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Optimal Control Problem for a Conformable Fractional Heat Conduction Equation

B.B. İskender Eroğlu*, D. Avci and N. Özdemir
Balıkesir University, Faculty of Science and Arts, Department of Mathematics, Balıkesir, Turkey

This paper presents an optimal boundary temperature control of thermal stresses in a plate, based on time-conformable fractional heat conduction equation. The aim is to find the boundary temperature that takes thermal stress under control. The fractional Laplace and finite Fourier *sine* transforms are used to obtain the fundamental solution. Then the optimal control is held by successive iterations. Numerical results are depicted by plots produced by MATLAB codes.

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1. Introduction

Heat conduction in the media with complex internal structures, such as porous, random and granular materials, semiconductors, polymers, glasses, etc., is more accurately modelled with fractional heat conduction equations, than with classical ones. The time-fractional heat conduction equation is defined by [1]

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \tag{1}$$

where T is temperature, a denotes the heat diffusivity coefficient and $\frac{\partial^\alpha}{\partial t^\alpha}$ represents the Caputo fractional derivative (see [2]).

Thermoelasticity theory, based on time fractional heat conduction equation, was first proposed by Povstenko [3], who investigated the physical behaviour of thermal stresses, by obtaining fundamental solutions of the Cauchy problems for fractional heat conduction equations, defined in one or multi-dimensional coordinate systems. The central-symmetric thermal stresses in an infinite medium with a spherical [4] and cylindrical [5] cavities were analyzed. In addition, the theory of thermal stresses for space-time fractional heat conduction equation was introduced [6]. Optimal control of thermal stresses, based on fractional heat conduction equation, was first proposed by Ozdemir et al. in [7], where boundary temperature control problem was studied for a sub-heat conduction process, defined in terms of Caputo fractional derivative. That paper was the generalization of boundary optimal control of a standard parabolic heat conduction equation, presented by Knopp [8].

In this paper, we aim to apply the optimal boundary control approach to a heat conduction equation with conformable fractional derivative, which has been recently defined by Khalil et al. [9]. It is a natural extension of usual derivative and it is named as conformable, because this operator preserves basic properties of classical derivative (see [9, 10]). Since conformable fractional derivative is a local and limit-based operator, it quickly takes a place in application problems [11–13].

2. Preliminaries

Until recently, many real world applications of fractional calculus have been confined to the well-known Riemann-Liouville, Caputo and Grünwald-Letnikov fractional operators (see [14, 15]). Although these fractional definitions display desired advantages, such as the description of memory and hereditary effects in natural phenomena, they unfortunately lead to computational complexities, requiring an improvement of numerical methods, because of their non-local descriptions with weakly singular kernels. Due to these complications, fractional researchers have shown increasing interest for the new local fractional definitions [16–20]. One of these definitions is the limit-based conformable fractional derivative, which is defined as follows.

Definition 1: [9] *For a given function $f:[0,\infty)\to\mathbb{R}$ the conformable fractional derivative of order $\alpha\in(0,1]$ is defined by*

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \tag{2}$$

for all $t>0$. If f is conformable fractional differentiable of order α, simply called as α-differentiable, in some $(0,a)$, $a>0$ and the $\lim_{t\to a^+}\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}$ exists, then

$$\frac{\mathrm{d}^\alpha f(a)}{\mathrm{d}t^\alpha} = \lim_{t\to a^+}\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}. \tag{3}$$

The following theorem shows that the fundamental properties of usual derivative are satisfied by conformable fractional derivative.

Theorem 1: [9] *Let $0<\alpha\le 1$ and $f,g:[0,\infty)\to\mathbb{R}$ be α-differentiable functions at a point $t>0$. Then*

1. $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(af+bg) = a\frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} + b\frac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha}$, *for all $a,b\in\mathbb{R}$,*

2. $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(t^p) = pt^{p-\alpha}$, *for all $p\in\mathbb{R}$,*

*corresponding author; e-mail: biskender@balikesir.edu.tr

3. $\dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(\lambda) = 0,$ *for all constant functions* $f(t) = \lambda$,

4. $\dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(fg) = f\dfrac{\mathrm{d}^\alpha g}{\mathrm{d}t^\alpha} + g\dfrac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha}$,

5. $\dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left(\dfrac{f}{g}\right) = \dfrac{g\,\mathrm{d}^\alpha f/\mathrm{d}t^\alpha - f\,\mathrm{d}^\alpha g/\mathrm{d}t^\alpha}{g^2}$,

6. *If f is a differentiable function,*

$$\text{then } \frac{\mathrm{d}^\alpha f}{\mathrm{d}t^\alpha} = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}.$$

Several papers were devoted to detailed investigation of the properties and the useful theorems related with this derivative [21]. Here, we deal with the fractional Laplace transform which was first defined by Abdeljawad [10].

