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Fractional optimal control problem of an axis-symmetric diffusion-wave propagation

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Abstract

This paper presents the formulation of an axis-symmetric fractional optimal control problem (FOCP). Dynamic characteristics of the system are defined in terms of the left and right Riemann–Liouville fractional derivatives (RLFDs). The performance index of a FOCP is described with a state and a control function. Furthermore, dynamic constraints of the system are given by a fractional diffusion-wave equation. It is preferred to use the method of separation of variables for finding the analytical solution of the problem. In this way, the closed form solution of the problem is obtained by a linear combination of eigenfunctions and eigencoordinates. For numerical evaluation, the Grünwald–Letnikov approximation is applied to the problem. Consequently, some simulation results show that analytical and numerical solutions overlap for $\alpha = 1$. This numerical approach is applicable and effective for such a kind of FOCP. In addition, the changing of some variables related to the problem formulation is analyzed.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

In recent years, researchers in the areas of engineering, science and mathematics have demonstrated that the dynamics of many physical systems are described more accurately by using fractional differential equations (FDEs). Therefore, the solution of FDEs with analytical and numerical schemes is of growing interest. If FDEs are used to describe the performance index and system dynamics, it leads to a fractional optimal control problem (FOCP). The fractional optimal control (FOC) of a distributed system is a FOC for which the system dynamics are described by a partial FDE (PFDE). Although a significant amount of work has been done in the area of integer order optimal control, very little work has been published in the area of FOCPs, especially FOC of the distributed system. Several papers dealing with fractional order control are presented in [1–5]. However, these papers do not mention FOCPs either.

Note that calculus of variations play an important role in the area of FOC such as classical optimal control. In this context, the papers [6] and [7] were the first to formulate a fractional variational calculus (FVC). The differential equations of fractionally damped systems and the fractional Euler–Lagrange equations for fractional variational problems were obtained in [8] and [9], respectively. In [10], a general formulation and a numerical scheme for FOCPs in terms of the Riemann–Liouville fractional derivatives (RLFDs) are presented. Similarly, Agrawal and Baleanu [12] formulate FOCPs in terms of the RLFDs, which are then solved by using the Grünwald–Letnikov approach. In [13], a formulation and a numerical scheme for the FOC of a distributed system whose dynamics are described by Caputo fractional derivatives are presented. The FVC is applied to a deterministic and stochastic analysis of FOCPs in [11].

Recently, an eigenfunction expansion-based scheme for the FOC of a two-dimensional (2D) distributed system has been shown in [17]. In addition, Özdemir *et al* [18] consider an axis-symmetric fractional diffusion-wave equation.

In this paper, we present a formulation of the FOC of a 2D distributed system that is defined in polar coordinates.

Therefore, the solutions of the problem are axis-symmetric. Polar coordinates are used to formulate the solution of fractional generalization of the Navier–Stokes equations in [14]. Fractional radial diffusion in a cylinder and in a sphere are presented by [15, 16]. In addition, a formulation and a numerical scheme for FOC for a class of distributed systems whose dynamic constraints are defined in the Caputo sense are considered in [11], whereas this paper considers the problem in terms of RLFDs and in polar coordinates. In other words, we present a formulation for an axis-symmetric FOCP. This paper also finds numerical solutions by using the Grünwald–Letnikov approach.

The paper is organized as follows. In section 2, we give the RLFD definitions and formulation of a FOCP. In section 3, the axis-symmetric FOCP is defined. In section 4, simulation results are analyzed for an initial condition and further analytical solution for $\alpha = 1$. Then, section 5 concludes this work.

2. Preliminaries

In this section, we first define RLFDs, then formulate a FOCP and obtain the necessary terminal conditions for optimality. Several definitions of fractional derivative, which include the Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud and Riesz fractional derivatives, could be used to formulate the problem. We begin with the definition of the left and right RLFDs (LRLFD and RRLFD), which are given as

