Result.Math. 55 (2009), 1–10

1422-6383/010001-10, *published online* July 14, 2009
DOI 10.1007/s00025-009-0395-8

Results in Mathematics

Hypersurfaces of an Almost r-Paracontact Riemannian Manifold with a Semi-Symmetric Non-Metric Connection

Mobin Ahmad and Cihan Özgür

Abstract. We define a semi-symmetric non-metric connection in an almost r-paracontact Riemannian manifold and consider invariant, non-invariant and anti-invariant hypersurfaces, respectively, of almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection.

Mathematics Subject Classification (2000). Primary 53D10; secondary 53B05.

Keywords. Almost r-paracontact Riemannian manifold, semi-symmetric non-metric connection, hypersurfaces of r-paracontact Riemannian manifolds.

1. Introduction

In [1], T. Adati studied hypersurfaces of an almost paracontact manifold. In [5], A. Bucki considered hypersurfaces of an almost r-paracontact Riemannian manifold. Some properties of invariant hypersurfaces of an almost r-paracontact Riemannian manifold were investigated in [6] by A. Bucki and A. Miernowski. Moreover, in [9], I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given by

$$T(X,Y) \equiv \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively. The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection of some metric g.

In [7,10], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection on a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form. In [13], K. Yano considered a semi-symmetric metric connection and studied some of its properties. In [2,3,8,11] and [12], different types of semi-symmetric non-metric connections were studied.

In this paper, we study a semi-symmetric non-metric connection in an almost r-paracontact Riemannian manifold. We consider invariant, non-invariant and anti-invariant hypersurfaces, respectively, of almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction about an almost r-paracontact Riemannian manifold. In Section 3, we show that the induced connection on a hypersurface of an almost r-paracontact Riemannian manifold with semi-symmetric non-metric connection with respect to the unit normal is also a semi-symmetric non-metric connection. We find the characteristic properties of invariant, non-invariant and anti-invariant hypersurfaces, respectively, of almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field φ of type (1,1), r vector fields $\xi_1, \xi_2, \ldots, \xi_r$ $(n > r)$, and r one-forms $\eta^1, \eta^2, \ldots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) := \{1,2,3,\ldots,r\}, \tag{2.1}$$

$$\varphi^2(X) = X - \eta^\alpha(X)\xi_\alpha, \tag{2.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r), \tag{2.3}$$

$$\eta^\alpha \circ \varphi = 0, \quad \alpha \in (r), \tag{2.4}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{2.5}$$

where X and Y are vector fields on M, then the structure $\sum := (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is said to be an *almost r-paracontact structure* and M is an *almost r-paracontact Riemannian manifold* [1]. From (2.1)–(2.5), we have

$$\varphi(\xi_\alpha) = 0, \quad \alpha \in (r),$$

$$\Psi(X,Y) \stackrel{def}{=} g(\varphi X, Y) = g(X, \varphi Y).$$

For a Riemannian connection $\overset{*}{\nabla}$ on M, the tensor N is given by

$$N(X,Y) := \left(\overset{*}{\nabla}_{\varphi Y}\varphi\right)X - \left(\overset{*}{\nabla}_{X}\varphi\right)\varphi Y - \left(\overset{*}{\nabla}_{\varphi X}\varphi\right)Y + \left(\overset{*}{\nabla}_{Y}\varphi\right)\varphi X + \eta^{\alpha}(X)\overset{*}{\nabla}_{Y}\xi_{\alpha} - \eta^{\alpha}(Y)\overset{*}{\nabla}_{X}\xi_{\alpha}\,.$$

An almost r-paracontact Riemannian manifold M with structure $\sum = (\varphi, \xi_{\alpha}, \eta^{\alpha}, g)_{\alpha\in(r)}$ is said to be of *paracontact type* [4] if

$$2\Psi(X,Y) = \left(\overset{*}{\nabla}_{X}\eta^{\alpha}\right)Y + \left(\overset{*}{\nabla}_{Y}\eta^{\alpha}\right)X\,, \quad \text{for all} \quad \alpha\in(r)\,. \tag{2.6}$$

