (see Fig. 1) leads to a stable closed-loop system which achieves asymptotic tracking of constant reference signals, provided that $|k|$ is sufficiently small and $k\mathbf{G}(0) > 0$.

One of the main issues in the design of low-gain controllers is the tuning of the integrator gain k. There have been two basic approaches to the tuning problem; either steady-state data from the plant is used *off-line* to determine suitable ranges for the gain k, or else simple *on-line* adaptive tuning of k is used. See Cook [2] and Miller and Davison [12], [13] for the results in the finite dimensional case and Logemann and Townley [9]–[11] in the infinite dimensional case.

Of particular relevance here are the results on integral control in the presence of input-nonlinearities see, Özdemir and Townley [14], [16], Logemann and Ryan [5], [6], and in the presence of actuator and in the sensor nonlinearities, see Coughlan and Logemann [3]. Note that no matter what the context is, it is necessary, in achieving tracking of constant reference signal, that $\mathbf{G}(0)$ is invertible.

There are two important issues in the literature:

- Choice of parameters, in particular the integrator gain k and, in a sampled-data context, the sampling period τ.
- Robustness with respect to parametric uncertainty (e.g. in estimating the steady-state gain) and input nonlinearity.

Inspired, to some extent, by the following result due to Åström [1], we adopt an alternative approach.

Manuscript received April 10, 2008; revised August 20, 2008, December 31, 2008, January 14, 2009, and January 15, 2009. First published May 27, 2009; current version published June 10, 2009. Recommended by Associate Editor D. Dochain.

The author is with the Department of Mathematics, Faculty of Science and Arts, Balikesir University, Balikesir, Turkey (e-mail: nozdemir@balikesir.edu.tr).

Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org

Digital Object Identifier 10.1109/TAC.2009.2015555

Proposition 1.1: (Åström [1]): Let a stable single-input, single-output system has a continuous and monotone increasing step-response $t \mapsto H(t)$. Choose a fixed sampling period τ so that

$$2H(\tau) > \mathbf{G}(0)$$

and a fixed integrator gain k so that

$$k\mathbf{G}(0) < 2.$$

Then the sampled-data integral controller, with current error integrator,

$$u(t) = u_n \quad \text{for } t \in [n\tau, (n+1)\tau)$$
$$u_{n+1} = u_n + k\left(r - y\left((n+1)\tau\right)\right)$$

achieves tracking of arbitrary constants r.

Remark 1.2: Proposition 1.1 can be proved by following the lines of the proof of [1, Theorem 1] readily, since the later is mainly based on Wiener's theorem, which gives a characterization of the invertibility of a transfer function as the z-transform of an absolutely summable sequence, i.e. in the algebra l^1.

Proposition 1.1 is appealing in the sense that the parameters τ and k are determined from available knowledge of the open-loop system, in this case knowledge of the system step-response.

We continue with the flavor of Proposition 1.1, but with the added features that there is input nonlinearity and we do not need to assume that the system has a monotone step-response. We find that k and the growth of a variable sampling period τ_n can be determined from knowledge of the steady-state gain and the decay rate of the system. The rest of the technical note is organized as follows: In Section II, we consider system (1) with divergent sampling period τ_n. This allows us to study first the stability of a much simpler system. The main result Theorem 2.2 is given and its effectiveness is shown in Example 2.6. In Section III, conclusions are given.

II. ROBUSTNESS TO INPUT NONLINEARITY

In [17], Özdemir and Townley considered robustness in the choice of k with respect to uncertainty in experimental measurement of the steady-state gain. Another common source of uncertainty in low-gain integral control is input saturation or more generally input nonlinearity. Low-gain integral control for infinite dimensional systems in the presence of input nonlinearity has been studied by Logemann, Ryan and Townley [7] (continuous time), Logemann and Ryan [5] (continuous time, adaptive), Logemann and Mawby [4] (continuous time, hysteresis nonlinearity) and Özdemir [15].

We consider sampled-data low-gain I-control with input nonlinearity and in particular the robustness of the design of k with respect to such input nonlinearity. Our basic system is given by

$$\dot{x}(t) = Ax(t) + B\Phi(u_n), \qquad x(0) \in X \tag{1a}$$
$$y(t) = Cx(t) \tag{1b}$$

here $\Phi(u_n)$ represents input nonlinearity and $u(t)$ is given by

$$u(t) = u_n \quad \text{for } t \in [t_n, t_{n+1}) \text{ with} \tag{2a}$$
$$u_{n+1} = u_n + ke(t), \quad e(t) = r - y(t_n) \tag{2b}$$
$$t_{n+1} = t_n + \tau_n \quad \text{with } t_0 = 0 \tag{2c}$$

where $y(t_n) = Cx(t_n)$ is the sampled output at the sampling time t_n. Typically $t_n = n\tau$ such that τ is a fixed sampling period and τ_n is an adaptive sampling period.