Definition 2: *Let $f : [0,\infty) \to \mathbb{R}$ is a function and $\alpha \in (0,1]$. Then the fractional Laplace transform of order α is defined by*

$$L_\alpha\{f(t)\} = \int_0^\infty \mathrm{e}^{-s\frac{t^\alpha}{\alpha}} f(t)\,\mathrm{d}\alpha(t) = \int_0^\infty \mathrm{e}^{-s\frac{t^\alpha}{\alpha}} f(t)\,t^{\alpha-1}\,\mathrm{d}t, \tag{4}$$

where s is the transform variable.

The relation between the usual and the fractional Laplace transforms is given below.

Lemma: [10] *Let $f : [0,\infty) \to \mathbb{R}$ be a function, such that $L_\alpha\{f(t)\}$ exists for $0 < \alpha \le 1$. Then*

$$L_\alpha\{f(t)\} = L\left\{f\left((\alpha t)^{1/\alpha}\right)\right\}, \tag{5}$$

where $L\{f(t)\} = \int\limits_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$.

As an example, the fractional Laplace transform of $\mathrm{e}^{\lambda\frac{t^\alpha}{\alpha}}$, $\lambda \in \mathbb{R}$, often resulting in the solutions of conformable fractional differential equations, can be easily computed as

$$L_\alpha\left\{\mathrm{e}^{\lambda\frac{t^\alpha}{\alpha}}\right\} = L\left\{\mathrm{e}^{\lambda t}\right\} = \frac{1}{s-\lambda}, \tag{6}$$

whereas the usual Laplace transform of such function is not easy to calculate. Similarly, the fractional Laplace transform of some certain functions can be simply taken by using the Lemma.

The following theorem gives the fractional Laplace transform of conformable fractional derivative.

Theorem 2: [10] *Let $f : [0,\infty) \to \mathbb{R}$ is an α-differentiable function of order, $\alpha \in (0,1]$ and $L_\alpha\{f(t)\}$ exists. Then*

$$L_\alpha\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = sL_\alpha\{f(t)\} - f(0). \tag{7}$$

As it is known, Laplace transform is a powerful tool to solve linear differential equations. Similarly, it is expected to solve conformable fractional differential equations by the fractional Laplace transform. At this stage, we give the following theorem, which is used to assign the inverse fractional Laplace transform of certain types of functions.

Theorem 3: *Let $f, g : [0,\infty) \to \mathbb{R}$ be real valued functions, such that f is the function of t^α for $0 < \alpha \le 1$. If $L_\alpha\{f(t^\alpha)\}$ and $L_\alpha\{g(t)\}$ exist, then*

$$L_\alpha\{f * g\} = L_\alpha\{f(t^\alpha)\}\,L_\alpha\{g(t)\}, \tag{8}$$

where

$$(f * g)(t) = \int_0^t f(t^\alpha - \tau^\alpha)\,g(\tau)\,\tau^{\alpha-1}\,\mathrm{d}\tau. \tag{9}$$

Proof: We first apply the fractional Laplace transform to Eq. (9)

$$L_\alpha\{(f*g)(t)\} = \int_0^\infty \mathrm{e}^{-st}\left(\int_0^t f(t^\alpha - \tau^\alpha)\,g(\tau)\,\tau^{\alpha-1}\,\mathrm{d}\tau\right) t^{\alpha-1}\,\mathrm{d}t.$$

By changing the order of integration we get

$$L_\alpha\{(f*g)(t)\} = \int_0^\infty\int_\tau^\infty \mathrm{e}^{-s\frac{t^\alpha}{\alpha}} f(t^\alpha - \tau^\alpha)\,g(\tau)\,t^{\alpha-1}\tau^{\alpha-1}\,\mathrm{d}t\,\mathrm{d}\tau.$$