If all η^{α} are closed then (2.6) reduces to

$$\Psi(X,Y) = \left(\overset{*}{\nabla}_{X}\eta^{\alpha}\right)Y\,, \quad \text{for all} \quad \alpha\in(r)\,, \tag{2.7}$$

and M satisfying this condition is called an *almost r-paracontact Riemannian manifold of s-paracontact type* [4]. An almost r-paracontact Riemann manifold M with a structure $\sum = (\varphi, \xi_{\alpha}, \eta^{\alpha}, g)_{\alpha\in(r)}$ is said to be *P-Sasakian* if (2.7) and

$$\left(\overset{*}{\nabla}_{Z}\Psi\right)(X,Y) = -\sum_{\alpha}\eta^{\alpha}(X)\left[g(Y,Z) - \sum_{\beta}\eta^{\beta}(Y)\eta^{\beta}(Z)\right] - \sum_{\alpha}\eta^{\alpha}(Y)\left[g(X,Z) - \sum_{\beta}\eta^{\beta}(Y)\eta^{\beta}(Z)\right] \tag{2.8}$$

yield for all vector fields X, Y and Z on M [4]. (2.7) and (2.8) are equivalent to

$$\varphi X = \overset{*}{\nabla}_{X}\xi_{\alpha}\,, \quad \text{for all} \quad \alpha\in(r)\,, \tag{2.9}$$

and

$$\left(\overset{*}{\nabla}_{Y}\varphi\right)X = -\sum_{\alpha}\eta^{\alpha}(X)\left[Y - \sum_{\beta}\eta^{\alpha}(Y)\xi_{\alpha}\right] - \left[g(X,Y) - \sum_{\alpha}\eta^{\alpha}(X)\eta^{\alpha}(Y)\right]\sum_{\beta}\xi_{\beta}\,, \tag{2.10}$$

respectively [5]. We define a semi-symmetric non-metric connection ∇ on M by

$$\nabla_{X}Y = \overset{*}{\nabla}_{X}Y + \eta^{\alpha}(Y)X\,, \tag{2.11}$$

for any $\alpha\in(r)$. Inserting (2.11) into (2.9) and (2.10), we get

$$\varphi X = \nabla_{X}\xi_{\alpha} - X \tag{2.12}$$

and

$$(\nabla_Y\varphi)X = -\sum_\alpha \eta^\alpha(X)\big[Y - \eta^\alpha(Y)\xi_\alpha\big] - \left[g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\right]\sum_\beta \xi_\beta\,. \tag{2.13}$$

3. Hypersurfaces of almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection

Let $\widetilde{M}^n$ be an almost r-paracontact Riemannian manifold with a positive definite metric g and M^{n-1} be a hypersurface in $\widetilde{M}^n$, given by an immersion $f : M^{n-1} \to \widetilde{M}^n$. If B denotes the differential of f then any vector field $\overline{X} \in \chi(M^{n-1})$ implies $B\overline{X} \in \chi(\widetilde{M}^n)$. In an obvious way we use a bar to mark objects belonging to M^{n-1}.

Let N be an oriented unit normal field to M^{n-1} and

$$\overline{g}(\overline{X},\overline{Y}) = g(\overline{X},\overline{Y})\,,$$

the induced metric on M^{n-1}. We have [6]

$$g(\overline{X},N) = 0 \quad \text{and} \quad g(N,N) = 1\,.$$

If $\overline{\overset{*}{\nabla}}$ is the connection, induced from $\overset{*}{\nabla}$ with respect to unit normal N on the hypersurface, then the Gauss equation is given by

$$\overset{*}{\nabla}_{\overline{X}}\overline{Y} = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N\,, \tag{3.1}$$

where h is the second fundamental tensor, satisfying

$$h(\overline{X},\overline{Y}) = h(\overline{Y},\overline{X}) = g\big(H(\overline{X}),Y\big)\,, \tag{3.2}$$

and H is the shape operator of M^{n-1} in $\widetilde{M}^n$. If $\overline{\nabla}$ is the connection, induced from the semisymmetric non-metric connection ∇ with respect to unit normal N on the hypersurface, then we have

