

CONTROL OF THERMAL STRESSES IN AXISSYMMETRIC PROBLEMS OF FRACTIONAL THERMOELASTICITY FOR AN INFINITE CYLINDRICAL DOMAIN

by

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In this paper, we study a control problem of thermal stresses in an infinite cylindrical body. The temperature distribution is defined by the time-fractional heat conduction equation with the Caputo derivative of the order $0 < \alpha \leq 2$. The problem is formulated for axisymmetric case. The sought-for heat source function is treated as a control of stress and displacement components. For this purpose, we find the control function which guarantees the distribution of the stress component in some section of a body and at some time at a prescribed level. Integral transform technique is applied to obtain the desired control function, stresses, and displacement components. Numerical results are illustrated graphically.

Key words: *control of thermal stresses, time-fractional heat conduction, Caputo derivative*

Introduction

The standard heat conduction equation is based on the classical Fourier law which relates the heat flux vector to the temperature gradient. The classical thermoelasticity theory deals with stresses caused by the temperature field obtained from the classical heat conduction equation. To have an extensive knowledge on the classical theory, the researchers are referred to the books by Noda *et al.* [1], Nowacki [2], and Parkus [3].

In non-classical theories, the Fourier law and the parabolic heat conduction equation are replaced by more general equations. The generalized Fourier law which relates the time non-local dependence between the heat flux vector and the temperature gradient with the *long-tail* power kernel is defined [4-8]:

$$\bar{q}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \text{grad } T(\tau) d\tau, \quad 0 < \alpha \leq 1 \quad (1)$$

$$\bar{q}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \text{grad } T(\tau) d\tau, \quad 1 < \alpha \leq 2 \quad (2)$$

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or can be interpreted in terms of the Riemann-Liouville fractional derivatives and integrals (essentials of the fractional calculus are presented in [9-12]):

$$\bar{q}(t) = -kD_{RL}^{1-\alpha} \text{grad } T(t), \quad 0 < \alpha \leq 1 \quad (3)$$

$$\bar{q}(t) = -kI_{RL}^{\alpha-1} \text{grad } T(t), \quad 1 < \alpha \leq 2 \quad (4)$$

The constitutive eqs. (3) and (4) result in the time-fractional heat conduction equation with the Caputo derivative:

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \quad 0 < \alpha \leq 2 \quad (5)$$

It is worthy of note that eq. (5) is also a mathematical model of important physical processes in fractals, biological systems, random and disordered media, colloid and porous materials, polymer chains, transport of molecules, and so on [5, 13-16]. Additional applications of fractional calculus to various problems of diffusion, heat conduction, and mechanics of can be found in [17-24]. The solutions of eq. (5) in Cartesian, cylindrical, and spherical co-ordinates under different initial and boundary conditions are presented in the book [25].

The theory of fractional thermoelasticity deals with mechanical and thermal effects such as stresses, strains and displacements in an elastic body in which anomalous heat conduction is observed [4-7]. By neglecting the mechanical oscillations, *i.e.* assuming that the relaxation time of mechanical oscillations is significantly less than the relaxation time of heat conduction process, a quasi-static uncoupled theory of thermal stresses based on eq. (5) was proposed by Povstenko [4]. Povstenko [8] sums up investigations in this field.

In the framework of the classical theory of thermoelasticity, various control problems were considered by Vigak [26, 27]. Solving optimal control problem of 1-D non-stationary temperature regimes with some restrictions on the control and the thermal stresses on a prescribed stressed surface of a solid body was researched in [28]. Vigak *et al.* [29] developed a method to solve the problems of optimally rapid control of heating elastic and viscoelastic isotropic bodies with the restrictions placed on thermal stresses. Vigak and Lisevich [30] formulated the problem of optimizing the rapidity of the control of a heating thick-walled cylinder when axial and angular thermal stresses are restricted at fixed points.

Recently, Ozdemir *et al.* [31] first formulate the boundary optimal control of temperature distribution defined by a 1-D time-fractional heat conduction equation. In the present work, we focus on the control problem of axisymmetric thermal stresses in an infinite cylindrical domain. For this purpose, we use the heat source as a control function which guarantees the distribution of stress component at a prescribed level. The temperature regime is determined by the time-fractional heat conduction equation. Integral transform technique is applied to obtain the temperature of a body under the effect of a heat source control as well as the stress and displacement components. After some numerical calculations, we illustrate the behavior of control function and the stress tensor components.

