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ORIGINAL RESEARCH

# Some new contractive mappings on $S$-metric spaces and their relationships with the mapping (S25)

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**Abstract** Recently, $S$-metric spaces are introduced as a generalization of metric spaces. In this paper, we consider the relationships between of an $S$-metric space and a metric space, and give an example of an $S$-metric which does not generate a metric. Then, we introduce new contractive mappings on $S$-metric spaces and investigate relationships among them by counterexamples. In addition, we obtain new fixed point theorems on $S$-metric spaces.

**Keywords** $S$-metric space · Fixed point theorem · Periodic point · Diameter

**Mathematics Subject Classification** 54E35 · 54E40 · 54E45 · 54E50

## Introduction

Recently, Sedghi, Shobe, and Aliouche have defined the concept of an $S$-metric space as a generalization of a metric space in [14] as follows:

**Definition 1** [14] Let $X$ be a nonempty set, and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$ :

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

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Then, $S$ is called an $S$-metric on $X$ and the pair $(X, S)$ is called an $S$ -metric space.

The fixed point theory on various metric spaces was studied by many authors. For example, A. Aghajani, M. Abbas, and J. R. Roshan proved some common fixed point results for four mappings satisfying generalized weak contractive condition on partially ordered complete $b$-metric spaces [1]; T. V. An, N. V. Dung, and V. T. L. Hang studied some fixed point theorems on $G$-metric spaces [2]; N. V. Dung, N. T. Hieu, and S. Radojevic proved some fixed point theorems on partially ordered $S$-metric spaces [6]. Gupta and Deep studied some fixed point results using mixed weakly monotone property and altering distance function in the setting of $S$-metric space [9]. The present authors investigated some generalized fixed point theorems on a complete $S$-metric space [11].

Motivated by the above studies, our aim is to obtain new fixed point theorems on $S$-metric spaces related to Rhoades' conditions.

We recall Rhoades' conditions in $(X, d)$ and $(X, S)$, respectively.

Let $(X, d)$ be a complete metric space and $T$ be a self-mapping of $X$. In [13], $T$ is called a Rhoades' mapping (**RN**), $(N = 25, 50, 75, 100, 125)$ if the following condition is satisfied, respectively:

(**R25**) $$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for each $x, y \in X$, $x \neq y$.

(**R50**) There exists a positive integer $p$, such that

$$d(T^p x, T^p y) < \max\{d(x, y), d(x, T^p x), d(y, T^p y), d(x, T^p y), d(y, T^p x)\},$$

for each $x, y \in X$, $x \neq y$.

(**R75**) There exist positive integers $p$, $q$, such that

$$d(T^p x, T^q y) < \max\{d(x,y), d(x,T^p x), d(y,T^q y), d(x,T^q y), d(y,T^p x)\},$$

for each $x, y \in X$, $x \neq y$.

(**R100**) There exists a positive integer $p(x)$, such that

$$d(T^{p(x)}x, T^{p(x)}y) < \max\{d(x,y), d(x,T^{p(x)}x), d(y,T^{p(x)}y), d(x,T^{p(x)}y), d(y,T^{p(x)}x)\},$$

for any given $x$, every $y \in X$, $x \neq y$.

(**R125**) There exists a positive integer $p(x, y)$, such that

$$d(T^{p(x,y)}x, T^{p(x,y)}y) < \max\{d(x,y), d(x,T^{p(x,y)}x), d(y,T^{p(x,y)}y), d(x,T^{p(x,y)}y), d(y,T^{p(x,y)}x)\},$$

for any given $x, y \in X$, $x \neq y$.

Let ($X$, $S$) be an $S$-metric space and $T$ be a self-mapping of $X$. In [12], the present authors defined Rhoades' condition (**S25**) on ($X$, $S$) as follows:

(**S25**) $$S(Tx,Tx,Ty) < max\{S(x,x,y), S(Tx,Tx,x), S(Ty,Ty,y), S(Ty,Ty,x), S(Tx,Tx,y)\},$$

for each $x, y \in X$, $x \neq y$.

In this paper, we consider some forms of Rhoades' conditions and give some fixed point theorems on $S$-metric spaces. In Sect. 2, we investigate relationships between metric spaces and $S$-metric spaces. It is known that every metric generates an $S$-metric, and in [10], it was given an example of an $S$-metric which is not generated by a metric. Here, we give a new example of an $S$-metric which is not generated by a metric and use this new $S$-metric in the next sections. In [8], it is mentioned that every $S$-metric defines a metric. However, we give a counterexample to this result. We obtain an example of an $S$-metric which does not generate a metric. We introduce new contractive mappings, such as (**S50**), (**S75**), (**S100**), and (**S125**), and also study relations among them by counterexamples. In Sect. 3, we investigate some new fixed point theorems using periodic index on $S$-metric spaces for the contractive mappings defined in Sect. 2. In Sect. 4, we define the condition (**Q25**) and give new fixed point theorems on $S$-metric spaces.

## New contractive mappings on $S$-metric spaces

In this section, we introduce new types of Rhoades' conditions on $S$-metric spaces, such as (**S50**), (**S75**), (**S100**), and (**S125**). At first, we recall some definitions and theorems.

**Definition 2** [14] Let ($X$, $S$) be an $S$-metric space and $A \subset X$.

1. A sequence $\{x_n\}$ in $X$ converges to $x$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ for each $\varepsilon > 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $\lim_{n\to\infty} S(x_n, x_n, x) = 0$.
2. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, there exists $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$, $S(x_n, x_n, x_m) < \varepsilon$ for each $\varepsilon > 0$.
3. The $S$-metric space ($X$, $S$) is called complete if every Cauchy sequence is convergent.

Lemma 1 [14] Let ($X$, $S$) be an $S$-metric space. Then,

$$S(x,x,y) = S(y,y,x). \tag{2.1}$$

The relation between a metric and an $S$-metric is given in [10] as follows:

Lemma 2 [10] Let ($X$, $d$) be a metric space. Then, the following properties are satisfied:

1. $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in X$ is an $S$-metric on $X$.
2. $x_n \to x$ in ($X$, $d$) if and only if $x_n \to x$ in $(X, S_d)$.
3. $\{x_n\}$ is Cauchy in ($X$, $d$) if and only if $\{x_n\}$ is Cauchy in $(X, S_d)$.
4. ($X$, $d$) is complete if and only if $(X, S_d)$ is complete.