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X},\overline{Y})N\,, \tag{3.3}$$

where m is a $(0,2)$-tensor field on M^{n-1}. From (2.11) we obtain

$$\nabla_{\overline{X}}\overline{Y} = \overset{*}{\nabla}_{\overline{X}}\overline{Y} + \eta^\alpha(\overline{Y})\overline{X}\,, \tag{3.4}$$

and the equations (3.1)–(3.4) give

$$\overline{\nabla}_{\overline{X}}\overline{Y} + m(\overline{X},\overline{Y})N = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N + \eta^\alpha(\overline{Y})\overline{X}\,.$$

Taking tangential and normal parts from both sides, we obtain

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\overset{*}{\nabla}}_{\overline{X}}\overline{Y} + \eta^\alpha(\overline{Y})\overline{X}$$

and

$$m(\overline{X},\overline{Y}) = h(\overline{X},\overline{Y})\,.$$

Thus we get the following theorem:

Theorem 3.1. *The connection, induced with respect to the unit normal on a hypersurface of an almost r-paracontact Riemannian manifold with semi-symmetric non-metric connection, is also a semi-symmetric non-metric connection.*

From (3.3) and (3) we have

$$\nabla_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} + h(\overline{X},\overline{Y})N\,, \tag{3.5}$$

which is the Gauss equation for a semi-symmetric non-metric connection. The equation of Weingarten with respect to the Riemannian connection $\overset{*}{\nabla}$ is given by

$$\overset{*}{\nabla}_{\overline{X}}N = -H\overline{X} \tag{3.6}$$

for every tangent vector field $\overline{X}$ in M^{n-1}. From equation (2.11) we have

$$\nabla_{\overline{X}}N = \overset{*}{\nabla}_{\overline{X}}N + a_\alpha\overline{X}\,, \tag{3.7}$$

where

$$\eta^\alpha(N) = a_\alpha = m(\xi_\alpha)\,.$$

The relations (3.6) and (3.7) give

$$\nabla_{\overline{X}}N = -M\overline{X}\,, \tag{3.8}$$

where $M = H - a_\alpha$, which is the Weingarten equation with respect to a semi-symmetric non-metric connection. Now suppose that $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is an almost r-paracontact Riemannian structure on $\widetilde{M}^n$, then every vector field X on $\widetilde{M}^n$ is decomposed as

$$X = \overline{X} + l(X)N\,,$$

where l is a one-form on $\widetilde{M}^n$; for any tangent vector field $\overline{X}$ on M^{n-1} and normal N we have

$$\varphi\overline{X} = \overline{\varphi}\overline{X} + b(\overline{X})N\,, \tag{3.9}$$

$$\varphi N = \overline{N} + KN\,, \tag{3.10}$$

where $\overline{\varphi}$ is a tensor field of type $(1,1)$ on the hypersurface M^{n-1}, b is a one-form and K a scalar function on M^{n-1}. For each $\alpha \in (r)$, we have

$$\xi_\alpha = \overline{\xi}_\alpha + a_\alpha N\,. \tag{3.11}$$

We define $\overline{\eta}^\alpha$ by

$$\overline{\eta}^\alpha(\overline{X}) = \eta^\alpha(\overline{X})\,, \quad \alpha \in (r)\,.$$

