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# Analysis of an axis-symmetric fractional diffusion-wave problem

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## Abstract

This paper presents an axis-symmetric fractional diffusion-wave problem which is considered in polar coordinates. The dynamic characteristics of the system are described with a partial fractional differential equation in terms of the Riemann–Liouville fractional derivative. This continuum problem is reduced to a countable infinite problem by using the method of separation of variables. In this way, the closed form solution of the problem is obtained. The Grünwald–Letnikov approach is applied to take a numerical evaluation. The compatibility and effectiveness of this approach are realized by some simulation results which are obtained by a MATLAB program. It can be seen that the analytical and numerical solutions overlap.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Fractional derivatives are a generalization of ordinary differentiation and integration to non-integer orders. Some of the definitions of fractional derivatives proposed include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Riesz and Marchaud derivatives [1–4]. Fractional derivatives arise in many physical problems, for example the frequency-dependent damping behavior of materials, relaxation functions for viscoelastic materials, fractional  $PI^{\lambda}D^{\mu}$  control of dynamical systems, motion of a plate in a Newtonian fluid and phenomena in electromagnetics and acoustics [4].

In this paper, we focus on axis-symmetric fractional diffusion wave equations (FDWEs). The FDWEs are obtained by replacing the integer order (time and/or space) derivatives in ordinary diffusion/wave equations with the fractional derivatives. The FDWEs have been considered by several investigators in recent years. Oldham and Spanier [5] considered a fractional diffusion equation that contained the first-order derivative in space and half-order derivative in time. Wyss [6] and Schneider and Wyss [7] presented the solutions of the time-fractional diffusion and wave equations in terms of Fox functions. Giona *et al* [8]

presented fractional diffusion equations describing the relaxation phenomena in complex viscoelastic materials. Giona and Roman [9] presented fractional diffusion equations for transport phenomena in random media. Giona and Roman [10] and Roman and Giona [11] used fractional diffusion equations to describe one- and three-dimensional cases of anomalous diffusions on fractals without external forces. Their work extends the expression of Oldham and Spanier [1].

Mainardi [12–14] used a Laplace transform method to obtain fundamental solutions for a FDWE and the solutions for fractional relaxation oscillations. Mainardi [14] and Mainardi and Paradisi [15] showed that as the order of a fractional derivative in a FDWE increases from 0 to 2, the process changes from slow diffusion to classical diffusion to diffusion-wave to classical wave processes. Agrawal [16] presented fundamental solutions for a FDWE where the diffusion equation contained a fourth-order space derivative and a fractional order time derivative. Zou *et al* [17] considered a fractional diffusion equation of higher dimension to describe anomalous diffusion processes involving external force fields by using Giona and Roman's heuristic argument. Hilfer [18] proposed the closed form solution of a fractional diffusion problem in terms of  $H$ -functions. More recently, Luchko [19] researched the uniqueness problem for the solution of the initial-boundary value problem which is defined by a generalized time-fractional diffusion equation.

Agrawal [20, 21] used modal/integral transform methods to find solutions of FDWEs defined in finite domains. The method is general and can be used to find closed form solutions to many problems in vibration analysis of fractional systems where modal/integral transform methods could be applied. Given that vibration of the continuous system is a vast field, these papers open up a multitude of possibilities and new problems in the field of fractional diffusion wave. Modal/integral transform methods have recently been used to solve fractional generalization of Navier–Stokes equations in El-Shahed and Salem [22] and a time fractional radial diffusion in a sphere [23]. Agrawal [25] presented a stochastic analysis of FDWEs defined in one dimension. Meerschaert *et al* [26] also analyzed fractional Cauchy problems with Dirichlet boundary on bounded domains from the stochastic point of view. It should be pointed out that very little work has been done in the area of stochastic analysis of fractional order engineering systems. Since the approach presented here also uses modal/integral transform methods, it could be extended to all fractional stochastic problems in multi-dimensions where the transform methods could be applied.

The exact solution of a fractional diffusion equation with an absorbent term and a linear external force appears in [27]. Exact solutions of generalized nonlinear fractional diffusion equations with external force and absorption are presented in [28].