We call the metric $S_d$ as the $S$-metric generated by $d$.

Note that there exists an $S$-metric $S$ satisfying $S \neq S_d$ for all metrics $d$ [10]. Now, we give an another example which shows that there exists an $S$-metric $S$ satisfying $S \neq S_d$ for all metrics $d$.

Example 1 Let $X = \mathbb{R}$ and define the function

$$S(x,y,z) = |x - z| + |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$. Then, ($X$, $S$) is an $S$-metric space. Now, we prove that there does not exist any metric $d$, such that $S = S_d$. Conversely, suppose that there exists a metric $d$, such that

$$S(x,y,z) = d(x,z) + d(y,z),$$

for all $x, y, z \in \mathbb{R}$. Then, we obtain

$$S(x,x,z) = 2d(x,z) = 2|x - z| \text{ and } d(x,z) = |x - z|$$

and

$$S(y,y,z) = 2d(y,z) = 2|y - z| \text{ and } d(y,z) = |y - z|,$$

for all $x, y, z \in \mathbb{R}$. Hence, we have

$$|x - z| + |x + z - 2y| = |x - z| + |y - z|,$$

which is a contradiction. Therefore, $S \neq S_d$.

Now, we give the relationship between the Rhoades' condition (**R25**) and (**S25**).

**Proposition 1** *Let (X, d) be a complete metric space, $(X, S_d)$ be the S-metric space obtained by the S-metric generated by d, and T be a self-mapping of X. If T satisfies the inequality* (**R25**)*, then T satisfies the inequality* (**S25**).

*Proof* Let the inequality (**R25**) be satisfied. Using the inequality (**R25**) and (2.1), we have

$$\begin{aligned}S_d(Tx,Tx,Ty) &= d(Tx,Ty)+d(Tx,Ty)=2d(Tx,Ty)\\ &< 2\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\\ &= \max\{2d(x,y),2d(x,Tx),2d(y,Ty),2d(x,Ty),2d(y,Tx)\}\\ &= \max\{S_d(x,x,y),S_d(x,x,Tx),S_d(y,y,Ty),S_d(x,x,Ty),\\ &\quad S_d(y,y,Tx)\}\\ &= \max\{S_d(x,x,y),S_d(Tx,Tx,x),S_d(Ty,Ty,y),S_d(Ty,Ty,x),\\ &\quad S_d(Tx,Tx,y)\},\end{aligned}$$

and so, the inequality (**S25**) is satisfied on $(X, S_d)$. □

Let (X, S) be any S-metric space. In [8], it was shown that every S-metric on X defines a metric $d_S$ on X as follows:

$$d_S(x,y) = S(x,x,y) + S(y,y,x), \tag{2.2}$$

for all $x, y \in X$. However, the function $d_S(x,y)$ defined in (2.2) does not always define a metric because of the reason that the triangle inequality does not satisfied for all elements of X everywhen. If the S-metric is generated by a metric d on X, then it can be easily seen that the function $d_S$ is a metric on X, especially we have $d_S(x,y) = 4d(x,y)$. However, if we consider an S-metric which is not generated by any metric, then $d_S$ can or cannot be a metric on X. We call this metric $d_S$ as the metric generated by S in the case $d_S$ is a metric.

More precisely, we can give the following examples.

Example 2 Let $X = \{1,2,3\}$ and the function $S : X \times X \times X \to [0,\infty)$ be defined as:

$$\begin{aligned}&S(1,1,2) = S(2,2,1) = 5,\\ &S(2,2,3) = S(3,3,2) = S(1,1,3) = S(3,3,1) = 2,\\ &S(x,y,z) = 0 \text{ if } x = y = z,\\ &S(x,y,z) = 1 \text{ if otherwise},\end{aligned}$$

for all $x, y, z \in X$. Then, the function S is an S-metric which is not generated by any metric and the pair (X, S) is an S-metric space. However, the function $d_S$ defined in (2.2) is not a metric on X. Indeed, for $x = 1$, $y = 2$, and $z = 3$, we get

$$d_S(1,2) = 10 \nleq d_S(1,3) + d_S(3,2) = 8.$$

Example 3 Let $X = \mathbb{R}$ and consider the S-metric defined in Example 1 which is not generated by any metric. Using the Eq. (2.2), we obtain

$$d_S(x,y) = 4|x-y|,$$

for all $x, y \in \mathbb{R}$. Then, $(\mathbb{R}, d_S)$ is a metric space on $\mathbb{R}$.

We give the following proposition.

**Proposition 2** *Let (X, S) be a complete S-metric space, $(X, d_S)$ be the metric space obtained by the metric generated by S , and T be a self-mapping of X. If T satisfies the inequality* (**S25**)*, then T satisfies the inequality* (**R25**).

*Proof* Let the inequality (**S25**) be satisfied. Using the inequality (**S25**) and (2.1), we have

$$\begin{aligned}d_S(Tx,Ty) &= S(Tx,Tx,Ty)+S(Ty,Ty,Tx)\\ &= S(Tx,Tx,Ty)+S(Tx,Tx,Ty)=2S(Tx,Tx,Ty)\\ &< 2\max\{S(x,x,y),S(Tx,Tx,x),S(Ty,Ty,y),S(Ty,Ty,x),\\ &\quad S(Tx,Tx,y)\}\\ &= \max\{2S(x,x,y),2S(Tx,Tx,x),2S(Ty,Ty,y),2S(Ty,Ty,x),\\ &\quad 2S(Tx,Tx,y)\}\\ &= \max\{S(x,x,y)+S(y,y,x),S(Tx,Tx,x)+S(x,x,Tx),\\ &\quad S(Ty,Ty,y)+S(y,y,Ty),\\ &\quad S(Ty,Ty,x)+S(x,x,Ty),S(Tx,Tx,y)+S(y,y,Tx)\}\\ &= \max\{d_S(x,y),d_S(x,Tx),d_S(y,Ty),d_S(x,Ty),d_S(y,Tx)\},\end{aligned}$$

and so, the inequality (**R25**) is satisfied on $(X, d_S)$. □

In [13], it was given another forms of (**R25**) as (**R50**), (**R75**), (**R100**), and (**R125**). Now, we extend the forms (**R50**) – (**R125**) for complete S-metric spaces. We can give the following definition.