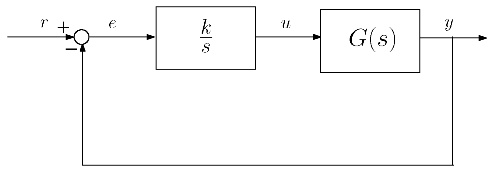

Fig. 1. Low-gain integral control.

In (1), X is a Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, A is the generator of an exponentially stable, strongly continuous (C_0) semi group $T(t)$, $t \geq 0$ on X thus, in particular, there exists $M \geq 1$ and $\omega > 0$ so that $\|T(t)x\| \leq Me^{-\omega t}\|x\|$ for all x. The input operator B is potentially unbounded but we assume $A^{-1}B$ is bounded, i.e. $B \in \mathcal{L}(\mathbb{R}, X_{-1})$ (where X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|A^{-1}x\|_X$) while we assume that the output operator C is bounded so that $C \in \mathcal{L}(X, \mathbb{R})$. It is necessary for tracking constant reference signals, we assume the invertibility of the steady-state gain

$$\mathbf{G}(0) := -CA^{-1}B \in \mathbb{R}.$$

Remark 2.1:

(a) The class of systems encompassed by (1) is large. Note that because we use piecewise-constant inputs arising from sampled-data control, well-posedness of the open-/closed-loop control system does not involve difficulty to check admissibility type assumptions. We only need $A^{-1}B$ to be bounded, see [18, Section 12.1]. In addition, we need C to be bounded because the output $y(\cdot)$, which is sampled directly, needs to be continuous. If C was not bounded, then usually the free output $y(\cdot)$ would not be continuous so that sampling would require pre-filters.

(b) We emphasize that while our results are valid for a very large class of infinite dimensional systems, they are new even in the finite dimensional case.

The closed-loop system is depicted in the block diagram given in Fig. 2, wherein S_τ is the sampling operator, $y(\cdot) \mapsto y(t_n)$, and H_τ is the hold operator, $u_n \mapsto u(t_{n+1}+\cdot)$, which is certainly not the same as (2a). After sampling the closed-loop system becomes

$$x_{n+1} = T(\tau_n)x_n + (T(\tau_n) - I)\,A^{-1}B\Phi(u_n) \tag{3a}$$
$$u_{n+1} = u_n + k(r - Cx_n) \tag{3b}$$

where $x_n = x(t_n)$. Here $k > 0$ is the scalar integrator gain. The operator $T(\tau_n)$ is $T(t)$ sampled at $t = \tau_n$.

We assume throughout this section that there exists v_r such that $\Phi(v_r) = \Phi_r$ where $\Phi_r = [\mathbf{G}(0)]^{-1}r \in clos(im\Phi)$ and $r \in \mathbb{R}$. We apply change of coordinates

$$z_n = x_n - x_r, \quad v_n = u_n - v_r$$

where $x_r = -A^{-1}B\Phi_r$ and $\Psi(v) = \Phi(v + v_r) - \Phi_r$ as in Özdemir and Townley [17], then (3) becomes

$$z_{n+1} = T(\tau_n)z_n + (T(\tau_n) - I)\,A^{-1}B\Psi(v_n) \tag{4a}$$
$$v_{n+1} = v_n - kCz_n. \tag{4b}$$

We use available step-response data. We consider an integral controller with fixed integrator gain, time-varying sampling and input nonlinearities. The main result of this note is the following theorem.

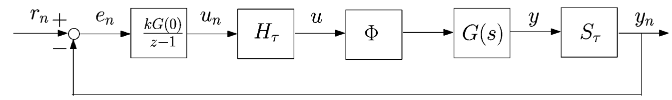

Fig. 2. Sampled data low-gain I-control with input nonlinearity.

Theorem 2.2: Consider the infinite dimensional system defined by (1a) and (1b) with input nonlinearity Φ, where Φ is monotone nondecreasing and globally Lipschitz, with Lipschitz constant λ. Define the control input by (2a) and (2b). If

$$k\lambda\mathbf{G}(0) \in \left(0, \frac{2}{3}\right) \quad \text{and} \quad \tau_n \geq \beta\log(n+2) \tag{5}$$

with $\beta w > 1$, where $w > 0$ is such that $\|T(t)\| \leq Me^{-wt}$, then

$$\begin{aligned}
&(i) \quad \lim_{n\to\infty}\|x_n - x_r\| = 0\\
&(ii) \ \lim_{t\to\infty}\Phi(u(t)) = \Phi_r := [\mathbf{G}(0)]^{-1}r\\
&(iii) \lim_{t\to\infty}x(t) = x_r := -A^{-1}B\Phi_r\\
&(iv) \ \lim_{t\to\infty}y(t) = r.
\end{aligned}$$