This brief review of formulations and methods for FDWEs is by no means complete. Other formulations, methods and solutions can be found, among others, in [29] and references therein.

In this paper, we present a numerical scheme for an axis-symmetric fractional diffusion-wave problem. We define the problem in terms of Riemann–Liouville fractional derivatives and use the modal/integral transform method presented in Agrawal [20, 21] to reduce the continuum problem to a countable infinite degrees-of-freedom problem for which a solution could be found in a closed form. Problems similar to that considered here have also been solved using similar techniques in [22, 23]. However, the formulation here differs with Agrawal [20–23] in two respects. First, Agrawal [20, 21] considered one-dimensional problems, whereas this paper considers an axis-symmetric problem. Second, Agrawal [20–23] found only closed form solutions, whereas this paper also finds numerical solutions using the Grünwald–Letnikov approximation. Özdemir and Karadeniz [24] also apply this numerical scheme to an axis-symmetric diffusion-wave problem defined in cylindrical coordinates. They show that the

numerical results taken by the Grünwald–Letnikov scheme are compatible with analytical results. In [30], we will present a formulation for an axis-symmetric fractional optimal control problem for which closed form solutions could not be found. Therefore, our numerical scheme used here will allow us to find numerical solutions to the problem.

In the following section, we present the formulation and the analytical solution to an axis-symmetric FDWE.

## 2. Axis-symmetric FDWE and its analytical solution

In this section, we define an axis-symmetric FDWE in the sense of the Riemann–Liouville fractional derivative (RLFD) and improve the analytical solution using the formula of the Laplace transform method of the Mittag–Leffler function with two parameters. Therefore, we firstly begin to give the basic definitions of problem formulation. Let  $f(t) \in C([a, b])$  be a time-dependent function,  $\alpha$  ( $n - 1 < \alpha < n$ ) is the order of a fractional derivative and  $n \in \mathbb{N}^+$ . The RLFD is defined as follows:

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau. \tag{1}$$

In the case of  $\alpha$  is an integer, the fractional derivative turns to an ordinary differentiation operator. Furthermore, if  $f$  is dependent on two or more variables, then the ordinary derivative in equation (1) is replaced with a partial derivative. A two-parameter Mittag–Leffler function, as we mentioned above, is defined as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0). \tag{2}$$

The axis-symmetric fractional diffusion-wave problem can now be defined as follows: find the response of the system

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \beta \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + u(r, t), \tag{3}$$

subjected to the following boundary and initial conditions:

$$w(R, t) = 0, \quad t > 0 \tag{4}$$

and

$$w(r, 0) = \dot{w}(r, 0) = 0, \tag{5}$$

where  $\alpha$  ( $0 < \alpha \leq 2$ ) is the order of the fractional derivative that the system defines classical diffusion and wave propagations for  $\alpha = 1$  and  $\alpha = 2$ , respectively. Moreover, in the case of  $0 < \alpha \leq 1$  the behavior of the system is called sub-diffusion and for  $1 < \alpha \leq 2$  super-diffusion process occurs. Other variables of the formulation are as follows:  $r$  is the radial space coordinate,  $\beta$  is a constant which depends on the physical properties of the system,  $R$  is the radius of the domain on which problem formulation is considered and  $u(r, t) \in C([0, R])$  is the external source term. The second condition in equation (5) is considered only if  $\alpha > 1$ . In the case of heat transfer,  $u(r, t)$  represents the rate of heat generation, and in the case of membrane vibration, it represents the external forcing function.

The separation of variables method is used to find the analytical solution of the problem. For this purpose, the solution is assumed as

$$w(r, t) = \sum_{j=1}^{\infty} \phi_j(r) q_j(t), \quad j = 1, 2, \dots, \infty, \tag{6}$$

where  $\phi_j(r)$  are called eigenfunctions. Substituting equation (6) into equation (3) and after some manipulations, Bessel and fractional differential equations are obtained. In fact,

eigenfunctions are solutions of Bessel differential equations:

$$\frac{1}{\phi} \frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \left(\frac{\mu}{R}\right)^2 \phi = 0. \tag{7}$$