Then we obtain (see [5]):

$$\begin{aligned}
b(\overline{N}) + K^2 &= 1 - \sum_{\alpha} (a_\alpha)^2 \,, \\
K a_\alpha + b(\overline{\xi}_\alpha) &= 0 \,, \quad \alpha \in (r) \,, \\
\Psi(\overline{X}, \overline{Y}) &= \overline{g}(\overline{\varphi}\overline{X}, \overline{Y}) = \overline{g}(\overline{X}, \overline{\varphi}\overline{Y}) = \overline{\Psi}(\overline{X}, \overline{Y}) \,, \\
g(\overline{X}, \overline{N}) &= b(\overline{X}) \,.
\end{aligned}$$

Differentiating covariantly (3.9), (3.10), (3.11) along M^{n-1} and making use of (3.5) and (3.8), we get

$$\begin{aligned}
(\nabla_{\overline{Y}}\varphi)\overline{X} &= (\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X} - h(\overline{X}, \overline{Y})\overline{N} - b(\overline{X})\big[H(\overline{Y}) - a_\alpha \overline{Y}\big] \\
&\quad + \big[h(\overline{\varphi}\overline{X}, \overline{Y}) + (\overline{\nabla}_{\overline{Y}} b)\overline{X} - K h(\overline{X}, \overline{Y})\big] N \,, \qquad (3.12) \\
(\nabla_{\overline{Y}}\varphi)N &= \overline{\nabla}_{\overline{Y}}\overline{N} + \overline{\varphi}\left(H(\overline{Y}) - a_\alpha \overline{Y}\right) - K\big(H(\overline{Y}) - a_\alpha \overline{Y}\big) \\
&\quad + \big[2h(\overline{Y}, \overline{N}) + \overline{Y}(K) + a_\alpha b(\overline{Y})\big] N \,, \qquad (3.13) \\
\nabla_{\overline{Y}}\xi_\alpha &= \overline{\nabla}_{\overline{Y}}\overline{\xi}_\alpha - a_\alpha H(\overline{Y}) + (a_\alpha)^2 \overline{Y} + \big[\overline{Y}(a_\alpha) + h(\overline{Y}, \overline{\xi}_\alpha)\big] N \,, \qquad (3.14) \\
(\nabla_{\overline{Y}}\eta^\alpha)\,\overline{X} &= \left(\overline{\nabla}_{\overline{Y}}\eta^\alpha\right)\overline{X} - h(\overline{Y}, \overline{X}) a_\alpha \,,
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_{\overline{Z}}\Psi)\left(\overline{X}, \overline{Y}\right) &= \left(\overline{\nabla}_{\overline{Z}}\overline{\Psi}\right)\left(\overline{X}, \overline{Y}\right) - h(\overline{X}, \overline{Z}) b(\overline{Y}) \\
&\quad - b(\overline{X}) h(\overline{Z}, \overline{Y}) + b(\overline{X}) a_\alpha \overline{g}(\overline{Z}, \overline{Y}) \,.
\end{aligned}$$

Theorem 3.2. *If M^{n-1} is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ endowed with a semi-symmetric non-metric connection with structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ then either:*

- (i) *all ξ_α are tangent to M^{n-1}, and M^{n-1} admits an almost r-paracontact Riemannian structure $\sum_1 = (\overline{\varphi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})_{\alpha \in (r)}$, $(n - r > 2)$ or*
- (ii) *one of the vectors ξ_α (say ξ_r) is normal to M^{n-1}, and the remaining ξ_α are tangent to M^{n-1} and M^{n-1} admits an almost $(r-1)$-paracontact Riemannian structure $\sum_2 = (\overline{\varphi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})_{\alpha \in (r)}$, $(n - r > 1)$.*

Proof. The computations are similar to the proof of Theorem 3.1 in [5]. □

Corollary 3.3. *If M^{n-1} is a hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ endowed with a semi-symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ then the following statements are equivalent:*

- (i) *M^{n-1} is invariant,*
- (ii) *the normal field N is an eigenvector of φ,*