Statement of the problem

The governing equations of the theory of fractional thermoelasticity are [4, 8]:

- the equilibrium equation in terms of displacements

$$\mu\Delta u + (\lambda + \mu) \text{grad div } \vec{u} = \beta_r K_T \text{grad } T \quad (6)$$

– the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (2\text{tr } \mathbf{e} - \beta_T K_T T)\mathbf{I} \quad (7)$$

– and the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T + Q, \quad 0 < \alpha \leq 2 \quad (8)$$

where \vec{u} is the displacement vector, $\boldsymbol{\sigma}$ – the stress tensor, \mathbf{e} – the linear strain tensor, T – the temperature, Q – the heat source term, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T – the thermal coefficient of volumetric expansion, a – the thermal diffusivity, \mathbf{I} – the unit tensor, and $\partial^\alpha/\partial t^\alpha$ is the Caputo fractional derivative [2, 4, 12]:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \quad n-1 < \alpha < n \quad (9)$$

where $\Gamma(\bullet)$ is the Euler gamma function.

In this paper, we will consider the axisymmetric fractional heat conduction equation:

$$\frac{d^\alpha T}{dt^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) + Q(r, z, t), \quad 0 < \alpha \leq 2 \quad (10)$$

in the finite domain $0 \leq r < \infty$, $-\infty < z < \infty$, $0 < t < \infty$ under the initial conditions:

$$\begin{aligned} t=0: \quad T &= f_1(r, z), \quad 0 < \alpha \leq 2, \\ t=0: \quad \frac{\partial T}{\partial t} &= f_2(r, z), \quad 1 < \alpha \leq 2 \end{aligned} \quad (11)$$

The zero conditions at infinity:

$$\lim_{r \rightarrow \infty} T(r, z, t) = 0, \quad \lim_{z \rightarrow \pm\infty} T(r, z, t) = 0 \quad (12)$$

are also assumed. The heat source term $Q(r, z, t)$ in the heat conduction eq. (10) is treated as a control that guarantees the fulfilment of restrictions on the displacement components or on the stress components, for example, guarantees the required distribution of one of the following quantities [27, 28]:

$$u_r(r, z, t) = \varphi(r, z, t), \quad \sigma_{zz}(r, z, t) = \psi(r, z, t) \quad (13)$$

where $\varphi(r, z, t)$ and $\psi(r, z, t)$ are given functions.

Representation of the displacement vector and stress tensor in the axisymmetric case

As in the classical theory of thermal stresses [2, 3], we can introduce the displacement potential Φ :

$$\vec{u} = \text{grad } \Phi \quad (14)$$

In the quasi-static case, from the equilibrium eq. (6) we get:

$$\Delta \Phi = mT, \quad m = \frac{1+\nu}{1-\nu} \frac{\beta_T}{3} \quad (15)$$

where ν is the Poisson ratio.

In cylindrical co-ordinates in the case of axial symmetry, we have:

$$u_r = \frac{\partial \Phi}{\partial r}, \quad u_z = \frac{\partial \Phi}{\partial z} \quad (16)$$

and

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = mT \quad (17)$$

$$\sigma_{rr} = 2\mu \left(\frac{\partial^2 \Phi}{\partial r^2} - \Delta \Phi \right) \quad (18)$$

$$\sigma_{\theta\theta} = 2\mu \left(\frac{1}{r} \frac{\partial^2 \Phi}{\partial r^2} - \Delta \Phi \right) \quad (19)$$

$$\sigma_{zz} = 2\mu \left(\frac{\partial^2 \Phi}{\partial z^2} - \Delta \Phi \right) \quad (20)$$

$$\sigma_{rz} = 2\mu \frac{\partial^2 \Phi}{\partial r \partial z} \quad (21)$$

In what follows we will use the Hankel transform of order n with respect to the radial co-ordinate r [25, 32]:

$$H_{(n)} \{f(r)\} = \int_0^\infty f(r) J_n(r\xi) r dr \quad (22)$$

$$f(r) = \int_0^\infty H_{(n)} \{f(r)\} J_n(r\xi) \xi d\xi \quad (23)$$

In the case $n = 0$, simultaneously with the notation $H_0\{f(r)\}$, the notation $H_0\{f(r)\} = \hat{f}(\xi)$ will be used. From eqs. (16)-(21), we have (see [2], [33]):

$$H_{(1)} \{u_r\} = -\xi \hat{\Phi} \quad (24)$$

$$H_{(0)} \{u_z\} = \frac{\partial \hat{\Phi}}{\partial z} \quad (25)$$

$$H_{(0)} \{\sigma_{rr} + \sigma_{\theta\theta}\} = 2\mu \xi^2 \hat{\Phi} - 4\mu \frac{\partial^2 \hat{\Phi}}{\partial z^2} \quad (26)$$

$$H_{(2)} \{\sigma_{rr} - \sigma_{\theta\theta}\} = 2\mu \xi^2 \hat{\Phi} \quad (27)$$

$$H_{(0)} \{\sigma_{zz}\} = 2\mu \xi^2 \hat{\Phi} \quad (28)$$

$$H_{(1)} \{\sigma_{rz}\} = -2\mu \xi \frac{\partial \hat{\Phi}}{\partial z} \quad (29)$$

