ON INVARIANT SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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الخلاصة:

نعتبر المانيفولدات الجزئية نصف المتوازية و 2–نصف المتوازية الثابتة لمانيفولدات Lorentzian para-Sasakian . سنثبت أنّ هذه المانيفولدات الجزئية متقاصرة كلياً. كما سنعتبر المانيفولدات الجزئية الثابتة من مانيفولدات Lorentzian para-Sasakian التي تحقق $Z(X,Y)\cdot\alpha=0$ و $Z(X,Y)\cdot\overline{\nabla}\alpha=0$ و $\tau\neq n(n-1)$. تحت هذه الشروط، نثبت أنّ هذه المانيفولدات الجزئية متقاصرة كلياً.

ABSTRACT

We consider semiparallel and 2-semiparallel invariant submanifolds of Lorentzian para-Sasakian manifolds. We show that these submanifolds are totally geodesic. We also consider invariant submanifolds of Lorentzian para-Sasakian manifolds satisfying the conditions $\mathcal{Z}(X,Y)\cdot\alpha=0$ and $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha=0$ with $\tau\neq n(n-1)$. Under these conditions, we prove that the submanifolds are totally geodesic.

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Key words: Lorentzian paracontact manifold, Lorentzian para-Sasakian manifold, semiparallel submanifold, 2-semiparallel submanifold.

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1. INTRODUCTION

Let M and $\tilde{M}$ be two Riemannian or semi-Riemannian manifolds, $f : M \longrightarrow \tilde{M}$ an isometric immersion, α the second fundamental form and $\overline{\nabla}$ the van der Waerden-Bortolotti connection of M. An immersion is said to be *semiparallel* if

$$\overline{R}(X,Y)\cdot\alpha = (\overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]})\alpha = 0 \tag{1}$$

holds for all vector fields X, Y tangent to M [1], where $\overline{R}$ denotes the curvature tensor of the connection $\overline{\nabla}$. Semiparallel immersions have been studied by various authors. See, for example, [2], [3], [4], [5] and [6].

In [7], K. Arslan, Ü. Lumiste and the present authors defined and studied submanifolds satisfying the condition

$$\overline{R}(X,Y)\cdot\overline{\nabla}\alpha = 0 \tag{2}$$

for all vector fields X, Y tangent to M. Submanifolds satisfying the condition (2) are called *2-semiparallel.*

Motivated by the studies of the above authors, in this study, we consider semiparallel and 2-semiparallel invariant submanifolds of Lorentzian para-Sasakian (briefly LP-Sasakian) manifolds. We also consider the conditions $\mathcal{Z}(X,Y)\cdot\alpha = 0$ and $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$ on an invariant submanifold of a Lorentzian para-Sasakian manifold, where $\mathcal{Z}$ denotes the concircular curvature tensor of the submanifold.

The paper is organized as follows: In Section 2, we give necessary details about submanifolds and the concircular curvature tensor. In Section 3, we give a brief account of Lorentzian para-Sasakian manifolds and their invariant submanifolds. In Section 4, we study semiparallel and 2-semiparallel invariant submanifolds of LP-Sasakian manifolds. We show that these type submanifolds are totally geodesic. In Section 5, we prove that for an n-dimensional invariant submanifold M of an LP-Sasakian manifold $\tilde{M}$ such that the scalar curvature $\tau \neq n(n-1)$, the conditions $\mathcal{Z}(X,Y)\cdot\alpha = 0$ and $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$ imply that M^n is totally geodesic.

2. BASIC CONCEPTS

Let (M, g) be an n-dimensional semi-Riemannian submanifold of an $(n+d)$-dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. We denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections of $\widetilde{M}$ and M, respectively. Then we have the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y), \tag{3}$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

where X, Y are vector fields tangent to M and N is a normal vector field on M, respectively. $\nabla^{\perp}$ is called *the normal connection* of M. We call α the *second fundamental form* of the submanifold M. If $\alpha = 0$ then the manifold is said to be *totally geodesic*. For the second fundamental form α, the covariant derivative of α is defined by

$$(\overline{\nabla}_X\alpha)(Y,Z) = \nabla^{\perp}_X(\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) \tag{4}$$

for any vector fields X, Y, Z tangent to M. Then $\overline{\nabla}\alpha$ is a normal bundle valued tensor of type (0,3) and is called the *third fundamental form* of M. $\overline{\nabla}$ is called the *van der Waerden-Bortolotti connection* of M, i.e., $\overline{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$. If $\overline{\nabla}\alpha = 0$, then M is said to have *parallel second fundamental form* [8]. From the Gauss and Weingarten formulas we obtain

$$\left(\widetilde{R}(X,Y)Z\right)^T = R(X,Y)Z + A_{\alpha(X,Z)}Y - A_{\alpha(Y,Z)}X. \tag{5}$$