(iii) *all ξ_α are tangent to M^{n-1} if and only if M^{n-1} admits an almost r-paracontact Riemannian structure $\sum_1$, or one of the vectors ξ_α is normal and the $(r-1)$ remaining vectors ξ_i are tangent to M^{n-1} if and only if M^{n-1} admits an almost $(r-1)$ paracontact Riemannian structure $\sum_2$.*

Theorem 3.4. *If M^{n-1} is an invariant hypersurface immersed in an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ of P-Sasakian type endowed with a semi-symmetric non-metric connection with structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ then the induced almost r-paracontact Riemannian structure $\sum_1$ or the $(r-1)$-paracontact Riemannian structure $\sum_2$ is also of P-Sasakian type.*

Proof. The proof is similar to the proof of Theorem 3.3 in [5]. □

Lemma 3.5 ([5]). $\overline{\nabla}_{\overline{X}}(trace\overline{\varphi}) = trace\left(\overline{\nabla}_{\overline{X}}\overline{\varphi}\right)$.

Theorem 3.6. *Let M^{n-1} be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ endowed with a semi-symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ satisfying $\nabla\varphi = 0$ along M^{n-1}. Then M^{n-1} is totally geodesic if and only if*

$$\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + b(\overline{X})a_\alpha\overline{Y} = 0\,.$$

Proof. From (3.12) we have

$$(\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X} - h(\overline{Y},\overline{X})\overline{N} - b(\overline{X})\big(H(\overline{Y}) - a_\alpha\overline{Y}\big) = 0 \tag{3.15}$$

and

$$h(\overline{\varphi}\overline{X},\overline{Y}) + (\overline{\nabla}_{\overline{Y}}b)\overline{X} - Kh(\overline{Y},\overline{X}) = 0\,.$$

If M^{n-1} is totally geodesic then $h = 0$ and $H = 0$. From (3.15) we get

$$\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + b(\overline{X})a_\alpha\overline{Y} = 0\,.$$

Conversely, if $\left(\overline{\nabla}_{\overline{Y}}\overline{\varphi}\right)\overline{X} + b(\overline{X})a_\alpha\overline{Y} = 0$ then

$$h(\overline{Y},\overline{X})\overline{N} + b(\overline{X})H(\overline{Y}) = 0\,. \tag{3.16}$$

Interchanging $\overline{Y}$ and $\overline{X}$ in (3.16) we have

$$h(\overline{Y},\overline{X})\overline{N} + b(\overline{Y})H(\overline{X}) = 0\,. \tag{3.17}$$

(3.16) and (3.17) give

$$b(\overline{X})H(\overline{Y}) = b(\overline{Y})H(\overline{X})\,, \tag{3.18}$$

and (3.18) and (3.1) imply

$$b(\overline{X})h(\overline{Y},\overline{Z}) = b(\overline{Y})h(\overline{X},\overline{Z})\,. \tag{3.19}$$

From (3.18) and (3.19) we get $b(\overline{Z})h(\overline{X},\overline{Y}) = 0$ which gives $h = 0$ since $b \neq 0$. Using $h = 0$ in (3.16), we arrive at $H = 0$. Thus $h = 0$ and $H = 0$. Hence M^{n-1} is totally geodesic. This completes the proof of the theorem. □

We also have the following:

Theorem 3.7. *Let M^{n-1} be a non-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ with a semi-symmetric non-metric connection, satisfying $\nabla\varphi = 0$ along M^{n-1}. If trace$\overline{\varphi}$ = constant then*

$$h(\overline{X}, \overline{N}) = \frac{1}{2} a_\alpha \sum_a b(e_a) g(e_a, \overline{X}) . \tag{3.20}$$

Proof. From (3.15) we have

$$\overline{g}\big((\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X}, \overline{X}\big) = 2h(\overline{X}, \overline{Y}) b(\overline{X}) - a_\alpha b(\overline{X}) g(\overline{X}, \overline{Y})$$

and

$$\overline{\nabla}_{\overline{X}}(trace\overline{\varphi}) = 2h(\overline{X}, \overline{N}) - a_\alpha \sum_a b(e_a) g(e_a, \overline{X}) .$$