Lemma 2.3: Define

$$W_n := \alpha\|z_n\|^2 + k^2\|Cz_n\|^2 + (v_n - kCz_n)^2.$$

If $\tau_n = \beta\log(n+2)$, then there exists $\alpha > 0$ small enough, $\tilde{M} > 1$ and $N \in \mathbb{N}$, so that $\forall n \geq N$

$$W_{n+1} - W_n \leq -\frac{\alpha}{2}\|z_n\|^2 - \frac{\varepsilon}{4}\Psi^2(v_n) + \tilde{M}e^{-w\tau_n}W_n \tag{6}$$

where $\varepsilon := (2k/\lambda)\mathbf{G}(0) - 3k^2\mathbf{G}(0)^2 > 0$. The proof of Lemma 2.3 is given in Appendix.

Remark 2.4: In [6], discrete-time integral controllers are developed for systems with input nonlinearities using time-varying integrator gains satisfying

$$\lim_{n\to\infty} k_n = 0.$$

This time-varying gain result could be applied to continuous time systems with sampled-inputs. In this result, the convergence of k_n is not linked to the decay rate of the system. In [8], explicit bounds are obtained for a fixed integrator gain in a continuous-time integral controller so that tracking is guaranteed. However, such explicit bound-like results only apply to systems whose step responses have no overshoot. To some extent our result, motivated by Proposition 1.1, is a hybrid of these two cases—k converging to 0 is replaced by τ diverging to ∞, but slowly as determined by the decay rate of the system, and the gain is obtained from steady-state information but we do not need a step-response without overshoot.

Proof of Theorem 2.2: Using (6) we have

$$W_{n+1} - W_n \leq \tilde{M}e^{-w\tau_n}W_n, \quad \text{for all } n \geq N$$

along solutions of (4). Hence

$$\begin{aligned}
0 \leq W_{n+1} &\leq (1 + \tilde{M}e^{-w\tau_n})W_n \quad \text{for } n \geq N\\
&\leq \prod_{i=N}^{n}(1 + \tilde{M}e^{-w\tau_i})W_N.
\end{aligned}$$

Taking logarithms gives

$$\log W_{n+1} \leq \sum_{i=N}^{n} \log(1+\tilde{M}e^{-w\tau_i}) + \log W_N \leq \sum_{i=N}^{n} \tilde{M}e^{-w\tau_i} + \log W_N.$$

Now $\tau_n \geq \beta \log(n+2)$ so that $e^{-w\tau_n} \leq 1/(n+2)^{\beta w}$. But $\beta w > 1$. Therefore

$$\log W_{n+1} \leq \sum_{i=N}^{\infty} \frac{\tilde{M}}{(n+2)^{\beta w}} + \log W_N < \infty.$$

Hence $\{W_n\} \in \ell_\infty$ and

$$|W_{n+1} - W_n| \leq \frac{\bar{M}}{(n+2)^{\beta w}} \quad \text{for some } \bar{M}>0 \text{ and all } n \geq N.$$

From these we get $W_n \longrightarrow W_\infty < \infty$ as $n \longrightarrow \infty$. So

$$\frac{\epsilon}{4}\sum_{n=N}^{\infty}\Psi^2(v_n) \leq \sum_{n=N}^{\infty}(W_n - W_{n+1}) + \tilde{M}\|W_k\|_\infty \sum_{n=N}^{\infty} e^{-w\tau_n} = (W_N - W_\infty) + \tilde{M}\|W_k\|_\infty \sum_{n=N}^{\infty} e^{-w\tau_n} < \infty.$$

This shows that $\Psi(v_n) \in \ell^2$. So $\lim_{n\to\infty}\Phi(u_n) = \Phi_r$ proving (ii). Now

$$\|(x_{n+1} - x_r)\| = \left\|T(\tau_n)(x_n - x_r) + (T(\tau_n) - I)A^{-1}B\Psi(v_n)\right\| \leq \frac{1}{2}\|x_n - x_r\| + \frac{3}{2}\|A^{-1}B\|\,\|\Psi(v_n)\|$$

for all $n \geq N_1$ where $N_1 \in \mathbb{N}_0$ is such that

$$\frac{M}{(N_1+2)^{\beta w}} \leq \frac{1}{2}.$$

So

$$\lim_{n\to\infty}(x_n - x_r) = 0$$

i.e (i) holds. Finally from

$$(x(t) - x_r) = T(t-t_n)(x_n - x_r) + (T(t-t_n) - I)A^{-1}B(\Phi(u_n) - \Phi_r)$$

we have

$$\|x(t) - x_r\| \leq \|T(t-t_n)\|\,\|x_n - x_r\| + \|(T(t-t_n) - I)\|\,\|A^{-1}B\|\,\|\Phi(u_n) - \Phi_r\|$$

so that (iii) holds. It is obvious that (iv) is provided from (iii).