The LRLFD:

$$ {}_a\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau. \quad (1) $$

The RRLFD:

$$ {}_t\mathrm{D}_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad (2) $$

where $f(.)$ is a time-dependent function, $\Gamma(.)$ is the Gamma function, and α is the order of the derivative such that $n-1 < \alpha < n$, where n is an integer. These derivatives will be denoted as the LRLFD and RRLFD, respectively. When α is an integer the left (forward) and the right (backward) derivatives are replaced with D and −D, respectively, where D is an ordinary differential operator. Note that, in the literature, the Riemann–Liouville fractional derivative generally means the LRLFD. Using the above definitions, a FOCP can be defined in the following: find the optimal control $u(t)$ that minimizes the performance index

$$ J(u) = \int_0^1 F(x, u, t)\,\mathrm{d}t \quad (3) $$

subject to the system dynamic constraints

$$ {}_0\mathrm{D}_t^{\alpha} x = G(x, u, t) \quad (4) $$

and the initial condition

$$ x(0) = x_0, \quad (5) $$

where $x(t)$ and $u(t)$ are the state and the control variables, respectively, and F and G are two arbitrary functions. For $\alpha = 1$, the above problem reduces to a standard optimal control problem. In the case of $\alpha > 1$, additional initial conditions could be necessary. In optimal control formulations, traditionally the differential equations governing the dynamics of the system are written in state space form, in which case the order of the derivatives turns out to be less than 1. For this reason, we consider $0 < \alpha < 1$. The necessary terminal conditions that are determined by using the Lagrange multiplier technique are

$$ {}_0\mathrm{D}_t^{\alpha} x = G(x, u, t), \quad (6) $$

$$ {}_t\mathrm{D}_1^{\alpha} \lambda = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial x}\lambda, \quad (7) $$

$$ \frac{\partial F}{\partial u} + \frac{\partial G}{\partial u}\lambda = 0, \quad (8) $$

where λ is the Lagrange multiplier also known as a co-state variable and

$$ x(0) = x_0 \qquad \text{and} \qquad \lambda(1) = 0. \quad (9) $$

Equations (6)–(8) are called the Euler–Lagrange equations for the FOCP such that these equations determine the necessary conditions for optimality of FOCP.

3. Formulation of an axis-symmetric FOCP

We consider the following problem: determine the control $u(t)$ that minimizes the performance index

$$ J(u) = \frac{1}{2}\int_0^1 \int_0^R r\left[Ax^2(r,t) + Bu^2(r,t)\right] \mathrm{d}r\,\mathrm{d}t \quad (10) $$

subject to the system dynamic constraints

$$ \frac{\partial^{\alpha} x}{\partial t^{\alpha}} = \beta\left(\frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\frac{\partial x}{\partial r}\right) + u(r,t), \quad (11) $$

the initial condition

$$ x(r, 0) = x_0(r), \quad (0 < r < R), \quad (12) $$

and the boundary condition

$$ x(R, t) = 0, \quad (t > 0), \quad (13) $$

where $x(r,t)$ and $u(r,t)$ are the state and the control functions subject to the variables r and t, which represent polar coordinates in the case of axis-symmetry. A and B are two arbitrary functions and R is the radius of the membrane on which the problem is defined. The upper limit of time t is 1 for convenience. This limit can be any positive number. We assume that $x(r,t)$ and $u(r,t)$ can be written as

$$ x(r,t) = \sum_{i=1}^{m} x_i(t) J_0\left(\mu_i \frac{r}{R}\right), \quad (14) $$

$$ u(r,t) = \sum_{i=1}^{m} u_i(t) J_0\left(\mu_i \frac{r}{R}\right), \quad (15) $$

where $J_0\left(\mu_i \frac{r}{R}\right)$, $i = 1, 2, \ldots, m$, are the eigenfunctions that are determined by the method of separation of variables; m is a finite positive integer that theoretically should go to infinity. By substituting equations (14) and (15) into equation (10), we obtain

$$J = \frac{R^2}{4}\int_0^1 \sum_{i=1}^{m} J_1^2(\mu_i)\left[Ax_i^2(t) + Bu_i^2(t)\right]\mathrm{d}t \qquad (16)$$

and substituting equations (14) and (15) into equation (11), and equating the coefficients of $J_0\left(\mu_i \frac{r}{R}\right)$, we get

$${}_0\mathrm{D}_t^\alpha x_i(t) = -\beta\left(\frac{\mu_i}{R}\right)^2 x_i(t) + u_i(t), \quad i = 1, 2, \ldots, m. \qquad (17)$$