Solution of the time-fractional heat conduction equation

Now, we consider the axisymmetric fractional heat conduction equation:

$$\frac{d^\alpha T}{dt^\alpha} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) + F(r) \delta(z - z_0) H(t), \quad 0 < \alpha \leq 2 \quad (30)$$

under the zero initial conditions:

$$\begin{aligned} t = 0: \quad T &= 0, \quad 0 < \alpha \leq 2, \\ t = 0: \quad \frac{\partial T}{\partial t} &= 0, \quad 1 < \alpha \leq 2 \end{aligned} \quad (31)$$

In the heat source term in eq. (30), $\delta(z)$ is the Dirac delta function and, $H(t)$ is the Heaviside unit step function. In the general case, the function $F(r)$ should be found as a control that guarantees the prescribed distribution of displacements or stresses at some section of space. We can also consider similar problem with $\delta(t)$ instead $H(t)$ in the source term in eq. (30). This can be treated as some fundamental solution.

The Laplace transform with respect to time, t , the Hankel transform with respect to the radial co-ordinate, r , and the exponential Fourier transform with respect to the spatial co-ordinate, z , give:

$$\widehat{\widetilde{T}}^*(\xi, \eta, s) = \frac{1}{\sqrt{2\pi}} \widehat{F}(\xi) e^{i z_0 \eta} \frac{1}{a(\xi^2 + \eta^2)} \left[\frac{1}{s} - \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2 + \eta^2)} \right] \quad (32)$$

where the asterisk denotes the Laplace transform, s – the transform variable, and the tilde marks the Fourier transform with η being the transform variable.

Let us recall the Laplace transform rules for the Caputo derivative [10,11,14]:

$$L \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n \quad (33)$$

Taking into account the following formula for the inverse Laplace transform:

$$L^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + b} \right\} = E_\alpha(-bt^\alpha) \quad (34)$$

where $E_\alpha(z)$ is the Mittag-Leffler function in one parameter α :

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C} \quad (35)$$

we get:

$$\widehat{\widetilde{T}}(\xi, \eta, t) = \frac{1}{\sqrt{2\pi}} \widehat{F}(\xi) e^{i z_0 \eta} \frac{1}{a(\xi^2 + \eta^2)} \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} \quad (36)$$

Application of the Hankel transform with respect to the radial co-ordinate, r , and the exponential Fourier transform with respect to the spatial co-ordinate, z , to eq. (15) gives:

$$\widehat{\Phi}(\xi, \eta, t) = -\frac{m}{\xi^2 + \eta^2} \widehat{T} \quad (37)$$

or

$$\widehat{\Phi}(\xi, \eta, t) = -\frac{1}{\sqrt{2\pi}} \frac{m}{a(\xi^2 + \eta^2)^2} \widehat{F}(\xi) e^{i z_0 \eta} \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} \quad (38)$$

and

$$\widehat{\Phi}(\xi, z, t) = -\frac{m}{\pi a} \widehat{F}(\xi) \int_0^\infty \frac{\cos[(z - z_0)\eta]}{(\xi^2 + \eta^2)^2} \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} d\eta \quad (39)$$

According to eq. (28), the Hankel transform of the stress component, σ_{zz} , has the form:

$$\widehat{\sigma}_{zz}(\xi, z, t) = -\frac{2\mu m \xi^2}{\pi a} \widehat{F}(\xi) \int_0^\infty \frac{\cos[(z - z_0)\eta]}{(\xi^2 + \eta^2)^2} \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} d\eta \quad (40)$$

In this paper, we consider the following control problem: find the control function $F(r)$ which guarantees the prescribed distribution of the stress component, σ_{zz} , at the section $z = 0$ at some time, t_0 :

$$\sigma_{zz}(r, 0, t_0) = -\sigma_0 e^{-\gamma\sqrt{r^2 + b^2}} \quad (41)$$

where γ and b are some constants. The Hankel transform applied to eq. (41) gives:

$$\widehat{\sigma}_{zz}(\xi, 0, t_0) = -\sigma_0 \int_0^\infty e^{-\gamma\sqrt{r^2 + b^2}} J_0(r\xi) r dr \quad (42)$$

or after evaluation of integral in the right-hand side of eq. (42) (see [34]):

$$\widehat{\sigma}_{zz}(\xi, 0, t_0) = -\sigma_0 \frac{\gamma}{(\xi^2 + \gamma^2)^{3/2}} \left(1 + b\sqrt{\xi^2 + \gamma^2} \right) e^{-b\sqrt{\xi^2 + \gamma^2}} \quad (43)$$