The solution to equation (7) is found as (see, e.g. [31])

$$\phi_j(r) = J_0\left(\mu_j \frac{r}{R}\right), \quad j = 1, 2, \dots, \infty, \tag{8}$$

where  $J_0(*)$  is the zero-order Bessel function of the first kind, and  $\mu_j, j = 1, 2, \dots, \infty$ , are the positive roots of the equation

$$J_0(\mu) = 0. \tag{9}$$

The analytical solution of the problem defined by equations (3)–(5) is rewritten as

$$w(r, t) = \sum_{j=1}^{\infty} q_j(t) J_0\left(\mu_j \frac{r}{R}\right) \tag{10}$$

and using equations (3), (4), (5) and (10), the orthogonality conditions

$$\int_0^1 x J_0(\mu_i x) J_0(\mu_j x) dx = \begin{cases} 0, & i \neq j \\ \frac{J_1^2(\mu_j)}{2}, & i = j \end{cases}. \tag{11}$$

We obtain

$$\frac{d^\alpha q_k(t)}{dt^\alpha} = -\beta \left(\frac{\mu_k}{R}\right)^2 q_k(t) + \bar{f}_k(t), \quad (0 < \alpha \leq 2) \tag{12}$$

and

$$q_k(0) = \dot{q}_k(0) = 0, \tag{13}$$

where  $J_1(*)$  is the first-order Bessel function of the first kind, and  $\bar{f}_k(*)$  is given as

$$\bar{f}_k(t) = \frac{2}{R^2 J_1^2(\mu_k)} \int_0^R r J_0\left(\mu_k \frac{r}{R}\right) u(r, t) dr. \tag{14}$$

Once again, the second condition in equation (13) is considered when  $\alpha > 1$ .

By applying the Laplace transform to equation (12), using equation (13), and then taking the inverse Laplace transform, we get

$$q_k(t) = \int_0^t Q_k(t - \tau) \bar{f}_k(\tau) d\tau \tag{15}$$

where

$$Q_k(t) = L^{-1} \left\{ \frac{1}{s^\alpha + \beta \left(\frac{\mu_k}{R}\right)^2} \right\} \tag{16}$$

is the fractional Green's function, which can be written in a closed form as [4]

$$Q_k(t) = t^{\alpha-1} E_{\alpha,\alpha} \left( -\beta \left(\frac{\mu_k}{R}\right)^2 t^\alpha \right). \tag{17}$$

Here  $L^{-1}$  is the inverse Laplace transform operator and  $E_{\alpha,\beta}$  is the two-parameter Mittag-Leffler function (see [4]).

Substituting equation (15) into equation (10), we obtain the closed form solution of the axis-symmetric fractional diffusion wave equation defined by equations (3)–(5) as

$$w(r, t) = \sum_{k=1}^{\infty} J_0\left(\mu_k \frac{r}{R}\right) \int_0^t Q_k(t - \tau) \bar{f}_k(\tau) d\tau. \tag{18}$$

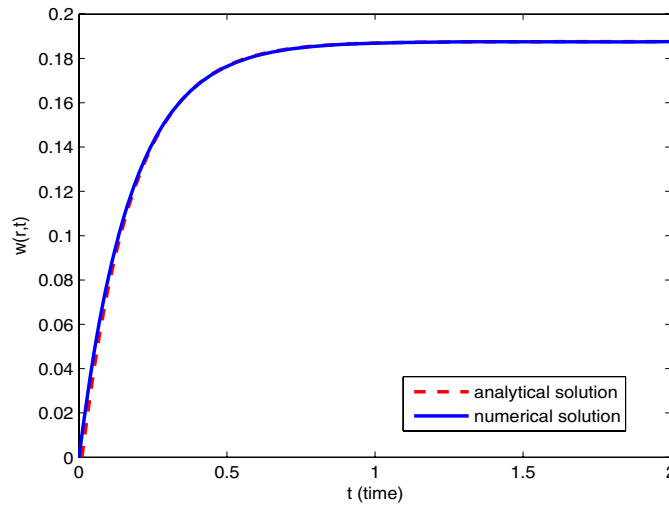


Figure 1. Comparison of the analytical and the numerical solution of  $w(r, t)$  for  $\alpha = 1$ ,  $r = 0.5$ ,  $M = 5$  and  $h = 0.01$ .