Substituting $x(r, t)$ in equation (14) into equation (12), multiplying both sides by $rJ_0\left(\mu_k \frac{r}{R}\right)$ and integrating from 0 to R, we obtain

$$x_i(0) = \frac{2}{R^2 J_1^2(\mu_i)}\int_0^R rJ_0\left(\mu_i \frac{r}{R}\right)x_0(r)\,\mathrm{d}r, \quad i = 1, 2, \ldots, m. \qquad (18)$$

Using equations (6)–(8), and replacing functions F and G in them with F and G in equations (16) and (17), the necessary conditions are written as

$${}_t\mathrm{D}_1^\alpha \lambda_i(t) - \frac{R^2}{2}AJ_1^2(\mu_i)x_i(t) + \beta\left(\frac{\mu_i}{R}\right)^2\lambda_i(t) = 0, \qquad (19)$$

$$\frac{R^2}{2}Bu_i(t)J_1^2(\mu_i) + \lambda_i(t) = 0, \qquad (20)$$

$${}_0\mathrm{D}_t^\alpha x_i(t) + \beta\left(\frac{\mu_i}{R}\right)^2 x_i(t) - u_i(t) = 0, \qquad (21)$$

where $\lambda_i(t)$, $i = 1, 2, \ldots, m$, are the Lagrange multipliers. Arranging the terms of equations (19)–(21), we can obtain

$${}_t\mathrm{D}_1^\alpha u_i(t) = -\frac{A}{B}x_i(t) - \beta\left(\frac{\mu_i}{R}\right)^2 u_i(t), \quad i = 1, 2, \ldots, m. \qquad (22)$$

Note that, for $\alpha = 1$, the fractional differential equations (17) and (22) reduce to the following form:

$$\dot{x}_i(t) = -\beta\left(\frac{\mu_i}{R}\right)^2 x_i(t) + u_i(t), \quad i = 1, 2, \ldots, m, \qquad (23)$$

$$\dot{u}_i(t) = \frac{A}{B}x_i(t) + \beta\left(\frac{\mu_i}{R}\right)^2 u_i(t), \quad i = 1, 2, \ldots, m. \qquad (24)$$

The closed form solution for this set of linear differential equations is given in the appendix.

4. Numerical examples

To solve the FOCP we use the Grünwald–Letnikov numerical scheme. For this purpose, the entire domain is divided into N equal subdomains, and the nodes are labeled as $0, 1, \ldots, N$. The size of each subdomain is $h = \frac{1}{N}$, and the time at node j is defined with $t_j = jh$. Consider the following FDEs corresponding to equations (17) and (22):

$${}_0\mathrm{D}_t^\alpha = ax + bu, \qquad (25)$$

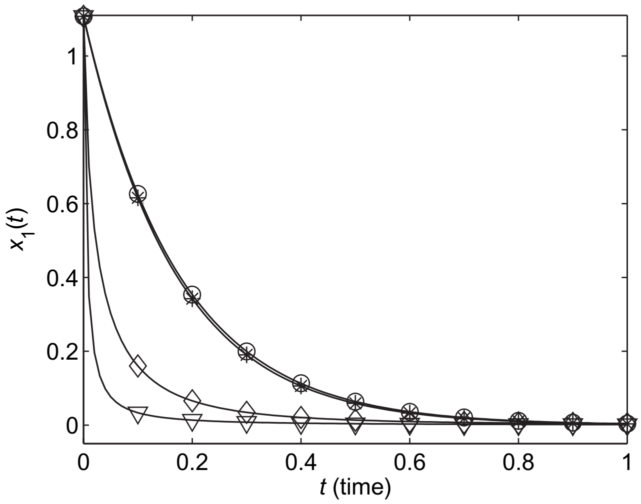

Figure 1. Evolution of state $x_1(t)$ for different values of α and $N = 100$ ($\triangledown$: $\alpha = 0.5$, $\diamond$: $\alpha = 0.75$. For $\alpha = 1$, $\circ$: numerical, $*$: analytical).