**Definition 3** Let (X, S) be an S-metric space and T be a self-mapping of X. We define (**S50**), (**S75**), (**S100**), and (**S125**), as follows :

(**S50**) There exists a positive integer p, such that

$$S(T^px,T^px,T^py) < \max\{S(x,x,y),S(T^px,T^px,x),S(T^py,T^py,y),\\ S(T^py,T^py,x),S(T^px,T^px,y)\},$$

for any $x, y \in X$, $x \neq y$.

(**S75**) There exist positive integers p, q, such that

$$S(T^px,T^px,T^qy) < \max\{S(x,x,y),S(T^px,T^px,x),S(T^qy,T^qy,y),\\ S(T^qy,T^qy,x),S(T^px,T^px,y)\},$$

for any $x, y \in X$, $x \neq y$.

(**S100**) For any given $x \in X$, there exists a positive integer $p(x)$, such that

$$S(T^{p(x)}x,T^{p(x)}x,T^{p(x)}y) < \max\{S(x,x,y),S(T^{p(x)}x,T^{p(x)}x,x),\\ S(T^{p(x)}y,T^{p(x)}y,y),S(T^{p(x)}y,T^{p(x)}y,x),\\ S(T^{p(x)}x,T^{p(x)}x,y)\},$$

for any $y \in X$, $x \neq y$.

(**S125**) For any given $x, y \in X$, $x \neq y$, there exists a positive integer $p(x, y)$, such that

$$S(T^{p(x,y)}x, T^{p(x,y)}x, T^{p(x,y)}y) < \max\{S(x,x,y), S(T^{p(x,y)}x, T^{p(x,y)}x, x), S(T^{p(x,y)}y, T^{p(x,y)}y, y), S(T^{p(x,y)}y, T^{p(x,y)}y, x), S(T^{p(x,y)}x, T^{p(x,y)}x, y)\}.$$

**Corollary 1** Let $(X, d)$ be a complete metric space, $(X, S_d)$ be the $S$-metric space obtained by the $S$-metric generated by $d$, and $T$ be a self-mapping of $X$. If $T$ satisfies the inequality (**R50**) [resp. (**R75**), (**R100**), and (**R125**)], then $T$ satisfies the inequality (**S50**) [resp. (**S75**), (**S100**), and (**S125**)].

**Corollary 2** Let $(X, S)$ be a complete $S$-metric space, $(X, d_S)$ be the metric space obtained by the metric generated by $S$, and $T$ be a self-mapping of $X$. If $T$satisfies the inequality (**S50**) [resp. (**S75**), (**S100**), and (**S125**)], then $T$ satisfies the inequality (**R50**) [resp. (**R75**), (**R100**), and (**R125**)].

The proof of following proposition is obvious, so it is omitted.

**Proposition 3** *Let (X, S) be an S-metric space and T be a self-mapping of X. We obtain the following implications by the Definition 3*:

(**S25**) $\Longrightarrow$ (**S50**) $\Longrightarrow$ (**S75**)and(**S50**) $\Longrightarrow$ (**S100**) $\Longrightarrow$ (**S125**).

The converses of above implications in Proposition 3 are not always true as we have seen in the following examples.

*Example 4* Let $\mathbb{R}$ be the real line. It can be easily seen that the following function defines an $S$-metric on $\mathbb{R}$ different from the usual $S$-metric defined in [15]:

$$S(x,y,z) = |x - z| + |x + z - 2y|$$

for all $x, y, z \in \mathbb{R}$. Let

$$Tx = \begin{cases} 0 & \text{if } x \in [0,1], x \neq \frac{1}{4} \\ 1 & \text{if } x = \frac{1}{4} \end{cases}.$$

Then, $T$ is a self-mapping on the $S$-metric space [0, 1].

For $x = \frac{1}{2}$, $y = \frac{1}{4}$, we have

$$S(Tx,Tx,Ty) = S(0,0,1) = 2,$$
$$S(x,x,y) = S\left(\frac{1}{2},\frac{1}{2},\frac{1}{4}\right) = \frac{1}{2},$$
$$S(Tx,Tx,x) = S\left(0,0,\frac{1}{2}\right) = 1,$$
$$S(Ty,Ty,y) = S\left(1,1,\frac{1}{4}\right) = \frac{3}{2}$$
$$S(Ty,Ty,x) = S\left(1,1,\frac{1}{2}\right) = 1,$$
$$S(Tx,Tx,y) = S\left(0,0,\frac{1}{4}\right) = \frac{1}{2}$$

and so

$$S(Tx,Tx,Ty) = 2 < \max\left\{\frac{1}{2},1,\frac{3}{2},1,\frac{1}{2}\right\} = \frac{3}{2},$$

which is a contradiction. Then, the inequality (**S25**) is not satisfied.

For each $x, y \in X$ $(x \neq y)$ and $p \geq 2$, $T$ is satisfied the inequality (**S50**).

Example 5 We consider the self-mapping $T$ in the example on page 105 in [3] and the usual $S$-metric defined in [15]. If we choose $x = (\frac{1}{n} + 1, 0)$, $y = (\frac{1}{n}, 0)$ for each $n$, then the inequality (**S50**) is not satisfied. A positive integer $p(x)$ can be chosen for any given $x \in X$, such that the inequality (**S100**) is satisfied.

*Example 6* Let $\mathbb{R}$ be the real line. Let us consider the $S$-metric defined in Example 4 on $\mathbb{R}$ and let

$$Tx = \begin{cases} 0 & \text{if } x \in \left[\frac{1}{2},1\right] \\ 1 & \text{if } x \in \left[0,\frac{1}{2}\right) \end{cases}.$$

Then, $T$ is a self-mapping on the $S$-metric space [0, 1].

Let us choose $x = 0$ and $y = 1$.