Remark 2.5: Theorem 2.2 is rather satisfying in that it applies if just two conditions are satisfied, namely $k\lambda\mathbf{G}(0) < 2/3$ and $\tau_n \geq \beta\log(n+2)$. These conditions involve two crucial system constants: The steady-state gain $\mathbf{G}(0)$ and the growth bound w.

Example 2.6: Consider a diffusion process (with diffusion coefficient $a > 0$ and Dirichlet boundary conditions), on the one-dimensional spatial domain [0, 1] with point actuation and sensing. This leads to the following controlled partial differential equation:

$$z_t(t,x) = az_{xx}(t,x) + \delta(x - x_b)\Phi(u(t))$$
$$y(t) = z(t, x_c)$$

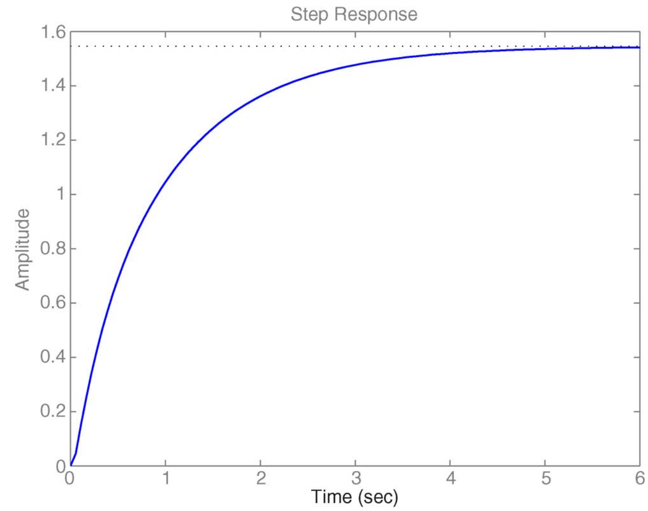

Fig. 3. Step response.

with boundary conditions

$$z(t,0) = 0 = z(t,1) \quad \text{for all } t > 0$$

and zero initial conditions

$$z(0,x) = 0,\ x \in [0,1].$$

Furthermore, we assume a nonlinearity Φ of saturation type, defined as follows:

$$u \mapsto \Phi(u(t)) := \begin{cases} 0, & u \leq 0 \\ u, & u \in (0,1) \\ 1, & u \geq 1. \end{cases}$$

In this case the transfer function is

$$\mathbf{G}(s) = \frac{sinh\left(x_b\sqrt{\frac{s}{a}}\right) sinh\left((1-x_c)\sqrt{\frac{s}{a}}\right)}{a\sqrt{\frac{s}{a}}sinh\sqrt{\frac{s}{a}}}. \quad (7)$$

With this input $u(t) \in \mathbb{R}$ and outputs $y(t) \in \mathbb{R}$, this diffusion system can be represented in the form of (1). Indeed, we have that $X = L^2(0,1)$, A has eigenvalues $-an^2\pi^2$, and eigenfunctions $sin(n\pi x)$, $n = 1, 2, 3 \ldots$. So we can write:

$$A = diag(-an^2\pi^2),\ B = (b_1, b_2, \ldots)$$
$$b_k = \frac{sin(k\pi x_b)}{\int_0^1 sin^2(k\pi x)dx} \quad (8)$$
$$C = (c_1, c_2, \ldots),\ c_k = sin(k\pi x_c),\ \text{where } k = 1, 2, \ldots \quad (9)$$

Then the boundedness of $A^{-1}B$ is equivalent to

$$\sum_{n=1}^{\infty}\frac{b_n^2}{a^2n^4\pi^4} < \infty \quad (10)$$

which holds. Furthermore, from (7), we have

$$\mathbf{G}(0) = \frac{1}{a}(x_b(1-x_c))$$

so that $x_b \neq 0$ and $x_c \neq 1$. For purposes of illustration, we adopt the following values:

$$a = 0.1, \quad x_b = \frac{1}{4}, \quad x_c = \frac{3}{8}.$$

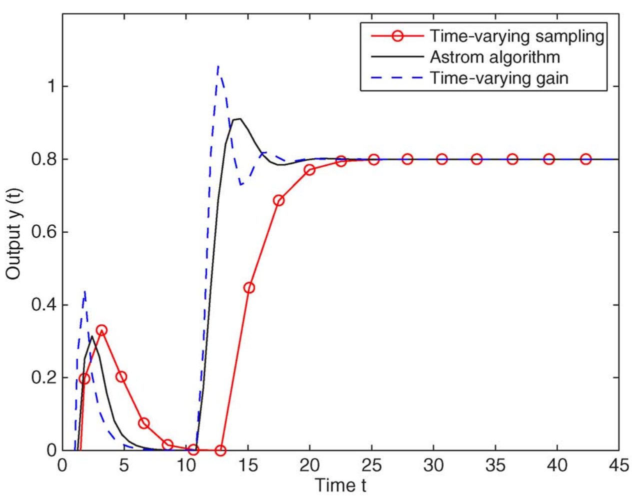

Fig. 4. Output $y(t_n)$, against t_n, for the system described in Example 2.6. *Our algorithm is time-varying sampling*.