By (1), we have

$$\left(\overline{R}(X,Y)\cdot\alpha\right)(U,V) = R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) \tag{6}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = \left[\nabla^{\perp}_X, \nabla^{\perp}_Y\right] - \nabla^{\perp}_{[X,Y]}$$

and $\overline{R}$ denotes the curvature tensor of $\overline{\nabla}$. Similarly, we have

$$\left(\overline{R}(X,Y)\cdot\overline{\nabla}\alpha\right)(U,V,W) = R^{\perp}(X,Y)(\overline{\nabla}\alpha)(U,V,W)$$
$$-(\overline{\nabla}\alpha)(R(X,Y)U,V,W) - (\overline{\nabla}\alpha)(U,R(X,Y)V,W) - (\overline{\nabla}\alpha)(U,V,R(X,Y)W) \quad (7)$$

for vector fields X, Y, U, V, W tangent to M, where $(\overline{\nabla}\alpha)(U,V,W)$ means $(\overline{\nabla}_U\alpha)(V,W)$ [7].

For an n-dimensional, $(n \geq 3)$, semi-Riemannian manifold (M^n, g), the *concircular curvature* tensor $\mathcal{Z}$ of M^n is defined by [9]

$$\mathcal{Z}(X,Y)V = R(X,Y)V - \frac{\tau}{n\,(n-1)}\left\{g(Y,V)X - g(X,V)Y\right\} \quad (8)$$

for vector fields X, Y and V on M^n, where τ is the scalar curvature of M^n. We observe immediately from the form of the concircular curvature tensor that semi-Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of the failure of a semi-Riemannian manifold to be of constant curvature.

Similar to (6) and (7) the tensors $\mathcal{Z}(X,Y)\cdot\alpha$ and $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha$ are defined by

$$\left(\mathcal{Z}(X,Y)\cdot\alpha\right)(U,V) = R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V) \quad (9)$$

and

$$\begin{aligned}\left(\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha\right)(U,V,W) &= R^{\perp}(X,Y)(\overline{\nabla}\alpha)(U,V,W) - (\overline{\nabla}\alpha)(\mathcal{Z}(X,Y)U,V,W) \\ &\quad -(\overline{\nabla}\alpha)(U,\mathcal{Z}(X,Y)V,W) - (\overline{\nabla}\alpha)(U,V,\mathcal{Z}(X,Y)W),\end{aligned} \quad (10)$$

respectively.

3. LORENTZIAN PARA-SASAKIAN MANIFOLDS

In this section, we give necessary details about Lorentzian para-Sasakian manifolds and their invariant submanifolds.

The notion of a Lorentzian para-Sasakian manifold was introduced by K. Matsumoto [10].

An n-dimensional differentiable manifold M^n is said to admit an *almost paracontact Riemannian structure* (φ, η, ξ, g), where φ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that (see [11])

$$\varphi^2 = I - \eta\otimes\xi, \quad \eta(\xi) = 1, \quad (11)$$
$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \quad (12)$$

for all vector fields X, Y. On the other hand, M is said to admit a *Lorentzian almost paracontact structure* (φ, η, ξ, g), if φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Lorentzian metric on M^n, which makes ξ a timelike unit vector field such that (see [10])

$$a)\quad \varphi^2 = I + \eta\otimes\xi, \quad b)\quad \eta(\xi) = -1, \quad (13)$$
$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y) \quad (14)$$

for all vector fields X, Y on M^n. For more details we refer to [12], [13], [14] and references cited therein.

For both of the structures mentioned above, it follows that

$$\varphi\xi = 0, \qquad \eta\circ\varphi = 0, \quad (15)$$
$$g(X,\xi) = \eta(X), \quad g(\varphi X, Y) = g(X, \varphi Y) \quad (16)$$

for all vector fields X, Y on M^n.