Using Lemma 3.5, we get (3.20), where $\overline{N} = \sum_a b(e_a)e_a$. Thus our theorem is proved. □

Let M^{n-1} be an almost r-paracontact Riemannian manifold of s-paracontact type, then from (2.9), (3.9) and (3.14), we get

$$\overline{\varphi}\overline{X} = \overline{\nabla}_{\overline{X}}\overline{\xi}_\alpha - \overline{X} - a_\alpha H(\overline{X}) + (a_\alpha)^2 \overline{X} , \quad \alpha \in (r) \tag{3.21}$$

$$b(\overline{X}) = \left[\overline{X}(a_\alpha) + h(\overline{X}, \overline{\xi}_\alpha)\right] , \quad \alpha \in (r) . \tag{3.22}$$

Making use of (3.22) and (3.11), if M^{n-1} is totally geodesic and all ξ_α are tangent to M^{n-1} then $a_\alpha = 0$ and $h = 0$. Hence $b = 0$, that is, M^{n-1} is invariant.

So we have the following Proposition:

Proposition 3.8. *If M^{n-1} is a totally geodesic hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ of s-paracontact type endowed with a semi-symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ and if all ξ_α are tangent to M^{n-1} then M^{n-1} is invariant.*

Theorem 3.9. *If M^{n-1} is an anti-invariant hypersurface of an almost r-paracontact Riemannian manifold $\widetilde{M}^n$ of s-paracontact type endowed with a semi-symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ then $\overline{\nabla}_{\overline{X}}\xi_\alpha = \overline{X}$ for all ξ_α tangent to M^{n-1}.*

Proof. If M^{n-1} is anti-invariant then $\overline{\varphi} = 0$, $a_\alpha = 0$ and from (3.21) we have

$$\overline{\nabla}_{\overline{X}}\xi_\alpha = -a_\alpha H(\overline{X}) + (a_\alpha)^2 \overline{X} + \overline{X} , \quad \alpha \in (r) .$$

That is

$$\overline{\nabla}_{\overline{X}}\xi_\alpha = \overline{X} .$$

This completes the proof of the theorem. □

Theorem 3.10. *Let $\widetilde{M}^n$ be an almost r-paracontact Riemannian manifold of P-Sasakian type endowed with a semi-symmetric non-metric connection with a structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, and let M^{n-1} be a hypersurface immersed in $\widetilde{M}^n$ such* *that none of the vectors* ξ_α *is tangent to* M^{n-1} *then* M^{n-1} *is totally geodesic if and only if*

$$(\overline{\nabla}_{\overline{Y}}\overline{\varphi})\overline{X} + a_\alpha b(\overline{X})\overline{Y} = -\sum_\alpha \overline{\eta}^\alpha(\overline{X})\left[\overline{Y} - \overline{\eta}^\alpha(\overline{Y})\bar{\xi}_\alpha\right] - \left[\overline{g}(\overline{X},\overline{Y}) - \sum_\alpha \overline{\eta}^\alpha(\overline{X})\overline{\eta}^\alpha(\overline{Y})\right]\sum_\beta \bar{\xi}_\beta \,.$$

Proof. The proof is similar to the proof of the Theorem 4.5 in [5]. □

Acknowledgements

We are grateful to the referee for a number of helpful suggestions.

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Mobin Ahmad
Department of Applied Mathematics
Integral University
Kursi Road
Lucknow-226026
India
e-mail: mobinahmad@rediffmail.com

Cihan Özgür
Department of Mathematics
Balıkesir University
10145, Çağış
Balıkesir
Turkey
e-mail: cozgur@balikesir.edu.tr

Received: May 12, 2008.
Revised: February 17, 2009.
Accepted: March 24, 2009.