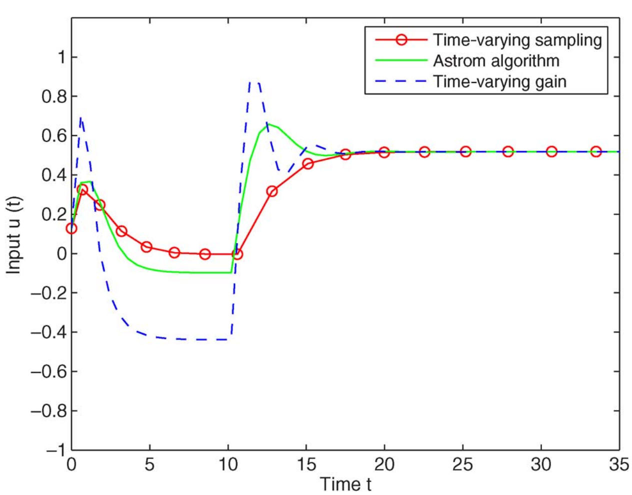

Fig. 5. Input u_n applied, via idealized hold to the system described in Example 2.6. *Our algorithm is time-varying sampling*.

This gives $w = 0.1\pi^2$ and $\mathbf{G}(0) = 1.5625$. For $\lambda = 1$, we choose the gain k so that $k\mathbf{G}(0) < 2/3$. One choice is $k = 0.4$. In the simulations, we assume a step-reference

$$r(t) := \begin{cases} 0, & t < 10 \\ 0.8, & t \geq 10 \end{cases}$$

and a variable sampling period $\tau_n = \beta \log(n+2)$, i.e. $\beta = 1.014$. To compare the results with the other algorithms, we choose the time-varying gain $k = 1/n$, $(n > 0)$.

In producing the simulations we use MATLAB 6.5 and a truncated eigenfunction expansion of order 10, adopted to model the diffusion process. Fig. 3 shows step-response which is non-decreasing. So, Proposition 1.1 can be applied. In Fig. 4, we show the plot of three different outputs and in Fig. 5 the corresponding three inputs. Fig. 6 depicts three controllers against to time t.

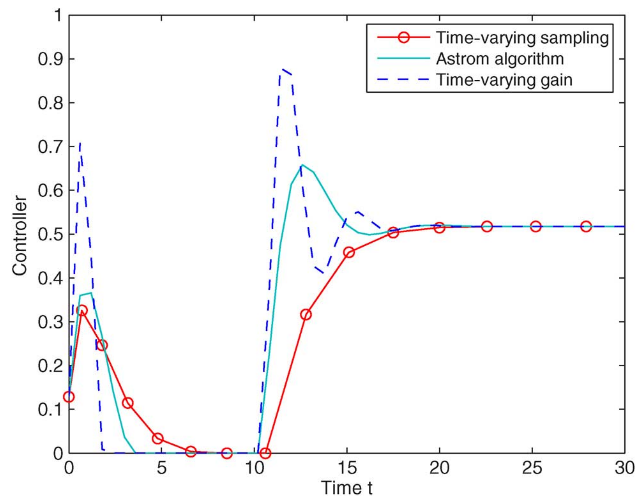

Fig. 6. Controller against t_n for the system described in Example 2.6. *Our algorithm is time-varying sampling*.

To apply Åström's controller we need to choose τ so that $2H(\tau) > \mathbf{G}(0)$, where $H(\tau)$ is the step-response of the system, and k so that $k\mathbf{G}(0) < 2$.

It is clearly seen from Fig. 4 and Fig. 5 that time-varying sampling algorithm is smoother, has no overshoot and converges at the same time with the others.

While theoretically our controller has a slowly diverging sampling period, closed-loop performance compared favorably to controllers which work with similar minimal system information.

III. Conclusion

We considered sampled-data integral control law for exponentially stable infinite dimensional linear systems subject to monotone non-decreasing and globally Lipschitz input nonlinearity with Lipschitz constant λ. Motivated by existing results in the literature, we focused on the important aspects: steady state data, adaptation of control parameters and input nonlinearity.