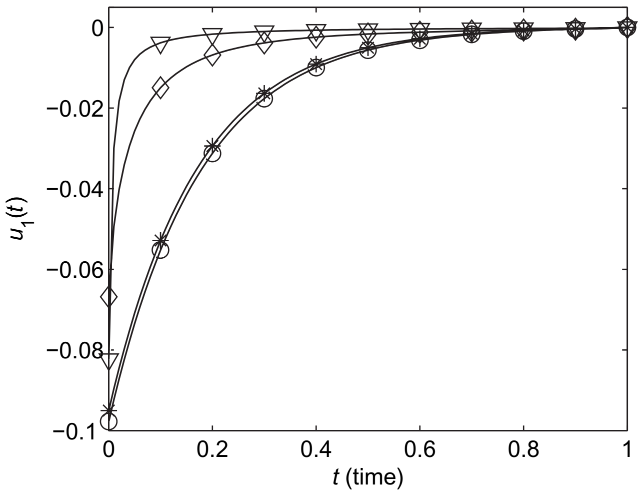

Figure 2. Evolution of control $u_1(t)$ for different values of α and $N = 100$ ($\triangledown$: $\alpha = 0.5$, $\diamond$: $\alpha = 0.75$. For $\alpha = 1$, $\circ$: numerical, $*$: analytical).

$${}_t\mathrm{D}_1^\alpha = cx + du, \qquad (26)$$

where a, b, c and d are arbitrary constants. The Grünwald–Letnikov approximation of the left and the right RLFDs at node M can be given in the following form:

$${}_0\mathrm{D}_t^\alpha x = \frac{1}{h^\alpha}\sum_{j=0}^{M} w_j^{(\alpha)} x(hM - jh), \qquad (27)$$

$${}_t\mathrm{D}_1^\alpha u = \frac{1}{h^\alpha}\sum_{j=0}^{N-M} w_j^{(\alpha)} u(hM + jh) \qquad (28)$$

where

$$w_0^\alpha = 1, \quad w_j^\alpha = \left(1 - \frac{\alpha + 1}{j}\right) w_{(j-1)}^\alpha. \qquad (29)$$

Therefore, the above equations reduce to

$$\frac{1}{h^\alpha}\sum_{j=0}^{M} w_j^{(\alpha)} x(hM - jh) = ax(Mh) + bu(Mh), \qquad (30)$$

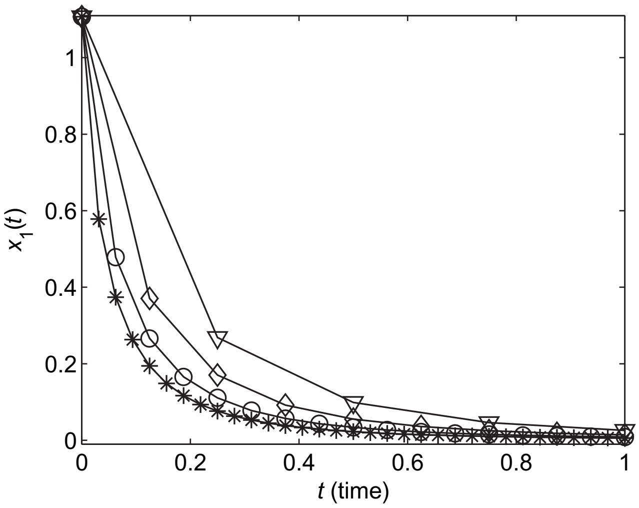

Figure 3. Evolution of state $x_1(t)$ for $\alpha = 0.75$ and different values of N ($\triangledown$: $N = 4$, $\diamond$: $N = 8$, $\circ$: $N = 16$, $*$: $N = 32$).

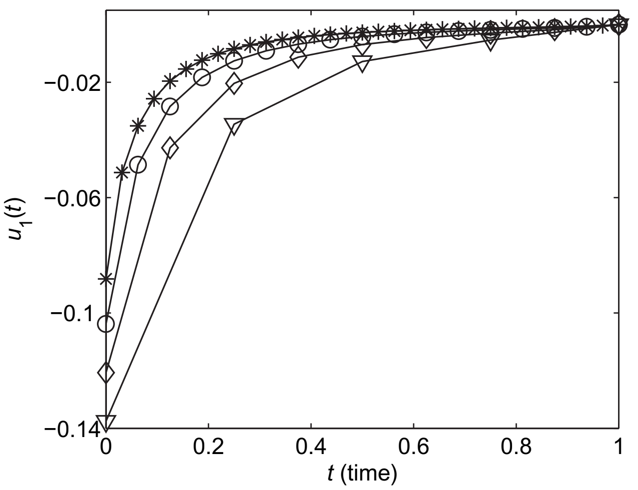

Figure 4. Evolution of control $u_1(t)$ for $\alpha = 0.75$ and different values of N ($\triangledown$: $N = 4$, $\diamond$: $N = 8$, $\circ$: $N = 16$, $*$: $N = 32$).