Thus,  $w(r, t)$  can be obtained provided  $u(r, t)$  is known. It will be shown in [30] that in the case of the fractional optimal control  $u(r, t)$  is not known *a priori*, and it is solved along with other variables. For this case, a numerical scheme is necessary.

In the following section, we present a numerical algorithm to solve the fractional differential equations defined by equations (12) and (13).

### 3. Numerical algorithm

The numerical algorithm presented here is based on an algorithm given in [4], and it relies on the Grünwald–Letnikov approximation of the fractional derivative. For simplicity in the discussion to follow, we drop the subscript  $k$  from equations (12) and (13), and rewrite them as

$$\frac{d^\alpha q(t)}{dt^\alpha} = -cq(t) + f(t) \tag{19}$$

and

$$q(0) = \dot{q}(0) = 0 \tag{20}$$

where  $c = \beta(\mu_k/R)^2$ .

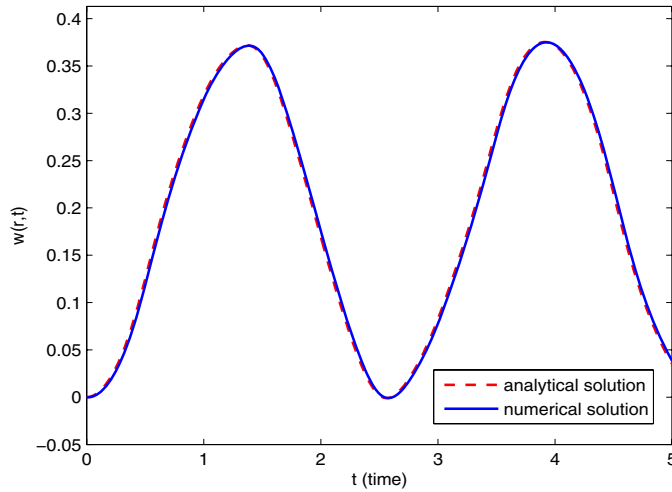
The algorithm can now be described as follows.

- (i) Divide the time interval into several subintervals of equal size  $h$  (also called the step size).
- (ii) Approximate  $d^\alpha q/dt^\alpha$  at node  $i$  using the Grünwald–Letnikov formula as [4]

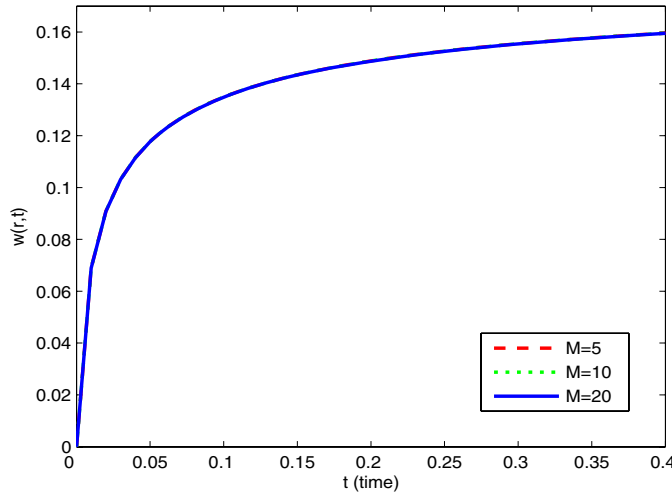
$$\frac{d^\alpha q}{dt^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^i w_j^{(\alpha)} q^{(i-j)}, \tag{21}$$

where  $q^{(j)}$  is the numerically computed value of  $q$  at node  $j$ , and  $w_j^{(\alpha)}$  are the coefficients defined as [4]

$$w_0^{(\alpha)} = 1, \quad w_j^{(\alpha)} = \left(1 - \frac{\alpha + 1}{j}\right) w_{j-1}^{(\alpha)}, \quad j = 1, \dots \tag{22}$$



**Figure 2.** Comparison of the analytical and the numerical solution of  $w(r, t)$  for  $\alpha = 2, r = 0.5, M = 5$  and  $h = 0.01$ .