The key idea was the use of the sampling period as a "control" parameter. We used an integral controller with fixed integrator gain, chosen on the basis of state gain information and a time-varying sampling period determined by the growth bound of the system. We compared this new algorithm with two other algorithms; one with fixed gain and sampling period, and the other with time-varying gain. It can be concluded from the simulation results that time-varying sampling is smoother, has no overshoot and converges at the same time with the other algorithms.

It would be interesting to investigate whether we can use more steady-state information, such as frequency response information that can be obtained experimentally. This might be helpful in problems such as non-constant disturbance rejection. An analogous open problem can be extended to multi-input, multi-output case.

Appendix

We compute $W_{n+1} - W_n$ along solutions of (4a) and (4b) with large adaptive sampling "$\tau_n = \infty$"

$$\begin{aligned} W_{n+1} - W_n = {} & \alpha\|z_{n+1}\|^2 + k^2\|Cz_{n+1}\|^2 + (v_{n+1} - kCz_{n+1})^2 \\ & - \alpha\|z_n\|^2 - k^2\|Cz_n\|^2 - (v_n - kCz_n)^2. \end{aligned}$$

For brevity set $\Psi(v_n) = \tilde{\Psi}$, then

$$\begin{aligned} W_{n+1} - W_n =& \alpha \left\| T(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) - A^{-1}B\tilde{\Psi} \right\|^2 \\ &+ k^2 \left\| CT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) + G(0)\tilde{\Psi} \right\|^2 \\ &+ \left[v_n - kCz_n - kCT(\tau_n) \right. \\ &\left. \times (z_n + A^{-1}B\tilde{\Psi}) - kG(0)\tilde{\Psi} \right]^2 \\ &- \alpha\|z_n\|^2 - k^2\|Cz_n\|^2 - (v_n - kCz_n)^2. \end{aligned}$$

So

$$\begin{aligned} W_{n+1} - W_n =& \alpha \left\| T(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) - A^{-1}B\tilde{\Psi} \right\|^2 \\ &+ 2k^2 \left\| CT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) + G(0)\tilde{\Psi} \right\|^2 \\ &- 2(v_n - kCz_n) \\ &\times \left(kCT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) + kG(0)\tilde{\Psi} \right) \\ &- \alpha\|z_n\|^2 - k^2\|Cz_n\|^2 \\ W_{n+1} - W_n =& I + II + III + IV \end{aligned}$$

where

$$\begin{aligned} I =& \alpha \left\| T(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) - A^{-1}B\tilde{\Psi} \right\|^2 \\ II =& 2k^2 \left\| CT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) + \mathbf{G}(0)\tilde{\Psi} \right\|^2 \\ III =& -2(v_n - kCz_n) \\ &\times \left(kCT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) + k\mathbf{G}(0)\tilde{\Psi} \right) \\ IV =& -\alpha\|z_n\|^2 - k^2\|Cz_n\|^2. \end{aligned}$$

Now consider

$$\begin{aligned} I \leq& 2\alpha\|A^{-1}B\tilde{\Psi}\|^2 + 4\alpha M^2 e^{-2w\tau_n}\|A^{-1}B\tilde{\Psi}\|^2 \\ &+ 4\alpha M^2 e^{-2w\tau_n}\|z_n\|^2 \end{aligned}$$

and

$$\begin{aligned} II \leq& 2k^2 \left\| \mathbf{G}(0)\tilde{\Psi} \right\|^2 + 2k^2\|C\|^2 M^2 e^{-2w\tau_n} \\ &\times \left[2\|A^{-1}B\tilde{\Psi}\|^2 + 2\|z_n\|^2 \right] + 2k^2\|C\|Me^{-w\tau_n} \\ &\times \left[2\left\| \mathbf{G}(0)\tilde{\Psi} \right\|^2 + \|z_n\|^2 + \|A^{-1}B\tilde{\Psi}^2\|^2 \right]. \end{aligned}$$

Finally

$$\begin{aligned} III =& -2(v_n - kCz_n)k\mathbf{G}(0)\tilde{\Psi} \\ &-2(v_n - kCz_n)\left(kCT(\tau_n)(z_n + A^{-1}B\tilde{\Psi}) \right) \end{aligned}$$

so that

$$\begin{aligned} III \leq& -2(v_n - kCz_n)k\mathbf{G}(0)\tilde{\Psi} + Me^{-w\tau_n}|v_n - kCz_n|^2 \\ &+ 2k^2 M\|C\|^2 e^{-w\tau_n}\left[\|z_n\|^2 + \|A^{-1}B\tilde{\Psi}\|^2 \right]. \end{aligned}$$

Now, we obtain

$$\begin{aligned} &2k^2 \left\| \mathbf{G}(0)\tilde{\Psi} \right\|^2 - 2(v_n - kCz_n)k\mathbf{G}(0)\tilde{\Psi} - k^2\|Cz_n\|^2 \\ &= 3k^2\mathbf{G}(0)^2\tilde{\Psi}^2 - 2kv_n\mathbf{G}(0)\tilde{\Psi} - k^2\left| Cz_n - \mathbf{G}(0)\tilde{\Psi} \right|^2. \quad (11) \end{aligned}$$