Comparing eqs. (40) and (43) allows us to obtain the Hankel transform of the control function (for simplicity, we consider $z_0 = 0$):

$$\widehat{F}(\xi) = \frac{\pi a \gamma \sigma_0}{2\mu m} \frac{1}{(\xi^2 + \gamma^2)^{3/2}} \frac{1}{\xi^2} \left(1 + b\sqrt{\xi^2 + \gamma^2} \right) e^{-b\sqrt{\xi^2 + \gamma^2}} \frac{1}{\Delta(\xi)} \quad (44)$$

where

$$\Delta(\xi) = \int_0^\infty (\xi^2 + \eta^2)^{-2} \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t_0^\alpha \right] \right\} d\eta \quad (45)$$

For the displacement vector components and the stress tensor components we finally get (with $z_0 = 0$):

$$u_r = \frac{m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\xi}{(\xi^2 + \eta^2)^2} \cos(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} J_1(r\xi) \xi d\xi d\eta \quad (46)$$

$$u_z = \frac{m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\eta}{(\xi^2 + \eta^2)^2} \sin(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2) t^\alpha \right] \right\} J_0(r\xi) \xi d\xi d\eta \quad (47)$$

$$\sigma_{rr} + \sigma_{\theta\theta} = -\frac{2\mu m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\xi^2 + 2\eta^2}{(\xi^2 + \eta^2)^2} \cos(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2)t^\alpha \right] \right\} J_0(r\xi) \xi \, d\xi \, d\eta \quad (48)$$

$$\sigma_{rr} - \sigma_{\theta\theta} = -\frac{2\mu m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\xi^2}{(\xi^2 + \eta^2)^2} \cos(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2)t^\alpha \right] \right\} J_2(r\xi) \xi \, d\xi \, d\eta \quad (49)$$

$$\sigma_{zz} = -\frac{2\mu m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\xi^2}{(\xi^2 + \eta^2)^2} \cos(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2)t^\alpha \right] \right\} J_0(r\xi) \xi \, d\xi \, d\eta \quad (50)$$

$$\sigma_{rz} = -\frac{2\mu m}{a\pi} \int_0^\infty \int_0^\infty \widehat{F}(\xi) \frac{\xi\eta}{(\xi^2 + \eta^2)^2} \sin(z\eta) \left\{ 1 - E_\alpha \left[-a(\xi^2 + \eta^2)t^\alpha \right] \right\} J_1(r\xi) \xi \, d\xi \, d\eta \quad (51)$$

Numerical results

For numerical calculations, the following non-dimensional quantities are introduced:

$$\bar{r} = \gamma r, \quad \bar{b} = \gamma b, \quad \tau_0 = a\gamma^2 t_0^\alpha, \quad \bar{F} = \frac{\mu m}{a\gamma\sigma_0}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0} \quad (52)$$

Figure 1 shows the dependence of the control function, $F(r)$, on the distance for various values of the order of fractional derivative. Figures 2-4 present the distribution of the stress

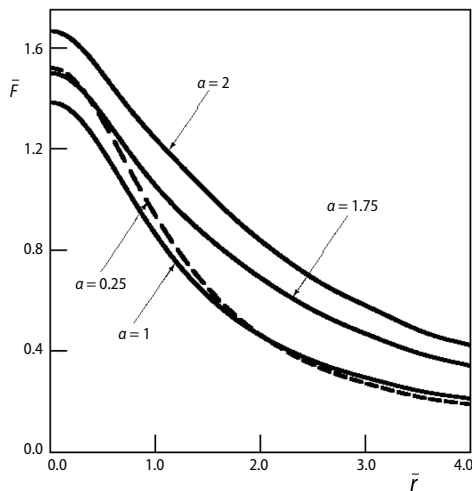


Figure 1. Dependence of the control function $F(r)$ on distance for different values of the order of fractional derivative ($z = 0, \bar{b} = 1, \tau_0 = 1$)