**Figure 3.** The solution of  $w(r, t)$  for  $\alpha = 0.5, r = 0.5$  and  $M = 5, 10, 20$ .

(iii) Approximate equation (19) at node  $i$ , and solve for  $q^{(i)}$  to obtain

$$q^{(i)} = \frac{1}{1 + ch^\alpha} (h^\alpha f(t_i) - \sum_{j=1}^{i-1} w_j^{(\alpha)} q^{(i-j)}), \quad i = i_\alpha, \dots, \quad (23)$$

where  $i_\alpha = 1$  if  $i_\alpha \leq 1$ , and  $i_\alpha = 2$  if  $i_\alpha > 1$ . In case  $\alpha$  is greater than 1,  $q^{(1)}$  is determined using equation (20) and taking linear approximation between  $q^{(0)}$  and  $q^{(1)}$  (see [4]).

(iv) Use equation (23) to find  $q^{(i)}$  at all nodes.

Thus, the numerical solution to equations (19) and (20) is obtained. To obtain the solution to equations (3)–(5), we replace  $\infty$  in equation (10) with an integer  $M$  (i.e. we truncate the series), solve equations (12) and (13) for  $k$  from 1 to  $M$ , and substitute the results in

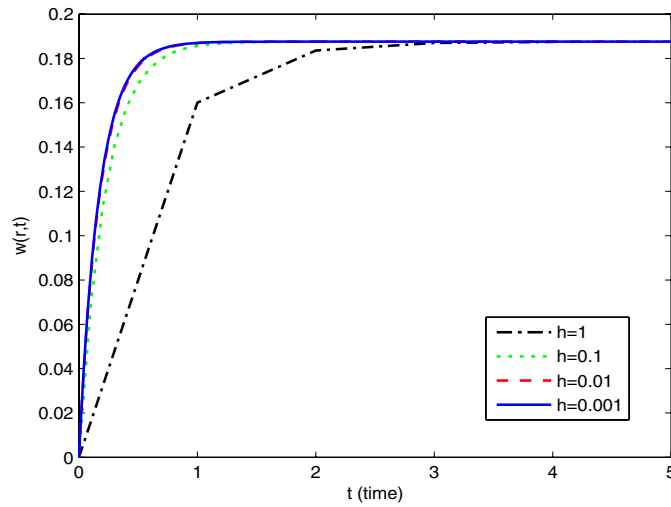


Figure 4. The solution of  $w(r, t)$  for  $\alpha = 1, r = 0.5, M = 5$  and  $h = 1, 0.1, 0.01, 0.001$ .

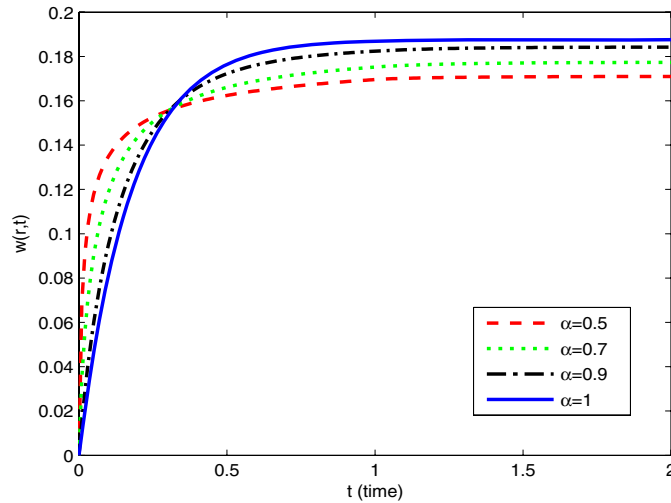


Figure 5. The solution of  $w(r, t)$  for  $r = 0.5, h = 0.01, M = 5$  and  $\alpha = 0.5, 0.7, 0.9, 1$ .

equation (10). Numerical studies show that only a small  $M$  is sufficient to find accurate results.