Using the estimates for II, III and IV obtained above and the boundedness of the operator $A^{-1}B$, we can find $\tilde{M} > 0$ so that

$$\begin{aligned} W_{n+1} - W_n \leq& -\alpha(1 - Me^{-w\tau_n})\|z_n\|^2 + \tilde{M}(\alpha + e^{-w\tau_n})\tilde{\Psi}^2 \\ &+ \tilde{M}e^{-w\tau_n}|v_n - kCz_n|^2 + 3k^2\mathbf{G}(0)^2\tilde{\Psi}^2 \\ &- 2kv_n\mathbf{G}(0)\tilde{\Psi} - k^2\left| Cz_n - \mathbf{G}(0)\tilde{\Psi} \right|^2. \end{aligned}$$

But $v_n\Psi(v_n) \geq (1/\lambda)\Psi^2(v_n)$. Substituting $\Psi(v_n)$ back in for $\tilde{\Psi}$ we obtain

$$\begin{aligned} &3k^2\mathbf{G}(0)^2\Psi(v_n)^2 - 2kv_n\mathbf{G}(0)\Psi(v_n) \\ &\leq -\left(\frac{2k}{\lambda}\mathbf{G}(0) - 3k^2\mathbf{G}(0)^2 \right)\Psi^2(v_n) \\ &= -\varepsilon\Psi^2(v_n). \end{aligned}$$

So

$$\begin{aligned} W_{n+1} - W_n \leq& -\alpha(1 - \tilde{M}e^{-w\tau_n})\|z_n\|^2 \\ &- \left(\varepsilon - \tilde{M}(\alpha + e^{-w\tau_n}) \right)\Psi^2(v_n) + \tilde{M}e^{-w\tau_n}|v_n - kCz_n|^2. \end{aligned}$$

Choosing α small enough so that $\tilde{M}\alpha < \varepsilon/4$, we can find $N \in \mathbb{N}$ large enough so that $\tilde{M}e^{-w\tau_n} < \max(1/2, \varepsilon/4)$ for all $n \geq \mathbb{N}$ and noting that $|v_n - kCz_n|^2 \leq W_n$ gives

$$W_{n+1} - W_n \leq -\frac{\alpha}{2}\|z_n\|^2 - \frac{\varepsilon}{4}\Psi^2(v_n) + \tilde{M}e^{-w\tau_n}W_n$$

as required.

ACKNOWLEDGMENT

The author would like to thank Dr. S. Townley for numerous discussions during the preparation of this research and the reviewers for carefully reading the manuscript and providing useful comments which led to more complete presentation.

REFERENCES

[1] K. J. Åström, "A robust sampled regulator for stable systems with monotone step responses," *Automatica*, vol. 16, pp. 313–315, 1980.

[2] P. A. Cook, "Controllers with universal tracking properties," in *Proc. Int. IMA Conf. Control: Modelling, Computation, Information*, Manchester, U.K., 1992, [CD ROM].

[3] J. J. Coughlan and H. Logemann, "Sampled-data low-gain control of linear systems in the presence of actuator and sensor nonlinearities," in *Proc. 17th Int. Symp. Math. Theory Networks Syst.*, Kyoto, Japan, Jul. 24–28, 2006, pp. 643–645.

[4] H. Logemann and A. D. Mawby, "Low-gain integral control of infinite-dimensional regular linear systems subject to input hysteresis," in *Advances in Mathematical Systems Theory*, F. Colonies, U. Helmke, D. Prätzel-Wolters, and F. Wirth, Eds. Boston, MA: Birkhäuser Verlag, 2000, pp. 255–293.

[5] H. Logemann and E. P. Ryan, "Time-varying and adaptive integral control of infinite-dimensional regular systems with input nonlinearities," *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1120–1144, 2000.

[6] H. Logemann and E. P. Ryan, "Time-varying and adaptive discrete-time control of infinite-dimensional linear systems with input nonlinearities," *Math. Control, Signals, Syst.*, vol. 13, pp. 293–317, 2000.

[7] H. Logemann, E. P. Ryan, and S. Townley, "Integral control of infinite-dimensional linear systems subject to input saturation," *SIAM J. Control Optim.*, vol. 36, no. 6, pp. 1940–1961, 1998.

[8] H. Logemann, E. P. Ryan, and S. Townley, "Integral control of linear systems with actuator nonlinearities: Lower bounds for the maximal regulating gain," *IEEE Trans. Automat. Control*, vol. 44, no. 6, pp. 1315–1319, Jun. 1999.

