Math Sci (2017) 11:231–240
DOI 10.1007/s40096-017-0226-0

ORIGINAL RESEARCH

New contractive conditions of integral type on complete S-metric spaces

Nihal Yilmaz Özgür[1] · Nihal Taş[1]

Received: 21 November 2016 / Accepted: 24 May 2017 / Published online: 6 June 2017

Abstract An S-metric space is a three-dimensional generalization of a metric space. In this paper our aim is to examine some fixed-point theorems using new contractive conditions of integral type on a complete S-metric space. We give some illustrative examples to verify the obtained results. Our findings generalize some fixed-point results on a complete metric space and on a complete S-metric space. An application to the Fredholm integral equation is also obtained.

Keywords Integral-type contractive conditions · Fixed point · S-metric

Mathematics Subject Classification Primary 47H10 · Secondary 54H25

Introduction

Recently, the notion of an S-metric has been introduced and studied as a generalization of a metric. This notion has been defined by Sedghi et al. [13] as follows:

Definition 1.1 [13] Let $X \neq \emptyset$ be any set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $u, v, z, a \in X$.

(S1) $S(u, v, z) = 0$ if and only if $u = v = z$.
(S2) $S(u, v, z) \leq S(u, u, a) + S(v, v, a) + S(z, z, a)$.

✉ Nihal Yilmaz Özgür
nihal@balikesir.edu.tr

Nihal Taş
nihaltas@balikesir.edu.tr

[1] Department of Mathematics, Balıkesir University, 10145 Balıkesir, Turkey

Then the function S is called an S-metric on X and the pair (X, S) is called an S-metric space.

Some fixed-point theorems have been given for self-mappings satisfying various contractive conditions on an S-metric space (see [4, 6, 8, 9, 13, 14]). One of the important results among these studies is the Banach's contraction principle on a complete S-metric space.

Theorem 1.2 [13] *Let (X, S) be a complete S-metric space, $h \in (0, 1)$ and $T : X \to X$ be a self-mapping of X such that*

$$S(Tu, Tu, Tv) \leq hS(u, u, v),$$

for all $u, v \in X$. Then T has a unique fixed point in X.

On the other hand some generalizations of the well-known Ćirić's and Nemytskii-Edelstein fixed-point theorems obtained on S-metric spaces via some new fixed point results (see [8, 9, 13, 14] for more details).

Later, different applications of some contractive conditions have been constructed on an S-metric space such as differential equations, complex valued functions etc. (see [5, 7, 10, 11]).

In recent years, fixed-point theory has been examined for various contractive conditions. For example, contractive conditions of integral type were adapted into some studied fixed-point results. So more general fixed-point theorems were obtained.

Through the whole paper we assume that $\varsigma : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, nonnegative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \varsigma(t)\mathrm{d}t > 0. \tag{1}$$

Branciari [1] studied a fixed-point theorem for a general contractive condition of integral type on a complete metric space as seen in the following theorem.

Theorem 1.3 [1] *Let (X, ρ) be a complete metric space, $h \in (0,1)$, the function $\varsigma : [0,\infty) \to [0,\infty)$ be defined as in (1) and $T : X \to X$ be a self-mapping of X such that*

$$\int_0^{\rho(Tu,Tv)} \varsigma(t)\mathrm{d}t \le h \int_0^{\rho(u,v)} \varsigma(t)\mathrm{d}t,$$

for all $u, v \in X$, then T has a unique fixed point $w \in X$ such that

$$\lim_{n\to\infty} T^n u = w,$$

for each $u \in X$.

After the study of Branciari, some researchers have investigated new generalized contractive conditions of integral type using different known inequalities on various metric spaces (see [2, 3, 12]).

The purpose of this paper is to give new contractive conditions of integral type satisfying some new generalized inequalities given in [6] on a complete S-metric space. Our results generalize some known fixed-point results on a complete metric space and on a complete S-metric space.

Fixed-point results under some contractive conditions of integral type

In this section we obtain new fixed-point theorems using some contractive conditions of integral type on a complete S-metric space. We construct three examples to show the validity of our results. At first we recall some basic results about S-metric spaces.