#### 4. Numerical example

In this section, we present some simulation results for the diffusion-wave problem defined by equations (3)–(5) for  $t > 0, 0 < \alpha \leq 2, r \in [0, R]$ . For simulation purposes, we take  $R = \beta = u(r, t) = 1$ , and vary  $M$  and  $h$ . We compute  $\tilde{f}_k(t)$  using equation (14) and solve equations (12) and (13) for  $k = 1, 2, \dots, M$  using the algorithm discussed in section 3. Finally, we use equation (10) to find the response. We also find the analytical solution using



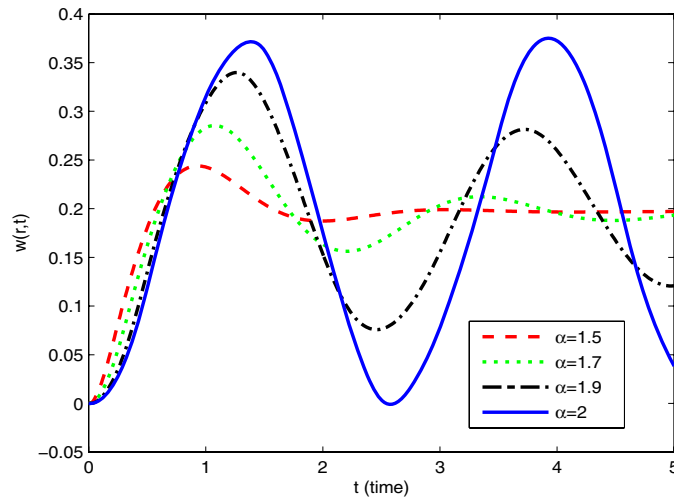


Figure 6. The solution of  $w(r, t)$  for  $r = 0.5$ ,  $h = 0.01$ ,  $M = 5$  and  $\alpha = 1.5, 1.7, 1.9, 2$ .

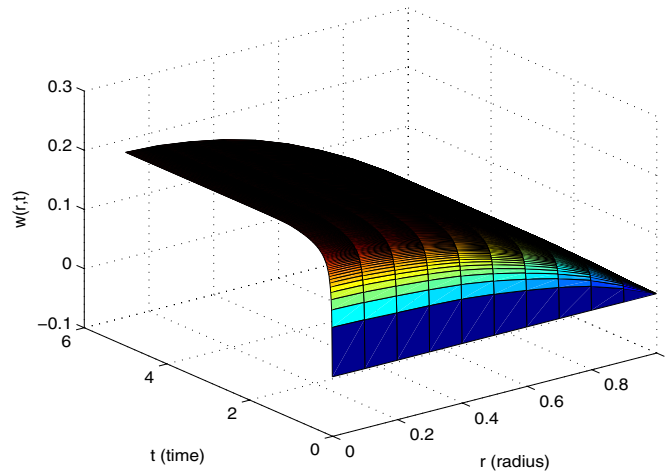


Figure 7. Three-dimensional figure of  $w(r, t)$  for  $\alpha = 0.5$ ,  $h = 0.01$  and  $M = 5$ .

equation (18) for comparison purposes. For computation purposes, this series is truncated after  $M$  terms. The results of this study are as follows.

Figures 1 and 2 show the analytical and numerical results for  $w(r, t)$  for  $\alpha = 1$  and  $\alpha = 2$ , respectively. In this study, we take  $r = 0.5$ ,  $M = 5$  and  $h = 0.01$ . Figure 1 shows that the diffusion reaches a steady state in a very short time, and figure 2 shows the undamped vibrational characteristics of the system. Note that in both cases, the analytical and numerical results are very close.

The rest of the figures shows numerical results for various  $M$ ,  $h$  and  $\alpha$ . Figure 3 shows  $w(r, t)$  at  $r = 0.5$  for  $\alpha = 0.5$  and  $M = 5, 10$  and  $20$ . All three curves are very close, indicating that only few terms are necessary to compute response of the system. Note that the diffusion process is slow. Figure 4 shows  $w(r, t)$  at  $r = 0.5$  for  $\alpha = 1.0$ ,  $M = 5$  and