For $p = 1$, we have

$$S(Tx,Tx,Ty) = S(1,1,0) = 2,$$
$$S(x,x,y) = S(0,0,1) = 2,$$
$$S(Tx,Tx,x) = S(1,1,0) = 2,$$
$$S(Ty,Ty,y) = S(0,0,1) = 2,$$
$$S(Ty,Ty,x) = S(0,0,0) = 0,$$
$$S(Tx,Tx,y) = S(1,1,1) = 0$$

and so

$$S(Tx,Tx,Ty) = 2 < \max\{2,2,2,0,0\} = 2,$$

which is a contradiction. Then, the inequality (**S50**) is not satisfied.

For $p = 2$, we have

$$S(T^2x,T^2x,T^2y) = S(0,0,1) = 2,$$
$$S(x,x,y) = S(0,0,1) = 2,$$
$$S(T^2x,T^2x,x) = S(0,0,0) = 0,$$
$$S(T^2y,T^2y,y) = S(1,1,1) = 0,$$
$$S(T^2y,T^2y,x) = S(1,1,0) = 2,$$
$$S(T^2x,T^2x,y) = S(0,0,1) = 2$$

and so

$$S(T^2x,T^2x,T^2y) = 2 < \max\{2,0,0,2,2\} = 2,$$

which is a contradiction. Then, the inequality (**S50**) is not satisfied.

For $p \geq 3$ using similar arguments, we can see that the inequality (**S50**) is not satisfied.

We now show that the inequality (**S75**) is satisfied under the following four cases:

*Case 1* We take $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1]$, $p = 2$, and $q = 1$. Then, the inequality (**S75**) is satisfied, since

$$S(T^2x, T^2x, Ty) = 0,$$

for $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1]$, $x \neq y$.

*Case 2* We take $y \in [0, \frac{1}{2})$, $x \in [\frac{1}{2}, 1]$, $p = 2$, and $q = 1$. Then, using similar arguments in Case 1, we can see that the inequality (**S75**) is satisfied.

*Case 3* We take $x, y \in [0, \frac{1}{2})$, $p = 2$, and $q = 2$. Then, the inequality (**S75**) is satisfied, since

$$S(T^2x, T^2x, T^2y) = 0,$$

for $x, y \in [0, \frac{1}{2})$, $x \neq y$.

*Case 4* We take $x, y \in [\frac{1}{2}, 1]$, $p = 2$, and $q = 2$. Then, using similar arguments in Case 3, we can see that the inequality (**S75**) is satisfied.

*Example 7* Let $\mathbb{R}$ be the $S$-metric space with the $S$-metric defined in Example 4 and let

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \in [0,1], x \neq \frac{1}{2}, x \neq \frac{1}{3} \\ \frac{1}{3} & \text{if } x = \frac{1}{2} \\ 3 & \text{if } x = \frac{1}{3} \\ \frac{1}{2} & \text{if } x = 3 \end{cases}.$$

Then, $T$ is a self-mapping on the $S$-metric space $[0, 1] \cup \{3\}$.

The inequality (**S100**) is not satisfied, since there is not a positive integer $p(x)$ for any given $x \in X$, such that $T$ is satisfied the inequality (**S100**) for any $y \in X$, $x \neq y$. However, for any given $x, y \in X$, $x \neq y$, there exists a positive integer $p(x, y)$, such that the inequality (**S125**) is satisfied.

*Remark 1* (**S75**) and (**S100**) are independent of each other by Examples 5 and 6.

## Some fixed point theorems on $S$-metric spaces

In this section, we give some fixed point theorems by means of periodic points on $S$-metric spaces for the contractive mappings defined in Sect. 3.

**Theorem 1** *Let $(X, S)$ be an $S$-metric space* and *$T$ be a self-mapping of $X$ which satisfies the inequality* (**S125**)*. If $T$ has a fixed point, then it is unique.*

Proof Suppose that $x$ and $y$ are fixed points of $T$, such that $x, y \in X$ $(x \neq y)$. Then, there exists a positive integer $p = p(x, y)$, such that

$$\begin{aligned} S(T^px, T^px, T^py) &< \max\{S(x,x,y), S(T^px, T^px, x), S(T^py, T^py, y), \\ &S(T^py, T^py, x), S(T^px, T^px, y)\} \\ &= \max\{S(x,x,y), 0, 0, S(y,y,x), S(x,x,y)\} \\ &= S(x,x,y), \end{aligned}$$

by the inequality (**S125**). Then, using Lemma 1 and the fact that $T^px = x$, $T^py = y$, we obtain

$$S(T^px, T^px, T^py) = S(x,x,y) < S(x,x,y).$$

Thus, the assumption that $x$ and $y$ are fixed points of $T$ has led to a contradiction. Consequently, the fixed point is unique. □

**Corollary 3** Let $(X, S)$ be an $S$-metric space, $T$ be a self-mapping of $X$, and the inequality (**S25**) [resp. $T \in$ (**S50**), $T \in$ (**S100**)] be satisfied. If $T$ has a fixed point, then it is unique.

*Proof* It can be seen from Proposition 3. □

**Corollary 4** Let $(X, S)$ be an $S$-metric space, $T$ be a self-mapping of $X$, and the inequality (**S75**) be satisfied. If $T$ has a fixed point, then it is unique.

*Proof* By a similar argument used in the proof of Theorem 1, the proof can be easily seen by the definition of (**S75**). □

Now, we recall the following definitions and corollary.

**Definition 4** [14] Let $(X, S)$ be an $S$-metric space and $A \subset X$. Then, $A$ is called $S$-bounded if there exists $r > 0$, such that $S(x,x,y) < r$ for all $x, y \in A$.

**Definition 5** [4] Let $(X, S)$ be an $S$-metric space, $T$ be a self-mapping of $X$, and $x \in X$. A point $x$ is called a periodic point of $T$, if there exists a positive integer $n$, such that

$$T^nx = x. \quad (3.1)$$

The least positive integer satisfying the condition (3.1) is called the periodic index of $x$.