A Lorentzian almost para-contact manifold is called *Lorentzian para-Sasakian* (briefly, LP-Sasakian) (see [10]) if

$$(\nabla_X\varphi)Y = g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2 X, \quad (17)$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In an *LP*-Sasakian manifold M with the structure (φ, η, ξ, g), it is easily seen that

$$\nabla_X \xi = \varphi X, \tag{18}$$
$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{19}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{20}$$
$$S(X,\xi) = (n-1)\eta(X) \tag{21}$$

for all vector fields X, Y on M^n [10], where S denotes the Ricci tensor of M^n. Moreover, from (8), we also have

$$\mathcal{Z}(\xi, X)Y = \left(1 - \frac{\tau}{n(n-1)}\right)(g(X,Y)\xi - \eta(Y)X) \tag{22}$$

and

$$\mathcal{Z}(\xi, X)\xi = \left(1 - \frac{\tau}{n(n-1)}\right)(\eta(X)\xi + X). \tag{23}$$

Example 3.1. Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining

$$\eta = ds - ydx - tdz, \qquad \xi = \tfrac{\partial}{\partial s},$$
$$g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2,$$
$$\varphi\left(\tfrac{\partial}{\partial x}\right) = -\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial s}, \qquad \varphi\left(\tfrac{\partial}{\partial y}\right) = -\tfrac{\partial}{\partial y},$$
$$\varphi\left(\tfrac{\partial}{\partial z}\right) = -\tfrac{\partial}{\partial z} - t\tfrac{\partial}{\partial s}, \quad \varphi\left(\tfrac{\partial}{\partial t}\right) = -\tfrac{\partial}{\partial t}, \quad \varphi\left(\tfrac{\partial}{\partial s}\right) = 0,$$

the structure (φ, η, ξ, g) becomes an *LP*-Sasakian structure in $\mathbb{R}^5$ [15].

Example 3.2. A Lorentzian unit sphere $S_1^n(1)$ is an Einstein *LP*-Sasakian manifold with scalar curvature $\tau = n(n-1)$.

A submanifold M of an *LP*-Sasakian manifold $\widetilde{M}$ is called an *invariant submanifold* of $\widetilde{M}$ if $\varphi(TM) \subset TM$. In an invariant submanifold of an *LP*-Sasakian manifold

$$\alpha(X, \xi) = 0, \tag{24}$$

for any vector field X tangent to M (see [16] and [17]). Now we give the following proposition:

Proposition 3.3. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then the following equalities hold on M^n.*

$$\nabla_X \xi = \varphi X, \tag{25}$$
$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{26}$$
$$Q\xi = (n-1)\xi, \tag{27}$$
$$S(X,\xi) = (n-1)\eta(X), \tag{28}$$
$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{29}$$
$$\alpha(X, \varphi Y) = \varphi\alpha(X,Y), \tag{30}$$

where Q denotes the Ricci operator of M^n defined by $S(X,Y) = g(QX,Y)$.

Proof. Since M^n is an invariant submanifold of an *LP*-Sasakian manifold $\widetilde{M}$

$$\widetilde{\nabla}_X \xi = \varphi X.$$

Using Gauss formula (3), we get

$$\varphi X = \widetilde{\nabla}_X \xi = \nabla_X \xi + \alpha(X, \xi),$$

which gives us

$$\nabla_X \xi = \varphi X,$$
$$\alpha(X, \xi) = 0,$$

so we get (25). Since $\widetilde{M}$ is *LP*-Sasakian, we get from (17),

$$(\widetilde{\nabla}_X \varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{31}$$

Then, in view of Gauss Formula, we have

$$(\widetilde{\nabla}_X\varphi)Y = \nabla_X\varphi Y + \alpha(X,\varphi Y) - \varphi\nabla_X Y - \varphi\alpha(X,Y). \tag{32}$$

Comparing the tangential and normal parts of (31) and (32), we get

$$(\nabla_X\varphi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$
$$\alpha(X,\varphi Y) = \varphi\alpha(X,Y).$$

So we obtain (29) and (30). From the Gauss equation (5) we have

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi + A_{\alpha(X,\xi)}Y - A_{\alpha(Y,\xi)}X.$$

Then using $\alpha(X,\xi) = 0$ we find

$$\widetilde{R}(X,Y)\xi = R(X,Y)\xi,$$

which, in view of (20), gives (26). A suitable contraction of (26) gives us (27) and (28). □