Lemma 2.1 [13] *Let (X, S) be an S-metric space. Then we have*

$$S(u,u,v) = S(v,v,u).$$

The above Lemma 2.1 can be considered as a symmetry condition on an S-metric space. The following definition is related to convergent sequences on an S-metric space.

Definition 2.2 [13] Let (X, S) be an S-metric space.

(1) A sequence $\{u_n\}$ in X converges to u if and only if $S(u_n, u_n, u) \to 0$ as $n \to \infty$. That is, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $S(u_n, u_n, u) < \varepsilon$ for each $\varepsilon > 0$. We denote this by

$$\lim_{n\to\infty} u_n = u \text{or} \lim_{n\to\infty} \mathrm{S}(\mathrm{u_n}, \mathrm{u_n}, \mathrm{u}) = 0.$$

(2) A sequence $\{u_n\}$ in X is called a Cauchy sequence if $S(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$. That is, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \ge n_0$, $S(u_n, u_n, u_m) < \varepsilon$ for each $\varepsilon > 0$.

(3) The S-metric space (X, S) is called complete if every Cauchy sequence is convergent.

In the following lemma we see the relationship between a metric and an S-metric.

Lemma 2.3 [4] *Let (X, ρ) be a metric space. Then the following properties are satisfied :*

(1) *$S_\rho(u,v,z) = \rho(u,z) + \rho(v,z)$ for all $u, v, z \in X$ is an S-metric on X.*

(2) *$u_n \to u$ in (X, ρ) if and only if $u_n \to u$ in (X, S_ρ).*

(3) *$\{u_n\}$ is Cauchy in (X, ρ) if and only if $\{u_n\}$ is Cauchy in (X, S_ρ).*

(4) *(X, ρ) is complete if and only if (X, S_ρ) is complete.*

We call the function S_ρ defined in Lemma 2.3 (1) as the S-metric generated by the metric ρ. It can be found an example of an S-metric which is not generated by any metric in [4, 9].

Now we give the following theorem.

Theorem 2.4 *Let (X, S) be a complete S-metric space, $h \in (0,1)$, the function $\varsigma : [0,\infty) \to [0,\infty)$ be defined as in (1) and $T : X \to X$ be a self-mapping of X such that*

$$\int_0^{S(Tu,Tu,Tv)} \varsigma(t)\mathrm{d}t \le h \int_0^{S(u,u,v)} \varsigma(t)\mathrm{d}t, \tag{2}$$

for all $u, v \in X$. Then T has a unique fixed point $w \in X$ and we have

$$\lim_{n\to\infty} T^n u = w,$$

for each $u \in X$.

Proof Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined as

$$T^n u_0 = u_n.$$

Suppose that $u_n \ne u_{n+1}$ for all n. Using the inequality (2), we obtain

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le h \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \le \cdots \le h^n \int_0^{S(u_0,u_0,u_1)} \varsigma(t)\mathrm{d}t. \tag{3}$$

If we take limit for $n \to \infty$, using the inequality (3) we get

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t = 0,$$

since $h \in (0,1)$. The condition (1) implies

$$\lim_{n\to\infty} S(u_n, u_n, u_{n+1}) = 0.$$

Now we show that the sequence $\{u_n\}$ is a Cauchy sequence. Assume that $\{u_n\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ such that $m_k < n_k < m_{k+1}$ with

$$S(u_{m_k}, u_{m_k}, u_{n_k}) \geq \varepsilon \tag{4}$$

and

$$S(u_{m_k}, u_{m_k}, u_{n_k-1}) < \varepsilon.$$

Hence using Lemma 2.1, we have

$$\begin{aligned} S(u_{m_k-1}, u_{m_k-1}, u_{n_k-1}) &\leq 2S(u_{m_k-1}, u_{m_k-1}, u_{m_k}) \\ &\quad + S(u_{n_k-1}, u_{n_k-1}, u_{m_k}) \\ &< 2S(u_{m_k-1}, u_{m_k-1}, u_{m_k}) + \varepsilon \end{aligned}$$

and

$$\lim_{k\to\infty} \int_0^{S(u_{m_k-1}, u_{m_k-1}, u_{n_k-1})} \varsigma(t)\mathrm{d}t \leq \int_0^{\varepsilon} \varsigma(t)\mathrm{d}t. \tag{5}$$