**Definition 6** [10] Let $(X, S)$ be an $S$-metric space, $T$, $F$ be two self-mappings of $X$, and $A \subset X$, $x \in X$. Then

1. $\delta(A) = \sup\{S(x,x,y) : x, y \in A\}$.
2. $O_{T,F}(x,n) = \{Tx, TFx, TF^2x, \ldots, TF^nx\}$.
3. $O_{T,F}(x,\infty) = \{Tx, TFx, TF^2x, \ldots, TF^nx, \ldots\}$.
4. If $T$ is identify, then $O_F(x,n) = O_{T,F}(x,n)$ and $O_F(x,\infty) = O_{T,F}(x,\infty)$.

Let $A$ be a nonempty subset of $X$. In [12], it was called $\delta(A)$ as the diameter of $A$ and we write

$$\delta(A) = diam\{A\} = sup\{S(x, x, y) : x, y \in A\}.$$

If $A$ is $S$-bounded, then we will write $\delta(A) < \infty$.

The following corollary is a generalization of [6, Theorem1] into the structure of $S$-metric in [5].

**Corollary 5** [10] Let ($X$, $S$) be an $S$-metric space and $T$ be a self-mapping of $X$, such that

(1) Every Cauchy sequence of the form $\{T^n x\}$ is convergent in $X$ for all $x \in X$;
(2) There exists $h \in [0, 1)$, such that

$$S(Tx, Tx, Ty) \leq h\max\{S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y), S(Ty, Ty, x), S(Ty, Ty, y)\},$$

for each $x, y \in X$.

Then

1. $\delta(T^i x, T^i x, T^j x) \leq h\delta[O_T(x, n)]$ for all $i, j \leq n$, $n \in \mathbb{N}$ and $x \in X$;
2. $\delta[O_T(x, \infty)] \leq \dfrac{2}{1-h} S(Tx, Tx, x)$ for all $x \in X$;
3. $T$ has a unique fixed point $x_0$;
4. $\lim\limits_{n\to\infty} T^n x = x_0$.

**Theorem 2** *Let ($X$, $S$) be an $S$-metric space, $T$ be a self-mapping of $X$, the inequality* (**S125**) *be satisfied, and $x \in X$. Assume that xis a periodic point of Twith periodic index m. Then, Thas a fixed point x in $\{T^n x\}(n \geq 0)$if and only if for any $T^{n_1}x$, $T^{n_2}x \in \{T^n x\}(n \geq 0)$, $T^{n_1}x \neq T^{n_2}x$ , there exist $T^{n_3}x$, $T^{n_4}x \in \{T^n x\}$, such that*

$$T^{p(T^{n_3}x, T^{n_4}x)}(T^{n_3}x) = T^{n_1}x \text{andT}^{p(T^{n_3}x, T^{n_4}x)}(T^{n_4}x) = T^{n_2}x.$$

*Then, the point x is the unique fixed point of T in X.*

*Proof* The proof of the if part of the theorem is obvious. Therefore, we prove the only if part. If $x$ is a periodic point of $T$ with periodic index $m$, then we have

$$\{T^n x\} = \{x, Tx, \ldots, T^{m-1}x\}.$$

If $x \neq Tx$, then there exist $T^{n_1}x$, $T^{n_2}x \in \{T^n x\}$, $T^{n_1}x \neq T^{n_2}x$, such that

$$\delta(\{T^n x\}) = \max_{0 \leq k, l \leq m-1, k \neq l}\{S(T^k x, T^k x, T^l x)\} = S(T^{n_1}x, T^{n_1}x, T^{n_2}x).$$

By the hypothesis, there exist $T^{n_3}x$, $T^{n_4}x \in \{T^n x\}$, such that

$$T^{p(T^n x, T^{n_4}x)}(T^{n_3}x) = T^{n_1}x \text{ and } T^{p(T^{n_3}x, T^{n_4}x)}(T^{n_4}x) = T^{n_2}x.$$

Since $T^{n_1}x \neq T^{n_2}x$, we obtain $T^{n_3}x \neq T^{n_4}x$. Hence, we have

$$\begin{aligned}\delta(\{T^n x\}) &= S(T^{n_1}x, T^{n_1}x, T^{n_2}x)\\ &= S(T^{p(T^{n_3}x, T^{n_4}x)}(T^{n_3}x), T^{p(T^{n_3}x, T^{n_4}x)}(T^{n_3}x), T^{p(T^{n_3}x, T^{n_4}x)}(T^{n_4}x))\\ &< \max\{S(T^{n_3}x, T^{n_3}x, T^{n_4}x), S(T^{n_1}x, T^{n_1}x, T^{n_3}x),\\ &\quad S(T^{n_2}x, T^{n_2}x, T^{n_4}x),\\ &\quad S(T^{n_2}x, T^{n_2}x, T^{n_3}x), S(T^{n_1}x, T^{n_1}x, T^{n_4}x)\}\\ &\leq \delta(\{T^n x\}),\end{aligned}$$

which is a contradiction, and so, we have $x = Tx$. It is obvious that $x$ is unique fixed point of $T$ in $X$ by Theorem 1. □

**Corollary 6** Let ($X$, $S$) be an $S$-metric space, $T$ be a self-mapping of $X$, the inequality (**S100**) be satisfied, and $x \in X$ be a periodic point of $T$. Then, the following conditions are equivalent:

(1) $T$ has a unique fixed point in $\{T^n x\}(n \geq 0)$,
(2) There exists $T^{n_0}x \in \{T^n x\}(n \geq 0)$, such that

$$T^{p(T^{n_0}x)}(T^{n_0}x) = T^{n_1}x,$$

for any $T^{n_1}x \in \{T^n x\}(n \geq 0)$, where $p(T^{n_0}x)$ is the positive integer.

Then, the point $x$ is the unique fixed point of $T$ in $X$.

**Corollary 7** Let ($X$, $S$) be an $S$-metric space, $T$ be a self-mapping of $X$, the inequality (**S75**) be satisfied, and $x \in X$ be a periodic point of $T$. Then, $x$ is the unique fixed point of $T$ if there exist $T^{n_3}x$, $T^{n_4}x \in \{T^n x\}(n \geq 0)$, and $T^{n_3}x \neq T^{n_4}x$, such that

$$T^p(T^{n_3}x) = T^{n_1}x \text{ and } T^q(T^{n_4}x) = T^{n_2}x,$$

for any $T^{n_1}x$, $T^{n_2}x \in \{T^n x\}(n \geq 0)$, $T^{n_1}x \neq T^{n_2}x$. Here, $p$ and $q$ are the positive integers.