Then we substitute $t^\alpha - \tau^\alpha = u^\alpha$ into the above integral and obtain

$$L_\alpha\{(f*g)(t)\} = \int_0^\infty\int_0^\infty \mathrm{e}^{-s\frac{u^\alpha+\tau^\alpha}{\alpha}} f(u^\alpha)\,g(\tau)\,u^{\alpha-1}\,\mathrm{d}u\,\tau^{\alpha-1}\,\mathrm{d}\tau = \int_0^\infty \mathrm{e}^{-s\frac{u^\alpha}{\alpha}} f(u^\alpha)\,u^{\alpha-1}\,\mathrm{d}u \int_0^\infty \mathrm{e}^{-s\frac{\tau^\alpha}{\alpha}} g(\tau)\,\tau^{\alpha-1}\,\mathrm{d}\tau = L_\alpha\{f(t^\alpha)\}\,L_\alpha\{g(t)\}.$$

Example: Consider the non-homogenous conformable fractional initial value problem:

$$\frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha} = Ay(t) + f(t),\; y(0) = y_0,\; t > 0,$$

where $y, f : [0,\infty) \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Application of fractional Laplace transform to both sides of the equation gives

$$sL_\alpha\{y(t)\} - y_0 = AL_\alpha\{y(t)\} + L_\alpha\{f(t)\}$$

and then

$$L_\alpha\{y(t)\} = (sI - A)^{-1} y_0 + (sI - A)^{-1} L_\alpha\{f(t)\},$$

in which I is the identity matrix. The solution is obtained by taking the inverse fractional Laplace transform L_α^{-1}. By using Eq. (6), we easily deduce that

$$L_\alpha^{-1}\left\{(sI - A)^{-1} y_0\right\} = \mathrm{e}^{A\frac{t^\alpha}{\alpha}} y_0.$$

According to the Theorem 3, we obtain the solution as

$$y(t) = \mathrm{e}^{A\frac{t^\alpha}{\alpha}} y_0 + \int_0^t \mathrm{e}^{A\left(\frac{t^\alpha - \tau^\alpha}{\alpha}\right)} f(\tau)\,\tau^{\alpha-1}\,\mathrm{d}\tau.$$

We use the fractional Laplace transform to solve our problem according to the time variable t. Also, we apply the finite Fourier *sine* transform to eliminate the spatial variable x, $x \in [0, L]$ in the problem. The finite Fourier *sine* transform of a function $f : [0, L] \to \mathbb{R}$ is

$$F\{f(x)\} = f_n^* = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,\ n = 1, 2, \ldots, \tag{10}$$

with the inverse transform

$$F^{-1}\{f_n^*\} = f(x) = \sum_{n=1}^{\infty} f_n^* \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{11}$$

If $f(x,t)$ is a function of two variables, then

$$F\{f(x,t)\} = f_n^*(t) = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{12}$$

$$F\left\{\frac{\partial^2 f\{x,t\}}{\partial x^2}\right\} = -\left(\frac{n\pi}{L}\right)^2 f_n^*(t) + \frac{2n\pi}{L^2}\left[f(0,t) + (-1)^{n+1} f(L,t)\right]. \tag{13}$$

Note, that for the rest of this paper we denote both the fractional Laplace and the finite Fourier sine transforms by asterisk, to avoid the confusion of the transforms notations.

3. Problem formulation

The theory of thermal stresses of a solid is governed by the equilibrium equation in terms of displacements [3]

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} = \beta_T K_T\,\mathrm{grad}\,T, \tag{14}$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = \mu\boldsymbol{e} + (\lambda\,\mathrm{tr}\boldsymbol{e} - \beta_T K_T T)\,\boldsymbol{I} \tag{15}$$

and the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T,\ 0 < \alpha \le 1, \tag{16}$$

where $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{e}$ is the linear strain tensor, a is the diffusivity coefficient, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T is the thermal coefficient of volumetric expansion, $\boldsymbol{I}$ denotes the unit tensor.

In the present work, we consider a centrally symmetric temperature distribution $T(x,t)$ on a line segment $0 \le x \le L$ at a time t. In this case, the thermoelastic stress $\sigma(x,t)$ is proportional to the deviation from the average temperature [19]:

$$\sigma_{yy}(x,t) = -\frac{\alpha_T E}{1-\upsilon}\left[T(x,t) - T_{\mathrm{average}}(t)\right], \tag{17}$$

where

$$T_{\mathrm{average}}(t) = \frac{1}{L}\int_0^L T(x,t)\,\mathrm{d}x. \tag{18}$$

Here, α_T is the linear thermal expansion coefficient, E is Young's modulus and υ denotes Poisson's ratio.

Consider the temperature field $T(x,t)$, which satisfies the time-fractional heat conduction equation

$$\frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = a\frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < L,\ 0 < t < \infty,\ 0 < \alpha \le 1, \tag{19}$$

in which $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes conformable fractional derivative.