Using the inequalities (2), (4) and (5) we obtain

$$\begin{aligned} \int_0^{\varepsilon} \varsigma(t)\mathrm{d}t &\leq \int_0^{S(u_{m_k}, u_{m_k}, u_{n_k})} \varsigma(t)\mathrm{d}t \leq h \int_0^{S(u_{m_k-1}, u_{m_k-1}, u_{n_k-1})} \varsigma(t)\mathrm{d}t \\ &\leq h \int_0^{\varepsilon} \varsigma(t)\mathrm{d}t, \end{aligned}$$

which is a contradiction with our assumption since $h \in (0,1)$. So the sequence $\{u_n\}$ is Cauchy. Using the completeness hypothesis, there exists $w \in X$ such that

$$\lim_{n\to\infty} T^n u_0 = w.$$

From the inequality (2) we find

$$\int_0^{S(Tw,Tw,u_{n+1})} \varsigma(t)\mathrm{d}t = \int_0^{S(Tw,Tw,Tu_n)} \varsigma(t)\mathrm{d}t \leq h \int_0^{S(w,w,u_n)} \varsigma(t)\mathrm{d}t.$$

If we take limit for $n \to \infty$, we get

$$\int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t = 0,$$

which implies $Tw = w$.

Now we show the uniqueness of the fixed point. Suppose that w_1 is another fixed point of T. Using the inequality (2) we have

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = \int_0^{S(Tw,Tw,Tw_1)} \varsigma(t)\mathrm{d}t \leq h \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t,$$

which implies

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = 0,$$

since $h \in (0,1)$. Using the inequality (1) we get $w = w_1$. Consequently, the fixed point w is unique. □

Remark 2.5

(1) If we set the function $\varsigma : [0,\infty) \to [0,\infty)$ in Theorem 2.4 as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$, then we obtain the Banach's contraction principle on a complete S-metric space.

(2) Since an S-metric space is a generalization of a metric space, Theorem 2.4 is a generalization of the classical Banach's fixed-point theorem.

(3) If we set the S-metric as $S : X \times X \times X \to \mathbb{C}$ and take the function $\varsigma : [0,\infty) \to [0,\infty)$ as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$ in Theorem 2.4, then we get Theorem 3.1 in [10] and Corollary 2.5 in [5] for $n = 1$.

Example 2.6 Let $X = \mathbb{R}$, $k > 1$ be a fixed real number and the function $S : X \times X \times X \to [0,\infty)$ be defined as

$$S(u,v,z) = \frac{k}{k+1}(|v-z| + |v+z-2u|),$$

for all $u,v,z \in \mathbb{R}$. It can be easily seen that the function S is an S-metric. Now we show that this S-metric can not be generated by any metric ρ. On the contrary, we assume that there exists a metric ρ such that

$$S(u,v,z) = \rho(u,z) + \rho(v,z), \tag{6}$$

for all $u,v,z \in \mathbb{R}$. Hence we find

$$S(u,u,z) = 2\rho(u,z) = \frac{2k}{k+1}|u-z|$$

and

$$\rho(u,z) = \frac{k}{k+1}|u-z|. \tag{7}$$

Similarly, we get

$$S(v,v,z) = 2\rho(v,z) = \frac{2k}{k+1}|v-z|$$

and

$$\rho(v,z)=\frac{k}{k+1}|v-z|. \tag{8}$$

Using the equalities (6), (7) and (8), we obtain

$$\frac{k}{k+1}(|v-z|+|v+z-2u|)=\frac{k}{k+1}|u-z|+\frac{k}{k+1}|v-z|,$$

which is a contradiction. Consequently, S is not generated by any metric and $(\mathbb{R}, S)$ is a complete S-metric space.