**Corollary 8** Let ($X$, $S$) be an $S$-metric space, $T$ be a self-mapping of $X$, and the inequality (**S50**) be satisfied. Then, the following conditions are equivalent:

(1) $T$ has a fixed point in $X$,
(2) There exists a periodic point $x \in X$ of $T$.

Then, the point $x$ is the unique fixed point of $T$ in $X$.

We give some sufficient conditions to guarantee the existence of fixed point for a self-mapping $T$ satisfying the inequality (**S75**) in the following theorem.

**Theorem 3** *Let ($X$, $S$) be an $S$-metric space, $T$ be a self-mapping of $X$, the inequality* (**S75**) *be satisfied, and $x \in X$be a periodic point of $T$ with periodic index m. Suppose that p and q are the positive integers and also the following conditions are satisfied:*

1. $p = p_1 m + p_2$, $q = q_1 m + q_{2,}$ $0 \leq p_2, q_2 < m$, and $p_1$ and $q_1$ are non-negative integers.
2. $2|p_2 - q_2| \neq m$.

*Then, the point x is the unique fixed point of T in X.*

*Proof* We now show that $x$ is the fixed point of $T$ in $X$. On the contrary, assume that $x$ is not the fixed point of $T$. Let

$$A = \{T^n x\} = \{x, Tx, T^2x, \ldots, T^n x, \ldots\}.$$

Since the periodic index of $x$ is $m$, we have

$$A = \{T^n x\} = \{x, Tx, T^2x, \ldots, T^{m-1}x\}$$

and the elements in $A$ are distinct. Therefore, there exist $i, j$, such that $0 \le i < j < m$ and

$$\delta(A) = \max_{0 \le k,l \le m-1, k \ne l} S(T^k x, T^k x, T^l x) = S(T^i x, T^i x, T^j x).$$

We can assume that $p_2 \ge q_2$. In addition, we have $T^n(A) = A$ for any non-negative integer $n$. Therefore, there exist $T^{n_1}x$ and $T^{n_2}x \in A$, such that

$$T^i x = T^{p_2}(T^{n_1}x) \text{ and } T^j x = T^{q_2}(T^{n_2}x). \tag{3.2}$$

Similarly, there exist $T^{n_3}x$ and $T^{n_4}x \in A$, such that

$$T^i x = T^{q_2}(T^{n_3}x) \text{ and } T^j x = T^{p_2}(T^{n_4}x). \tag{3.3}$$

We prove that at least one of the statements $n_1 \ne n_2$ and $n_3 \ne n_4$ is true.

Suppose that $n_3 = n_4$. Since

$$0 \le i, j, p_2, q_2, n_1, n_2, n_3, n_4 < m,$$

using (3.2) and (3.3), there exist $a, b, c, d \in \{0, 1\}$, such that

$$p_2 + n_1 = am + i, q_2 + n_2 = bm + j, \tag{3.4}$$

$$q_2 + n_3 = cm + i, p_2 + n_4 = dm + j. \tag{3.5}$$

If $n_1 = n_2$, we have $am + i \ge bm + j$, since $p_2 \ge q_2$. Since $i < j$, we have $a = 1$, $b = 0$. It follows from (3.4) that

$$(p_2 - q_2) + (j - i) = m. \tag{3.6}$$

Using the condition (3.5) and $n_3 = n_4$, we obtain

$$(p_2 - q_2) = (d - c)m + (j - i). \tag{3.7}$$

Since $0 \le p_2 - q_2 \le m - 1$, $0 \le j - i < m$, we have $d - c = 0$ using the condition (3.7), and so, $p_2 - q_2 = j - i$.

By the condition (3.6), we have

$$2(p_2 - q_2) = m,$$

which is a contradiction. Hence, it should be $n_1 \ne n_2$. Then, $T^{n_1}x \ne T^{n_2}x$. Using $T^{p_2}(x) = T^p x$ and $T^{q_2}(x) = T^q x$, we obtain

$$\begin{aligned}\delta(A) &= S(T^i x, T^i x, T^j x)\\ &= S(T^{p_2}(T^{n_1}x), T^{p_2}(T^{n_1}x), T^{q_2}(T^{n_2}x)) = S(T^p(T^{n_1}x), T^p(T^{n_1}x), T^q(T^{n_2}x))\\ &< \max\{S(T^{n_1}x, T^{n_1}x, T^{n_2}x), S(T^p(T^{n_1}x), T^p(T^{n_1}x), T^{n_1}x),\\ &\quad S(T^q(T^{n_2}x), T^q(T^{n_2}x), T^{n_2}x), S(T^q(T^{n_2}x), T^q(T^{n_2}x), T^{n_1}x),\\ &\quad S(T^p(T^{n_1}x), T^p(T^{n_1}x), T^{n_2}x)\} \le \delta(A),\end{aligned}$$

which is a contradiction. Consequently, $x = Tx$.

Similarly, it can be seen that if $n_1 = n_2$, then it should be $n_3 \ne n_4$, and hence, we get $x = Tx$.

It is obvious that $x$ is the unique fixed point of $T$ in $X$ by Corollary 4. □

## Some applications of contractive mappings on S-metric spaces

The following corollary was given in [15] on page 123 by Sedghi and Dung.

**Corollary 9** [15] *Let* $(X, S)$ *be a complete S -metric space, T be a self-mapping of X, and*

$$\begin{aligned}S(Tx, Tx, Ty) \le h \max\{&S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y),\\ &S(Ty, Ty, x), S(Ty, Ty, y)\},\end{aligned} \tag{4.1}$$

*for some* $h \in [0, \frac{1}{3})$ *and each* $x, y \in X$. *Then, T has a unique fixed point in X. In addition, T is continuous at this fixed point.*

We call the inequality (4.1) as (**Q25**) in Corollary as follows:

There exists a number $h$ with $h \in [0, \frac{1}{3})$, such that

(**Q25**) $$\begin{aligned}S(Tx, Tx, Ty) \le h \max\{&S(x, x, y), S(Tx, Tx, x), S(Tx, Tx, y),\\ &S(Ty, Ty, x), S(Ty, Ty, y)\},\end{aligned}$$

for any $x, y \in X$.