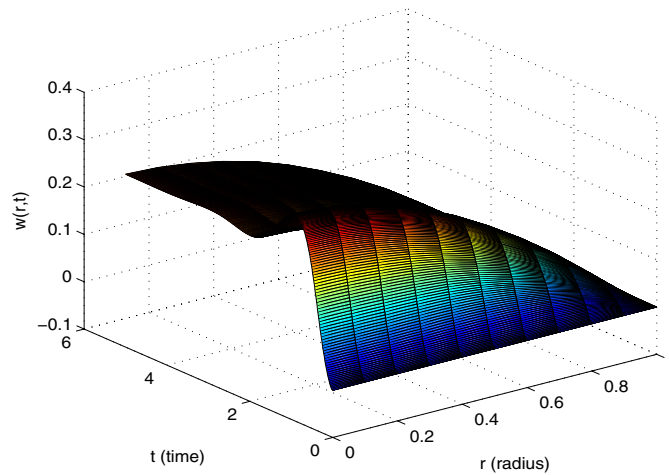


Figure 8. Three-dimensional figure of  $w(r, t)$  for  $\alpha = 1.5$ ,  $h = 0.01$  and  $M = 5$ .

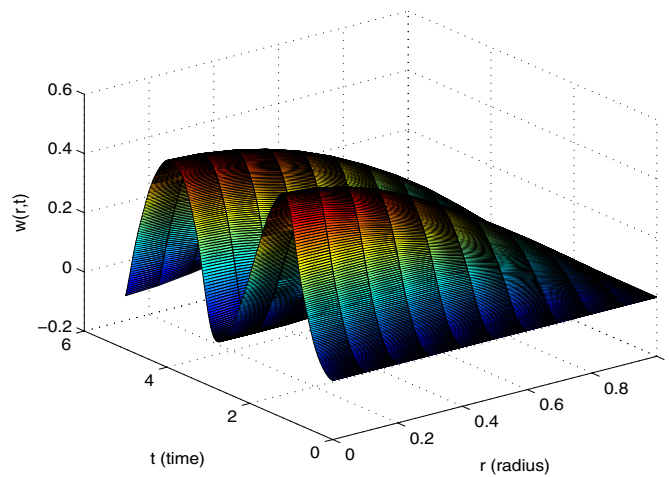


Figure 9. Three-dimensional figure of  $w(r, t)$  for  $\alpha = 2$ ,  $h = 0.01$  and  $M = 5$ .

$h = 1, 0.1, 0.01$  and  $0.001$ . Note that the solutions converge as the step size is reduced, which indicates that the algorithm may be stable.

Figure 5 shows  $w(r, t)$  for  $\alpha = 0.5, 0.7, 0.9$  and  $1$ , and figure 6 shows the same for  $\alpha = 1.5, 1.7, 1.9$  and  $2$ . In both cases, the following values are used:  $r = 0.5$ ,  $M = 5$  and  $h = 0.01$ . Both figures show that as  $\alpha$  approaches an integer value, the solution for the integer order system is recovered. These two figures also show that as  $\alpha$  changes from  $0.5$  to  $2$ , the response changes from sub-diffusion to diffusion to diffusion-wave to wave solution.

Figures 7–9 show the whole field response  $w(r, t)$  for  $\alpha = 0.5, 1.5$  and  $2$ , respectively. For these simulations, we took  $M = 5$  and  $h = 0.01$ . These figures also show the changing behavior of  $w(r, t)$  as  $\alpha$  changes from  $0.5$  to  $2$ . Note that  $w(r, t)$  reaches a steady state for  $\alpha = 0.5$  and  $1.5$ , but it continues to oscillate for  $\alpha = 2.0$ . Further, in all three cases,  $\partial w / \partial r$  is  $0$  at  $r = 0$ , as expected.

## 5. Conclusions

An axis-symmetric fractional diffusion-wave problem was defined in terms of the Riemann–Liouville fractional derivative, and a modal/integral transform method was presented to reduce the continuum problem to a countable infinite degrees-of-freedom problem. A Laplace transform-based technique was used to find closed form solutions. The Grünwald–Letnikov approximation was used to develop an algorithm for numerical solution of the problem. Results show that both the analytical and the numerical results agree well. As the time step size is reduced, the solutions converge and only few terms in the series are necessary to find a solution close to the exact solution. The response of the system changes from sub-diffusion to diffusion to diffusion-wave to wave solutions as  $\alpha$  changes from 0.5 to 2. The numerical algorithm developed here will allow us to find numerical solutions for axis-symmetric fractional optimal control problems.

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