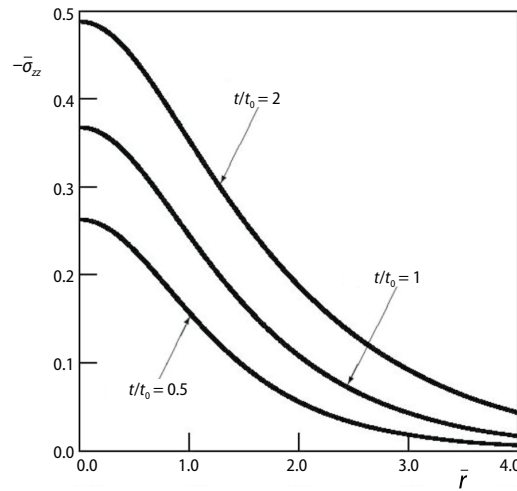


Figure 2. Dependence of the stress component on distance for different values of time ($\alpha = 1, z = 0$)

tensor, σ_{zz} , in the section $z = 0$ for different values of time. In all this figures, the curves corresponding to the values $t/t_0 = 1$ are the same and represent the restriction eq. (41). Similar results can be obtained for other components of the stress tensor.

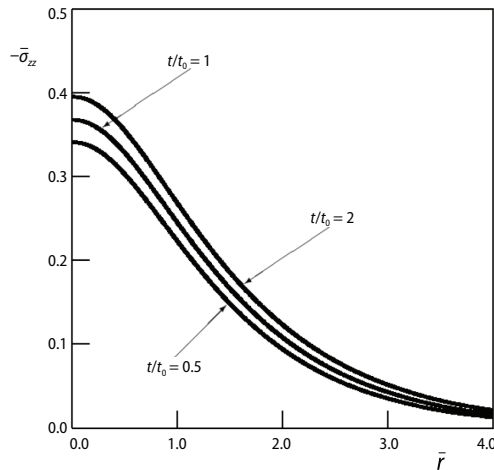


Figure 3. Dependence of the stress component on distance for different values of time ($\alpha = 0.25, z = 0$)

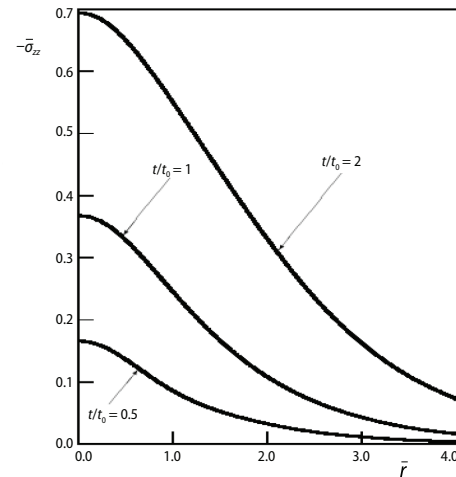


Figure 4. Dependence of the stress component on distance for different values of time ($\alpha = 1.75, z = 0$)

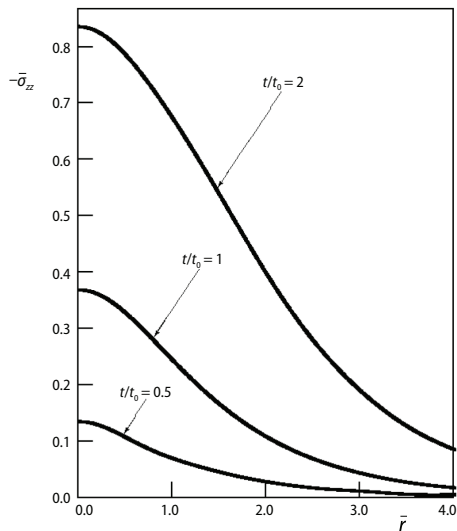


Figure 5. Dependence of the stress component on distance for different values of time ($\alpha = 2, z = 0$)

that fractional thermoelasticity in the case $1 < \alpha < 2$ interpolates the classical theory of elasticity and thermoelasticity without energy dissipation proposed by Green and Naghdi [35].

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Conclusion

We have considered the control problem for an infinite solid in cylindrical co-ordinates in the framework of fractional thermoelasticity. The control function was obtained from the condition that the stress component, σ_{zz} , at the section $z = 0$ and at the time, t_0 , coincides with the given distribution. The numerical calculations show the significant dependence of the obtained results on the order of time-fractional derivative. In fig. 1, for better distinguishing between the curves for $\alpha = 0.25$ and $\alpha = 1$ we have used the dashed line for the curve corresponding to the order of fractional derivative $\alpha = 0.25$. In figs. 2-4, we have used the same scale to show the difference in the behavior of the solution for different values of α , whereas in fig. 5 the scale is slightly reduced. For $t < t_0$, increase of α leads to smaller values of stresses, and for $t > t_0$ the dependence is inverse: increase of α results in increase of stresses. It should be also emphasized

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