$$\frac{1}{h^\alpha}\sum_{j=0}^{N-M} w_j^{(\alpha)} u(hM + jh) = cx(Mh) + du(Mh) \tag{31}$$

and

$$x(0) = x_0, \quad u(1) = 0. \tag{32}$$

In this section, we analyze some simulation results for the axis-symmetric FOCP defined by equations (3)–(9) for $t > 0$, $0 < \alpha < 1$ and $r \in [0, R]$. An example is given to demonstrate the applicability of the numerical approach. For different values of α, we compare the numerical results obtained by the Grünwald–Letnikov approach. For simulation purposes, we take the following data: $\beta = R = A = B = 1$ and the term number $m = 5$. The comparison of the analytical and numerical results for $\alpha = 1$ is obtained. Let us consider the following initial condition:

$$x_0(r) = 1 - \left(\frac{r}{R}\right)^2. \tag{33}$$

Substituting equation (33) into (18), we obtain

$$x_i(0) = \frac{8}{\mu_i^3 J_1(\mu_i)}, \quad i = 1, 2, \ldots, m. \tag{34}$$

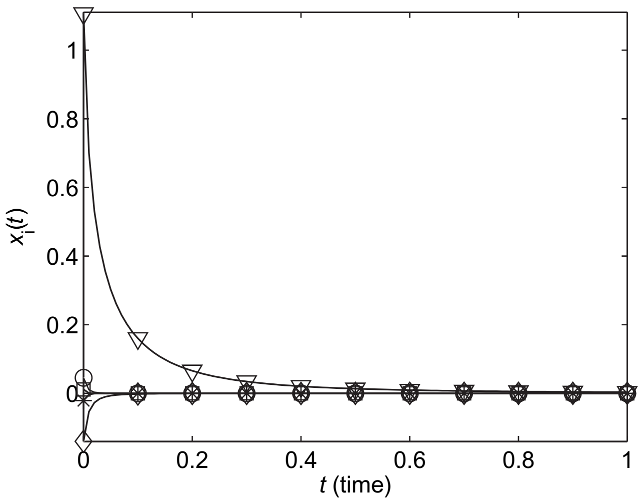

Figure 5. Evolution of states $x_i(t)$ for $m = 5$, $\alpha = 0.75$ and $N = 100$ ($\triangledown$: $x_1(t)$, $\diamond$: $x_2(t)$, $\circ$: $x_3(t)$, $*$: $x_4(t)$, $\square$: $x_5(t)$).

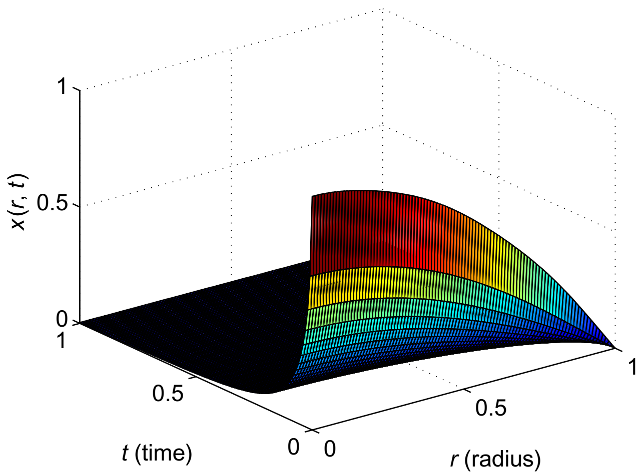

Figure 6. Evolution of state function $x(r, t)$ as a function of r and t for $\alpha = 0.75$ and $N = 100$.