So we can state the following theorem:

Theorem 3.4. *An invariant submanifold M^n of an LP-Sasakian manifold $\widetilde{M}$ is an LP-Sasakian manifold.*

4. SEMIPARALLEL AND 2-SEMIPARALLEL INVARIANT SUBMANIFOLDS OF *LP*-SASAKIAN MANIFOLDS

In this section, we consider semiparallel and 2-semiparallel invariant submanifolds of *LP*-Sasakian manifolds. Now we prove the following theorem:

Theorem 4.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then M^n is semiparallel if and only if M^n is totally geodesic.*

Proof. Since M is semiparallel $\overline{R}\cdot\alpha = 0$. Then, from (6), we have

$$R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V) = 0. \tag{33}$$

Taking $X = V = \xi$ in (33) we get

$$R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(R(\xi,Y)U,\xi) - \alpha(U,R(\xi,Y)\xi) = 0.$$

So, using (24), the last equation is reduced to

$$\alpha(U,R(\xi,Y)\xi) = 0.$$

Then by the use of (20), we have $\alpha(U,\eta(Y)\xi + Y) = 0$. Hence, in view of (24), we obtain $\alpha(Y,U) = 0$, which gives us M^n is totally geodesic.

The converse statement is trivial. This completes the proof of the theorem. □

Theorem 4.2. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then M^n has parallel second fundamental form if and only if M^n is totally geodesic.*

Proof. Since M^n has parallel second fundamental form, it follows from (4) that

$$(\overline{\nabla}_X\alpha)(Y,Z) = \nabla^{\perp}_X(\alpha(Y,Z)) - \alpha(\nabla_X Y,Z) - \alpha(Y,\nabla_X Z) = 0.$$

So, taking $Z = \xi$ in the above equation and using (24), we get

$$\alpha(Y,\nabla_X\xi) = 0.$$

Hence, in view of (25), we have

$$\alpha(Y,\varphi X) = 0. \tag{34}$$

Replacing X with φX in (34) and using (13)(a) and (24) we get

$$\alpha(X,Y) = 0,$$

whence M^n is totally geodesic.

The converse statement is trivial. Hence our theorem is proved. □

Theorem 4.3. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then M^n is 2-semiparallel if and only if M^n is totally geodesic.*

Proof. Since M^n is 2-semiparallel $\overline{R}\cdot\overline{\nabla}\alpha = 0$. Hence, it follows from (7) that

$$R^{\perp}(X,Y)(\overline{\nabla}\alpha)(U,V,W) - (\overline{\nabla}\alpha)(R(X,Y)U,V,W) - (\overline{\nabla}\alpha)(U,R(X,Y)V,W) - (\overline{\nabla}\alpha)(U,V,R(X,Y)W) = 0. \quad (35)$$

Taking $X = V = \xi$ in (35) we have

$$R^{\perp}(\xi,Y)(\overline{\nabla}\alpha)(U,\xi,W) - (\overline{\nabla}\alpha)(R(\xi,Y)U,\xi,W) - (\overline{\nabla}\alpha)(U,R(\xi,Y)\xi,W) - (\overline{\nabla}\alpha)(U,\xi,R(\xi,Y)W) = 0. \quad (36)$$

Then, in view of (4), (19) and (24) we have the following equalities:

$$\begin{aligned}(\overline{\nabla}\alpha)(U,\xi,W) &= (\overline{\nabla}_U\alpha)(\xi,W)\\ &= \nabla^{\perp}_U(\alpha(\xi,W)) - \alpha(\nabla_U\xi,W) - \alpha(\xi,\nabla_U W)\\ &= -\alpha(\varphi U,W),\end{aligned} \quad (37)$$

$$\begin{aligned}(\overline{\nabla}\alpha)(R(\xi,Y)U,\xi,W) &= (\overline{\nabla}_{R(\xi,Y)U}\alpha)(\xi,W)\\ &= \nabla^{\perp}_{R(\xi,Y)U}(\alpha(\xi,W)) - \alpha(\nabla_{R(\xi,Y)U}\xi,W) - \alpha(\xi,\nabla_{R(\xi,Y)U}W)\\ &= \eta(U)\alpha(\varphi Y,W),\end{aligned} \quad (38)$$