Let us assume the following initial

$$T(x,0) = 0, \tag{20}$$

and boundary conditions

$$x = 0 : T = g(t)\,T_0, \quad x = L : T = g(t)\,T_0, \tag{21}$$

where $g(t)$ is the boundary control function, which we use to find the optimal temperature regime, to keep the thermal stress under the intended values. Here, we first introduce the following non-dimensional quantities

$$\overline{x} = \frac{x}{L},\ \tau = \frac{t}{t_0},\ \overline{T} = \frac{T}{T_0},\ \kappa^2 = \frac{at_0^2}{L^2}, \tag{22}$$

where t_0 is the characteristic time. Hence, the problem reduces to

$$\frac{\partial^\alpha \overline{T}(\overline{x},\tau)}{\partial \tau^\alpha} = a\frac{\partial^2 \overline{T}(\overline{x},\tau)}{\partial \overline{x}^2}, \quad 0 < \overline{x} < 1,\ 0 < \tau < \infty,\ 0 < \alpha \le 1, \tag{23}$$

$$\tau = 0 : \overline{T} = 0, \tag{24}$$

$$\overline{x} = 0 : \overline{T} = g(\tau), \tag{25}$$

$$\overline{x} = 1 : \overline{T} = g(\tau). \tag{26}$$

Using the fractional Laplace transform with respect to time τ and the finite Fourier *sine* transform with respect to the spatial coordinate $\overline{x}$, we obtain

$$\overline{T}^{**} = \frac{2\kappa^2\xi_n}{s + \kappa^2\xi_n^2}g^*(s)\left[1 - (-1)^n\right], \tag{27}$$

where $\xi_n = n\pi$. Taking the inverse Fourier and the inverse fractional Laplace transforms leads to

$$\overline{T}(\overline{x},\tau) = 2\kappa^2\sum_{n=1}^{\infty}\xi_n\left[1-(-1)^n\right] \times \sin(\xi_n\overline{x})\int_0^\tau \mathrm{e}^{-\kappa^2\xi_n^2\frac{\tau^\alpha - u^\alpha}{\alpha}} g(u)\,u^{\alpha-1}\,\mathrm{d}u. \tag{28}$$

Similarly, we calculate the average value $\overline{T}_{\mathrm{average}}(\tau)$ using Eqs. (18) and (28)

$$\overline{T}_{average}(\tau) = 2\kappa^2\sum_{n=1}^{\infty}\left[1-(-1)^n\right]^2 \times \int_0^\tau \mathrm{e}^{-\kappa^2\xi_n^2\frac{\tau^\alpha - u^\alpha}{\alpha}} g(u)\,u^{\alpha-1}\,\mathrm{d}u. \tag{29}$$

Now, the associated non-dimensional thermal stress can be calculated as

$$\overline{\sigma}_{yy}(\overline{x},\tau) = \frac{1-\nu}{\alpha_T E T_0}\sigma_{yy}(\overline{x},\tau) \tag{30}$$

or

$$\overline{\sigma}_{yy}(\overline{x},\tau) = -\left[\overline{T}(\overline{x},\tau) - \overline{T}_{\mathrm{average}}(\tau)\right]. \tag{31}$$

Let $\overline{\sigma}_{yy}(1,\tau)$ represent the thermal stress at the boundary of line segment. We call

$$|\overline{\sigma}_{yy}(1,\tau)| = \overline{\sigma}_{\rm crit} \tag{32}$$

and also assume that maximal temperature and the resulting maximal thermal stress are reached at the boundary: $|\overline{\sigma}_{\max}(\tau)| = |\overline{\sigma}_{yy}(1,\tau)|$. Taking into account Eqs. (28)–(32), we get

$$g(\tau) = \overline{\sigma}_{\rm crit} + 2\kappa^2 \int\limits_0^\tau \sum_{n=1}^{\infty} [1-(-1)^n]^2 \times \mathrm{e}^{-\kappa^2 \xi_n^2 \frac{\tau^\alpha - u^\alpha}{\alpha}} g(u)\, u^{\alpha-1}\, \mathrm{d}u. \tag{33}$$

To find the temperature control function $g(\tau)$, we apply a numerical approach that solves the integral Eq. (33). It is worth noting, that numerous numerical methods of analysis of thermal and mechanical components, arising from heat conduction, have recently been improved [22–24].