Figures 1 and 2 show the analytical results (for $\alpha = 1$) and the numerical results (for $\alpha = 0.5, 0.75$ and 1) for the state $x_1(t)$ and the control $u_1(t)$ for $N = 100$. In these figures, it can be observed that the analytical and numerical solutions overlap. Thus, as α approaches 1, the solution for the integer-order system is recovered. Figures 3 and 4 show the numerical results for the state $x_1(t)$ and the control $u_1(t)$, respectively, for $\alpha = 0.75$ and different values of N ($N = 4, 8, 16$ and 32). Note that the solutions converge as step number N is increased, which indicates that the algorithm is stable. For computation aspects in equation (3), this series is truncated after m terms. From figure 5, it can be seen that the first term $x_1(t)$ is different from the rest of the terms, which are very close to 0. Therefore, we only need a few terms to calculate. Figures 6 and 7 show the surface of the state $x(r, t)$ and the control $u(r, t)$ for $\alpha = 1$, respectively. These surfaces are plotted for $\alpha = 0.75$ and $N = 100$.

5. Conclusions

An axis-symmetric FOCP was defined in terms of the RLFDs. The performance index of the FOCP was considered as

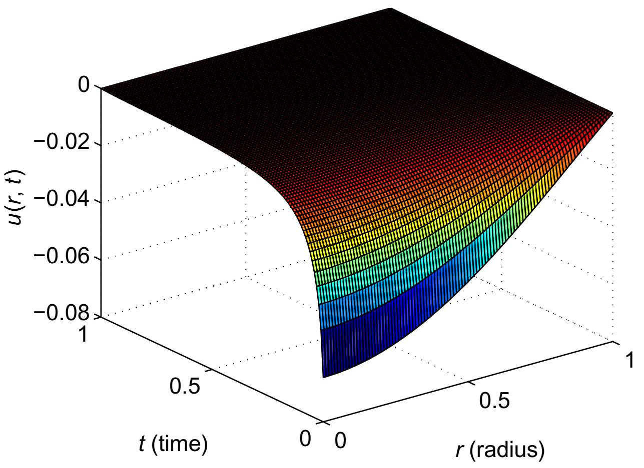

Figure 7. Evolution of control function $u(r, t)$ as a function of r and t for $\alpha = 0.75$ and $N = 100$.

a function of the state and the control variables, and the system dynamic constraints are described as PFDEs. The method of separation of variables was used to find the closed form solution of the problem. Eigenfunctions determined as Bessel functions were used to eliminate the space parameter terms. The Grünwald–Letnikov approach was used to develop an algorithm for the numerical solution of the problem. Simulation results showed that analytical and numerical solutions overlap for $\alpha = 1$. Numerical results were compared for different values of α and N.

Appendix A.

The FDW system reduces to the following form for $\alpha = 1$:

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + u_i(t), \\ \dot{u}_i(t) = e_0 x_i(t) + a_i u_i(t), \end{cases} \quad i = 0, 1, \ldots, m, \tag{A.1}$$

and terminal conditions are rewritten as

$$\begin{cases} x_i(0) = x_{i0}, \\ u_i(1) = 0, \end{cases} \quad i = 0, 1, \ldots, m, \tag{A.2}$$

where

$$a_i = \beta \left(\frac{\mu_i}{R}\right)^2, \tag{A.3}$$

$$e_0 = \frac{A}{B}. \tag{A.4}$$

Equation (A.1) leads to

$$\ddot{x}_i - k_i^2 x_i = 0, \tag{A.5}$$

where

$$k_i = \sqrt{e_0 + a_i^2}.$$

Using equations (A.1) and (A.2), the solution of equation (A.5) is given as

$$x_i(t) = x_{i0} \left[\frac{k_i \cosh(k_i(1-t)) + a_i \sinh(k_i(1-t))}{k_i \cosh(k_i) + a_i \sinh(k_i)}\right]. \tag{A.6}$$

Using equations (A.1) and (A.6), we obtain

$$u_i(t) = x_{i0} \left[\frac{(a_i^2 - k_i^2) \sinh(k_i(1-t))}{k_i \cosh(k_i) + a_i \sinh(k_i)}\right]. \tag{A.7}$$

Finally, $x(r, t)$ and $u(r, t)$ are given by equations (14) and (15), where $x_i(t)$ and $u_i(t)$ are obtained by (A.6) and (A.7).

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