In this section, we study fixed point theorems using the inequality (**Q25**) on $S$-metric spaces. Finally, we obtain a fixed point theorem for a self-mapping $T$ of a compact $S$-metric space $X$ satisfying the inequality (**S25**).

Now, we give the definition of $T_S$-orbitally complete space.

**Definition 7** Let $(X, S)$ be an $S$-metric space and $T$ be a self-mapping of $X$. Then, an $S$-metric space $X$ is said to be $T_S$-orbitally complete if and only if every Cauchy sequence which is contained in the sequence $\{x, Tx, \ldots, T^n x, \ldots\}$ for some $x \in X$ converges in $X$.

**Theorem 4** *Let (X, S) be $T_S$-orbitally complete, T be a self-mapping of X, and the inequality* (**Q25**) *be satisfied. Then, T has a unique fixed point in X.*

*Proof* It is obvious from Corollary 5. □

Now, we will extend the definition (**Q25**) on an $S$-metric space as follows:

(**Q25a**) $S(T^px,T^px,T^qy)\le h\max\{S(T^{r_1}x,T^{r_1}x,T^{s_1}y),S(T^{r_1}x,T^{r_1}x,T^{r_2}x),S(T^{s_1}y,T^{s_1}y,T^{s_2}y):0\le r_1,r_2\le p \text{ and } 0\le s_1,s_2\le q\},$

for each $x,y\in X$, some fixed positive integers $p$ and $q$. Here, $h\in[0,\frac{1}{2})$.

The following theorems are the generalizations of the fixed point theorems given in [7] to an $S$-metric space $(X, S)$.

**Theorem 5** *Let (X, S) be a complete S-metric space, Tbe a continuous self-mapping of X, and the inequality* (**Q25a**) *be satisfied. Then, T has a unique fixed point in X.*

*Proof* Without loss of generality, we assume that $h\in[\frac{1}{3},\frac{1}{2})$. Then, we have $\frac{h}{1-2h}\ge 1$. Suppose that $p\ge q$.

Let $x\in X$ and assume that the sequence $\{T^nx:n=1,2,\ldots\}$ is unbounded. Then, clearly, the sequence

$$\{S(T^nx,T^nx,T^qx):n=1,2,\ldots\}$$

is unbounded. Hence, there exists an integer $n$, such that

$$S(T^nx,T^nx,T^qx)>\frac{h}{1-2h}\max\{S(T^ix,T^ix,T^qx):0\le i\le p\}.$$

Suppose that $m$ is the smallest such $n$. Clearly, we have $m>p\ge q$. Therefore

$$\begin{aligned}S(T^mx,T^mx,T^qx)&>\frac{h}{1-2h}\max\{S(T^ix,T^ix,T^qx):0\le i\le p\}\\&\ge\max\{S(T^{r_1}x,T^{r_1}x,T^qx):0\le r_1<m\}.\end{aligned}\tag{4.2}$$

Using (4.2), we obtain

$$\begin{aligned}(1-2h)S(T^mx,T^mx,T^qx)&>h\max\{S(T^ix,T^ix,T^qx):0\le i\le p\}\\&\ge h\max\{S(T^ix,T^ix,T^{r_1}x)-2S(T^{r_1}x,T^{r_1}x,T^qx):0\le i\le p\text{ and }0\le r_1<m\}\\&\ge h\max\{S(T^ix,T^ix,T^{r_1}x)-2S(T^mx,T^mx,T^qx):0\le i\le p\text{ and }0\le r_1<m\}\end{aligned}$$

and then

$$S(T^mx,T^mx,T^qx)>h\max\{S(T^ix,T^ix,T^{r_1}x):0\le i\le p\text{and}0\le r_1<m\}.\tag{4.3}$$

Now, we prove that

$$S(T^mx,T^mx,T^qx)>h\max\{S(T^ix,T^ix,T^{r_1}x):0\le i,r_1<m\}.\tag{4.4}$$

For if not

$$S(T^mx,T^mx,T^qx)\le h\max\{S(T^ix,T^ix,T^{r_1}x):0\le i,r_1<m\}$$

and so using (4.3)

$$S(T^mx,T^mx,T^qx)\le h\max\{S(T^ix,T^ix,T^{r_1}x):p<i,r_1<m\}.\tag{4.5}$$

Using the inequality (**Q25a**), we can write

$$S(T^mx,T^mx,T^qx)\le h^k\max\{S(T^ix,T^ix,T^{r_1}x):p<i,r_1<m\}$$

for $k=1,2,\ldots$, since we can omitted the terms of the form as $S(T^ix,T^ix,T^{r_1}x)$ with $0\le i\le p$ by (4.3).

Now, we get $S(T^mx,T^mx,T^qx)=0$ for $k\to\infty$, which is a contradiction by our assumption. Therefore, we obtain the inequality (4.4).

However, using the inequality (**Q25a**), we have

$$\begin{aligned}S(T^mx,T^mx,T^qx)&\le h\max\{S(T^{r_1}x,T^{r_1}x,T^{s_1}x),S(T^{r_1}x,T^{r_1}x,T^{r_2}x),S(T^{s_1}x,T^{s_1}x,T^{s_2}x):m-p\le r_1,r_2\le m\text{ and }0\le s_1,s_2\le q\}\\&\le h\max\{S(T^{r_1}x,T^{r_1}x,T^{s_1}x):0\le r_1,s_1\le m\},\end{aligned}$$

which is a contradiction from (4.4). Then, the sequence $\{T^nx:n=1,2,\ldots\}$ should be $S$-bounded.