4. Numerical algorithm

We obtain the optimal boundary control of $g(\tau)$ by using the following iteration formula

$$g_{m+1}(\tau) = \overline{\sigma}_{\rm crit} + 2\kappa^2 \int\limits_0^\tau \sum_{n=1}^{\infty} [1-(-1)^n]^2 \times \mathrm{e}^{-\kappa^2 \xi_n^2 \frac{\tau^\alpha - u^\alpha}{\alpha}} g_m(u)\, u^{\alpha-1}\, \mathrm{d}u,\ m = 0,1,2,\ldots, \tag{34}$$

where we assume the initial values $g_0(\tau) = \overline{\sigma}_{\rm crit} = 1$. The integration of the iteration is numerically solved with cumulative trapezoidal rule. The numerical results are achieved by dividing the chosen time interval [0,1] into N equal parts. The obtained results are illustrated in figures under some variation of problem parameters. We plot all the figures for $N = 300$ and for the upper limit of the sum in Eq. (34) of 10. The dependence of the 10th iteration value of control function $g_{10}(\tau)$ on α, the order of conformable fractional derivative, is analyzed in Fig. 1. We estimate contribution of the iteration number to the solution in Fig. 2. Note, that the results overlap for the iteration number $m \geq 8$. In Fig. 3, we show the dependency of the boundary optimal control on the non-dimensional parameter κ.

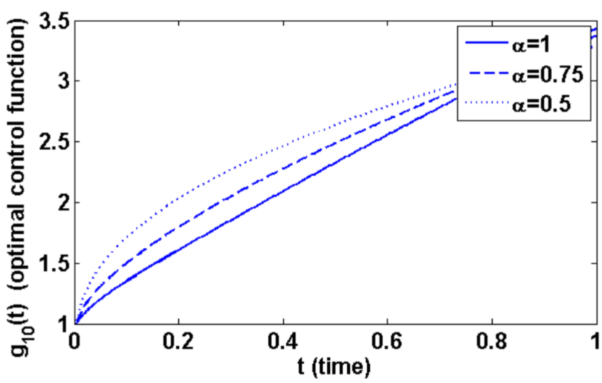

Fig. 1. Dependence of optimal control on the variation of α for $N = 300$ and $\kappa = 0.5$.

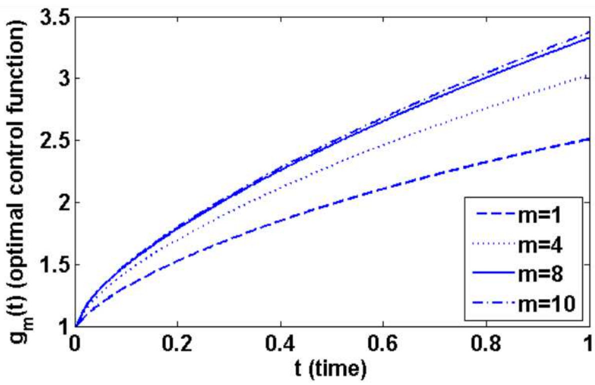

Fig. 2. Dependence of optimal control on iteration number m for $\alpha = 0.75$, $N = 300$ and $\kappa = 0.5$.

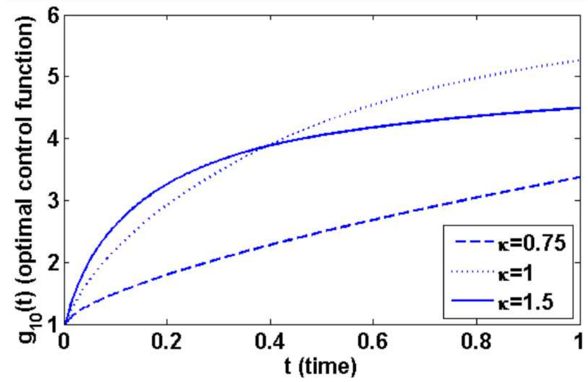

Fig. 3. Dependence of optimal control on the variation of κ for $\alpha = 0.75$ and $N = 300$.

5. Conclusions

In this study, optimal control problem of a sub-heat conduction process, defined by a time-conformable partial fractional differential equation, which is a local generalization of the problem from [8], has been considered. The boundary temperature has been studied as a control function that is used to bring the thermal stress within the desired range. To find the optimal boundary condition, the fractional Laplace transform has been initially applied with respect to the time variable. In line with the requirements, a useful theorem has been given, which can be used to attain the inverse fractional Laplace transform of convenient types of functions. Then the finite Fourier sine transform has been applied to the problem and an integral equation has been obtained for the boundary control. Finally, this integral equation has been solved by successive iterations and the optimal boundary control has been achieved numerically. Influence of the parameters on the solution has been shown using plots produced by MATLAB codes.

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