$$\begin{aligned}(\overline{\nabla}\alpha)(U,R(\xi,Y)\xi,W) &= (\overline{\nabla}_U\alpha)(R(\xi,Y)\xi,W)\\ &= \nabla^{\perp}_U(\alpha(R(\xi,Y)\xi,W)) - \alpha(\nabla_U R(\xi,Y)\xi,W) - \alpha(R(\xi,Y)\xi,\nabla_U W)\\ &= \nabla^{\perp}_U\alpha(Y,W) - \alpha\left(\nabla_U\left(\eta(Y)\xi+Y\right),W\right) - \alpha\left(Y,\nabla_U W\right)\end{aligned} \quad (39)$$

and

$$\begin{aligned}(\overline{\nabla}\alpha)(U,\xi,R(\xi,Y)W) &= (\overline{\nabla}_U\alpha)(\xi,R(\xi,Y)W)\\ &= \nabla^{\perp}_U\alpha(\xi,R(\xi,Y)W) - \alpha(\nabla_U\xi,R(\xi,Y)W) - \alpha(\xi,\nabla_U R(\xi,Y)W)\\ &= \eta(W)\alpha(\varphi U,Y).\end{aligned} \quad (40)$$

Then substituting (37)-(40) into (36) we obtain

$$\begin{aligned}&-R^{\perp}(\xi,Y)\alpha(\varphi U,W) - \eta(U)\alpha(\varphi Y,W) - \nabla^{\perp}_U\alpha(Y,W)\\ &+\alpha\left(\nabla_U\left(\eta(Y)\xi+Y\right),W\right) + \alpha\left(Y,\nabla_U W\right) - \eta(W)\alpha(\varphi U,Y) = 0.\end{aligned} \quad (41)$$

So, taking $W = \xi$ in (41) and using (24), we find

$$\alpha\left(Y,\nabla_U\xi\right) + \alpha(\varphi U,Y) = 0,$$

which yields, from (18),

$$\alpha(\varphi U,Y) = 0.$$

So, analogous to the proof of Theorem 4.2, we obtain $\alpha(U,Y) = 0$.

The converse statement is trivial. This proves the theorem. □

5. INVARIANT SUBMANIFOLDS OF LP-SASAKIAN MANIFOLDS SATISFYING $\mathcal{Z}(X,Y)\cdot\alpha = 0$ AND $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$

In this section, we consider invariant submanifolds of LP-Sasakian manifolds satisfying the conditions $\mathcal{Z}(X,Y)\cdot\alpha = 0$ and $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$. Firstly we have:

Theorem 5.1. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$ such that $\tau \neq n(n-1)$. The condition $\mathcal{Z}(X,Y)\cdot\alpha = 0$ holds on M^n if and only if M^n is totally geodesic.*

Proof. Since M satisfies the condition $\mathcal{Z}(X,Y)\cdot\alpha = 0$, it follows from (9) that

$$R^{\perp}(X,Y)\alpha(U,V) - \alpha(\mathcal{Z}(X,Y)U,V) - \alpha(U,\mathcal{Z}(X,Y)V) = 0. \tag{42}$$

Taking $X = V = \xi$ in (42) we get

$$R^{\perp}(\xi,Y)\alpha(U,\xi) - \alpha(\mathcal{Z}(\xi,Y)U,\xi) - \alpha(U,\mathcal{Z}(\xi,Y)\xi) = 0.$$

So, using (24), the last equation is reduced to

$$\alpha(U,\mathcal{Z}(\xi,Y)\xi) = 0.$$

Then, making use of (23), we have

$$\left(1 - \frac{\tau}{n(n-1)}\right)\alpha(U,\eta(Y)\xi + Y) = 0.$$

From the assumption, since $\tau \neq n(n-1)$, in view of (24), we obtain $\alpha(Y,U) = 0$, which gives us that M^n is totally geodesic.