Now, we put $N=\sup\{S(T^{r_1}x,T^{r_1}x,T^{s_1}x):r_1,s_1=0,1,2,\ldots\}<\infty$. Therefore, for arbitrary $\varepsilon>0$, choose $M$, so that $h^MN<\varepsilon$. For $m,n\ge M\max\{p,q\}$ and using the inequality (**Q25a**) $M$ times, we have

$$S(T^mx,T^mx,T^nx)\le h^MN<\varepsilon.$$

Hence, the sequence $\{T^nx:n=1,2,\ldots\}$ is a Cauchy sequence in the complete $S$-metric space $(X, S)$ and so has a limit $x_0$ in $X$. Since $T$ is continuous, we have $Tx_0=x_0$ and then $x_0$ is a fixed point of $T$. It can be easily seen that the point $x_0$ is a unique fixed point of $T$. Then, the proof is completed. □

From the inequality (**Q25a**), for $q=1$ (or $p=1$), we define the following generalization of (**Q25**):

(**Q25b**) $S(T^px,T^px,Ty)\le h\max\{S(T^{r_1}x,T^{r_1}x,T^sy),S(T^{r_1}x,T^{r_1}x,T^{r_2}x),S(Ty,Ty,y):0\le r_1,r_2\le p\text{ and }s=0,1\},$

for each $x,y\in X$, some fixed positive integer $p$. Here, $h\in[0,\frac{1}{2})$.

The condition that the self-mapping $T$ be continuous is not necessary when the inequality (**Q25b**) is satisfied as we have seen in the following theorem.

**Theorem 6** *Let (X, S)be a complete S-metric space and T be a self-mapping of X satisfying the inequality* (**Q25b**). *Then, T has a unique fixed point in X.*

*Proof* Let $x \in X$. Then, the sequence $\{T^n x : n = 1, 2, \ldots\}$ is a Cauchy sequence in the complete $S$-metric space $X$ as we have seen in the proof of Theorem 5. Hence, the sequence has a limit $x_0$ in $X$. For $n \geq p$, we obtain

$$S(T^n x, T^n x, Tx_0) \leq h \max\{S(T^{r_1}x, T^{r_1}x, T^s x_0), S(T^{r_1}x, T^{r_1}x, T^{r_2}x), S(Tx_0, Tx_0, x_0) : n-p \leq r_1, r_2 \leq n \text{ and } s = 0, 1\}.$$

Then, by (2.1), we have

$$\begin{aligned} S(x_0, x_0, Tx_0) &= S(Tx_0, Tx_0, x_0) \\ &\leq h \max\{S(T^s x_0, T^s x_0, x_0) : s = 0, 1\} \\ &= hS(Tx_0, Tx_0, x_0), \end{aligned}$$

when $n$ goes to infinity. Since $h < 1$, we have $Tx_0 = x_0$. Then, the proof is completed. □

**Corollary 10** *Let (X, S) be a complete S-metric space and T be a self-mapping of X satisfying the inequality* (**Q25**). *Then, Thas a unique fixed point in X.*

*Remark 2* The condition that $T$ be continuous when $p, q \geq 2$ is necessary in Theorem 5. The following example shows that Theorem 5 cannot be always true when $T$ is a discontinuous self-mapping of $X$.

Example 8 Let $\mathbb{R}$ be the real line. Let us consider the $S$-metric defined in Example on $\mathbb{R}$ and let

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ \frac{x}{4} & \text{if } x \neq 0 \end{cases}.$$

Then, $T$ is a discontinuous self-mapping on the complete $S$-metric space [0, 1]. For each $x, y \in X$, we obtain

$$S(T^p x, T^p x, T^q y) = \frac{1}{4} S(T^{p-1}x, T^{p-1}x, T^{q-1}y)$$

and so the inequality (**Q25a**) is satisfied with $h = \frac{1}{4}$. However, $T$ has not a fixed point.

Now, we consider compact $S$-metric spaces and prove the following theorem.

**Theorem 7** *Let (X, S) be a compact S-metric space and T be a continuous self-mapping of X satisfying*

$$S(T^p x, T^p x, T^q y) < \max\{S(T^{r_1}x, T^{r_1}x, T^{s_1}y), S(T^{r_1}x, T^{r_1}x, T^{r_2}x), S(T^{s_1}y, T^{s_1}y, T^{s_2}y) : 0 \leq r_1, r_2 \leq p \text{ and } 0 \leq s_1, s_2 \leq q\} \tag{4.6}$$

*for each $x, y \in X$. Here, the right-hand side of (4.6) is positive. Then, T has a unique fixed point in X.*

*Proof* Let the inequality (**Q25a**) be satisfied. Then, $T$ has a unique fixed point in $X$ from Theorem 5.

Let the inequality (**Q25a**) be not satisfied. If $\{h_n : n = 1, 2, \ldots\}$ is a monotonically increasing sequence of numbers converging to 1, then there exist sequences $\{x_n : n = 1, 2, \ldots\}$ and $\{y_n : n = 1, 2, \ldots\}$ in $X$, such that

$$S(T^p x_n, T^p x_n, T^q y_n) > h_n \max\{S(T^{r_1}x_n, T^{r_1}x_n, T^{s_1}y_n), S(T^{r_1}x_n, T^{r_1}x_n, T^{r_2}x_n), S(T^{s_1}y_n, T^{s_1}y_n, T^{s_2}y_n) : 0 \leq r_1, r_2 \leq p \text{ and } 0 \leq s_1, s_2 \leq q\}$$

for $n = 1, 2, \ldots$. Using compactness of $X$, there exist subsequences $\{x_{n_k} : k = 1, 2, \ldots\}$ and $\{y_{n_k} : k = 1, 2, \ldots\}$ of $\{x_n\}$ and $\{y_n\}$ converging to $x$ and $y$, respectively. Since $T$ is continuous self-mapping, for $k \to \infty$, we have

$$S(T^p x, T^p x, T^q y) \geq \max\{S(T^{r_1}x, T^{r_1}x, T^{s_1}y), S(T^{r_1}x, T^{r_1}x, T^{r_2}x), S(T^{s_1}y, T^{s_1}y, T^{s_2}y) : 0 \leq r_1, r_2 \leq p \text{ and } 0 \leq s_1, s_2 \leq q\},$$

which is a contradiction unless $Tx = x = y$. Then, $T$ has a fixed point $x$. It can be easily seen that $x$ is the unique fixed point. □

We have the following corollary for $p = q = 1$.

**Corollary 11** *Let (X, S) be a compact S-metric space and T be a continuous self-mapping of X satisfying the inequality* (**S25**). *Here, the right-hand side of the inequality* (**S25**) *is positive. Then, T has a unique fixed point in X.*

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