[9] H. Logemann and S. Townley, "Low-gain control of uncertain regular linear systems," *SIAM J. Control Optim.*, vol. 35, no. 1, pp. 78–116, 1997.

[10] H. Logemann and S. Townley, "Discrete-time low-gain control of uncertain infinite-dimensional systems," *IEEE Trans. Automat. Control*, vol. 42, no. 1, pp. 22–37, Jan. 1997.

[11] H. Logemann and S. Townley, "Adaptive low-gain integral control of multi variable well-posed linear systems," *SIAM J. Control Optim.*, vol. 41, no. 6, pp. 1722–1732, 2002.

[12] D. E. Miller and E. J. Davison, "The self-tuning robust servomechanism problem," *IEEE Trans. Automat. Control*, vol. AC-34, no. 5, pp. 511–523, May 1989.

[13] D. E. Miller and E. J. Davison, "An adaptive tracking problem with a control input constraint," *Automatica*, vol. 29, no. 4, pp. 877–887, 1993.

[14] N. Özdemir and S. Townley, "Adaptive low-gain control of infinite dimensional systems by means of sampling time adaptation," in *Proc. 5th Int. Symp. Methods Models Automat. Robot.*, Miedzyzdroje, Poland, 1998, pp. 63–68.

[15] N. Özdemir, "Robust and Adaptive Sampled Data I-Control," Ph.D. dissertation, Dept. Mathematics, University of Exeter, Exeter, U.K., 2000.

[16] N. Özdemir and S. Townley, "Variable sampling integral control of infinite dimensional systems," in *Proc. 39th IEEE Conf. Decision Control*, Sydney, Australia, 2000, pp. 3284–3287.

[17] N. Özdemir and S. Townley, "Integral control by variable sampling based on steady-state data," *Automatica*, vol. 39, pp. 135–140, 2003.

[18] O. J. Staffans, *Well-Posed Linear Systems*. Cambridge, U.K.: Cambridge University Press, 2005.

Compensating a String PDE in the Actuation or Sensing Path of an Unstable ODE

Miroslav Krstic

***Abstract*—How to control an unstable linear system with a long pure delay in the actuator path? This question was resolved using 'predictor' or 'finite spectrum assignment' designs in the 1970s. Here we address a more challenging question: How to control an unstable linear system with a wave partial differential equation (PDE) in the actuation path? Physically one can think of this problem as having to stabilize a system to whose input one has access through a string. The challenges of overcoming string/wave dynamics in the actuation path include their infinite dimension, finite propagation speed of the control signal, and the fact that all of their (infinitely many) eigenvalues are on the imaginary axis. In this technical note we provide an explicit feedback law that compensates the wave PDE dynamics at the input of an linear time-invariant ordinary differential equation and stabilizes the overall system. In addition, we prove robustness of the feedback to the error in *a priori* knowledge of the propagation speed in the wave PDE. Finally, we consider a dual problem where the wave PDE is in the sensing path and design an exponentially convergent observer.**

***Index Terms*—Linear time-invariant (LTI), ordinary differential equation (ODE), partial differential equation (PDE).**

I. Introduction

The "Smith predictor" and its extensions developed since the 1970s [1], [3]–[9], [13]–[19], [21]–[31] are important tools in several application areas. They allow to compensate a pure delay of arbitrary length in either the actuation or sensing path of a linear system, even when the system is unstable. Several results in adaptive control for unknown ODE parameters have been published [2], [20]. Extensions to nonlinear systems are also beginning to emerge [10].

In [11] we presented a first attempt of compensating infinite-dimensional actuator dynamics of more complex type than pure delay. We presented a design for diffusion-dominated partial differential equation (PDE) dynamics (such as the heat equation). While these dynamics do not have a finite speed of propagation, they are 'low-pass' and "phase-lag" to the extreme, as they have infinitely many (stable) poles and no zeros.

In this technical note we tackle a problem from a different class of PDE dynamics in the actuation or sensing path—the wave/string equation. The wave equation is challenging due to the fact that all of its (infinitely many) eigenvalues are on the imaginary axis, and due to the fact that it has a finite (limited) speed of propagation (large control doesn't help).

The problem studied here is more challenging than in [11] due to another difficulty—the PDE system is second order in time, which means that the state is 'doubly infinite dimensional' (distributed displacement and distributed velocity). This is not so much of a problem dimensionally, as it is a problem in constructing the state transformations for compensating the PDE dynamics. One has to deal with the coupling of two infinite-dimensional states.

Manuscript received September 10, 2008; revised November 24, 2008, January 20, 2009, and January 21, 2009. First published May 27, 2009; current version published June 10, 2009. This work was supported by the National Science Foundation (NSF) and Bosch. Recommended by Associate Editor R. D. Braatz.

The author is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).

Digital Object Identifier 10.1109/TAC.2009.2015557