The converse statement is trivial. This completes the proof of the theorem. □

Theorem 5.2. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$ such that $\tau \neq n(n-1)$. Then the condition $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$ holds on M^n if and only if M^n is totally geodesic.*

Proof. Since M satisfies the condition $\mathcal{Z}(X,Y)\cdot\overline{\nabla}\alpha = 0$, we have by (10):

$$\begin{aligned} &R^{\perp}(X,Y)(\overline{\nabla}\alpha)(U,V,W) - (\overline{\nabla}\alpha)(\mathcal{Z}(X,Y)U,V,W) \\ &-(\overline{\nabla}\alpha)(U,\mathcal{Z}(X,Y)V,W) - (\overline{\nabla}\alpha)(U,V,\mathcal{Z}(X,Y)W) = 0. \end{aligned} \tag{43}$$

Taking $X = V = \xi$ in (43) we have

$$\begin{aligned} &R^{\perp}(\xi,Y)(\overline{\nabla}\alpha)(U,\xi,W) - (\overline{\nabla}\alpha)(\mathcal{Z}(\xi,Y)U,\xi,W) \\ &-(\overline{\nabla}\alpha)(U,\mathcal{Z}(\xi,Y)\xi,W) - (\overline{\nabla}\alpha)(U,\xi,\mathcal{Z}(\xi,Y)W) = 0. \end{aligned} \tag{44}$$

Then, in view of (4), (22), (23) and (24), we have the following equalities:

$$\begin{aligned} (\overline{\nabla}\alpha)(\mathcal{Z}(\xi,Y)U,\xi,W) &= (\overline{\nabla}_{\mathcal{Z}(\xi,Y)U}\alpha)(\xi,W) = \nabla^{\perp}_{\mathcal{Z}(\xi,Y)U}(\alpha(\xi,W)) \\ &\quad -\alpha(\nabla_{\mathcal{Z}(\xi,Y)U}\xi,W) - \alpha(\xi,\nabla_{\mathcal{Z}(\xi,Y)U}W) \\ &= \left(1 - \frac{\tau}{n(n-1)}\right)\eta(U)\alpha(\varphi Y,W), \end{aligned} \tag{45}$$

$$\begin{aligned} (\overline{\nabla}\alpha)(U,\mathcal{Z}(\xi,Y)\xi,W) &= (\overline{\nabla}_U\alpha)(\mathcal{Z}(\xi,Y)\xi,W) = \nabla^{\perp}_U(\alpha(\mathcal{Z}(\xi,Y)\xi,W)) \\ &\quad -\alpha(\nabla_U\mathcal{Z}(\xi,Y)\xi,W) - \alpha(\mathcal{Z}(\xi,Y)\xi,\nabla_UW) \\ &= \nabla^{\perp}_U\left(\left(1 - \frac{\tau}{n(n-1)}\right)\alpha(Y,W)\right) \\ &\quad -\alpha\left(\nabla_U\left[\left(1 - \frac{\tau}{n(n-1)}\right)(\eta(Y)\xi + Y)\right],W\right) \\ &\quad -\left(1 - \frac{\tau}{n(n-1)}\right)\alpha\left(Y,\nabla_UW\right) \end{aligned} \tag{46}$$

and

$$(\overline{\nabla}\alpha)(U,\xi,\mathcal{Z}(\xi,Y)W) = (\overline{\nabla}_U\alpha)(\xi,\mathcal{Z}(\xi,Y)W)$$

$$= \nabla^{\perp}_{U}\alpha(\xi, \mathcal{Z}(\xi, Y)W) - \alpha(\nabla_{U}\xi, \mathcal{Z}(\xi, Y)W) - \alpha(\xi, \nabla_{U}\mathcal{Z}(\xi, Y)W)$$
$$= \left(1 - \frac{\tau}{n(n-1)}\right)\eta(W)\alpha(\varphi U, Y). \tag{47}$$

Then, substituting (37) and (45)-(47) into (44), we obtain

$$\begin{aligned} &-R^{\perp}(\xi, Y)\alpha(\varphi U, W) - \left(1 - \frac{\tau}{n(n-1)}\right)\eta(U)\alpha(\varphi Y, W) \\ &-\nabla^{\perp}_{U}\left(\left(1 - \frac{\tau}{n(n-1)}\right)\alpha(Y, W)\right) \\ &+\alpha\left(\nabla_{U}\left(1 - \frac{\tau}{n(n-1)}\right)(\eta(Y)\xi + Y), W\right) \\ &+\left(1 - \frac{\tau}{n(n-1)}\right)\alpha(Y, \nabla_{U}W) \\ &\quad -\left(1 - \frac{\tau}{n(n-1)}\right)\eta(W)\alpha(\varphi U, Y) = 0. \end{aligned} \tag{48}$$

So, taking $W = \xi$ in (48) and using (24) and (25), we get

$$\left(1 - \frac{\tau}{n(n-1)}\right)(\alpha(Y, \nabla_{U}\xi) + \alpha(\varphi U, Y)) = 0,$$

which yields, from (18) and the assumptions of $\tau \neq n(n-1)$,

$$\alpha(\varphi U, Y) = 0. \tag{49}$$

So, analogous to the proof of Theorem 4.2, we obtain $\alpha(U, Y) = 0$. Whence M^n is totally geodesic.

The converse statement is trivial. Hence we get the result as required. □

In view of Theorems 4.1, 4.2, 4.3, 5.1 and 5.2 we can state:

Corollary 5.3. *Let M^n be an invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. Then the following statements are equivalent:*

(1) *M^n is semiparallel;*
(2) *M^n has parallel second fundamental form;*
(3) *M^n is 2-semiparallel;*
(4) *M^n satisfies the condition $\mathcal{Z}(X, Y) \cdot \alpha = 0$ with $\tau \neq n(n-1)$;*
(5) *M^n satisfies the condition $\mathcal{Z}(X, Y) \cdot \overline{\nabla}\alpha = 0$ with $\tau \neq n(n-1)$;*
(6) *M^n is totally geodesic.*

REFERENCES

[1] J. Deprez, "Semi-parallel Surfaces in the Euclidean Space", *Journal of Geometry* **25** (1985), pp. 192-200.
[2] J. Deprez, "Semi-parallel Hypersurfaces", *Rend. Sem. Mat. Univ. Politechn. Torino* **44** (1986), pp. 303-316.
[3] F. Dillen, "Semi-parallel Hypersurfaces of a Real Space Form", *Israel J. Math.* **75** (1991), pp. 193-202.
[4] H. Endo, "Invariant Submanifolds in Contact Metric Manifolds", *Tensor (N.S.)* **43 (1)** (1986), pp. 83-87.
[5] M. Kon, "Invariant Submanifolds of Normal Contact Metric Manifolds", *Kodai Math. Sem. Rep.* **25** (1973), pp. 330-336.
[6] Ü. Lumiste, "Semi-symmetric Submanifolds as the Second Order Envelope of Symmetric Submanifolds", *Proc. Estonian Acad. Sci. Phys. Math.* **39** (1990), pp. 1-8.
[7] K. Arslan, Ü. Lumiste, C. Murathan and C. Özgür, "2-Semiparallel Surfaces in Space Forms. I. Two Particular Cases", *Proc. Estonian Acad. Sci. Phys. Math.* **49 (3)** (2000), pp. 139-148.
[8] B. Y. Chen, *Geometry of Submanifolds*", Pure and Applied Mathematics, Vol. **22**, Marcel Dekker, Inc., New York, 1973.
[9] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Math., Vol. **3**, World Sci., 1984.

[10] K. Matsumoto, "On Lorentzian Paracontact Manifolds", *Bull. Yamagata Univ. Nat. Sci.* **12 (2)** (1989), pp. 151-156.
[11] I. Satō, "On a Structure Similar to the Almost Contact Structure", *Tensor (N. S.)* **30 (3)** (1976), pp. 219-224.
[12] I. Mihai and R. Rosca, "On Lorentzian P-Sasakian Manifolds", Classical Analysis, World Scientific Publ., Singapore (1992), pp. 155-169.
[13] C. Murathan, A. Yıldız, K. Arslan and U. C. De, "On a Class of Lorentzian Para-Sasakian Manifolds", *Proc. Estonian Acad. Sci. Phys. Math.* **55 (4)** (2006), pp. 210-219.
[14] M. M. Tripathi and U. C. De, "Lorentzian Almost Paracontact Manifolds and their Submanifolds", *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* **8** (2001), pp. 101-105.
[15] K. Matsumoto, I. Mihai and R. Roşca, "ξ-Null Geodesic Gradient Vector Fields on a Lorentzian Para-Sasakian Manifold", *J. Korean Math. Soc.* **32 (1)** (1995), pp. 17-31.
[16] U. C. De, A. Al-Aqeel and A. A. Shaikh, "Submanifolds of a Lorentzian Para-Sasakian Manifold", *Bull. Malays. Math. Sci. Soc.* **2 (28)** (2005), pp. 223-227.
[17] M. M. Tripathi and S. S. Shukla, "On Submanifolds of Lorentzian Almost Paracontact Manifolds", *Publ. Math. Debrecen* **59 (3-4)** (2001), pp. 327-338.