Let us define the self-mapping $T:\mathbb{R}\to\mathbb{R}$ as

$$Tu=\frac{u}{6},$$

for all $u\in\mathbb{R}$ and the function $\varsigma:[0,\infty)\to[0,\infty)$ as

$$\varsigma(t)=3t^2,$$

for all $t\in[0,\infty)$. Then we get

$$\int_0^{\varepsilon}\varsigma(t)\mathrm{d}t=\int_0^{\varepsilon}3t^2\mathrm{d}t=\varepsilon^3>0,$$

for each $\varepsilon>0$. Therefore T satisfies the inequality (2) in Theorem 2.4 for $h=\frac{1}{2}$. Indeed, we have

$$\frac{k^3}{27(k+1)^3}|u-v|^3\le\frac{4k^3}{(k+1)^3}|u-v|^3,$$

for all $u,v\in\mathbb{R}$. Consequently, T has a unique fixed point $u=0$.

Now we give the first generalization of Theorem 2.4.

Theorem 2.7 *Let (X, S) be a complete S-metric space, the function $\varsigma:[0,\infty)\to[0,\infty)$ be defined as in (1) and $T:X\to X$ be a self-mapping of X such that*

$$\begin{aligned}\int_0^{S(Tu,Tu,Tv)}\varsigma(t)\mathrm{d}t\le{}& h_1\int_0^{S(u,u,v)}\varsigma(t)\mathrm{d}t+h_2\int_0^{S(Tu,Tu,v)}\varsigma(t)\mathrm{d}t\\&+h_3\int_0^{S(Tv,Tv,u)}\varsigma(t)\mathrm{d}t\\&+h_4\int_0^{\max\{S(Tu,Tu,u),S(Tv,Tv,v)\}}\varsigma(t)\mathrm{d}t,\end{aligned} \tag{9}$$

for all $u,v\in X$ with nonnegative real numbers h_i $(i\in\{1,2,3,4\})$ satisfying $\max\{h_1+3h_3+2h_4, h_1+h_2+h_3\}<1$. Then T has a unique fixed point $w\in X$ and we have

$$\lim_{n\to\infty}T^nu=w,$$

for each $u\in X$.

Proof Let $u_0\in X$ and the sequence $\{u_n\}$ be defined as

$$T^nu_0=u_n.$$

Suppose that $u_n\ne u_{n+1}$ for all n. Using the inequality (9), the condition $(S2)$ and Lemma 2.1 we get

$$\begin{aligned}\int_0^{S(u_n,u_n,u_{n+1})}\varsigma(t)\mathrm{d}t&=\int_0^{S(Tu_{n-1},Tu_{n-1},Tu_n)}\varsigma(t)\mathrm{d}t\le h_1\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t\\&\quad+h_2\int_0^{S(u_n,u_n,u_n)}\varsigma(t)\mathrm{d}t+h_3\int_0^{S(u_{n+1},u_{n+1},u_{n-1})}\varsigma(t)\mathrm{d}t\\&\quad+h_4\int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}}\varsigma(t)\mathrm{d}t\\&=h_1\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t+h_3\int_0^{S(u_{n+1},u_{n+1},u_{n-1})}\varsigma(t)\mathrm{d}t\\&\quad+h_4\int_0^{\max\{S(u_n,u_n,u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}}\varsigma(t)\mathrm{d}t\\&\le h_1\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t+h_3\int_0^{2S(u_{n+1},u_{n+1},u_n)}\varsigma(t)\mathrm{d}t\\&\quad+h_3\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t+h_4\int_0^{S(u_n,u_n,u_{n-1})}\varsigma(t)\mathrm{d}t\\&\quad+h_4\int_0^{S(u_{n+1},u_{n+1},u_n)}\varsigma(t)\mathrm{d}t\\&=(h_1+h_3+h_4)\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t\\&\quad+(2h_3+h_4)\int_0^{S(u_n,u_n,u_{n+1})}\varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(u_n,u_n,u_{n+1})}\varsigma(t)\mathrm{d}t\le\left(\frac{h_1+h_3+h_4}{1-2h_3-h_4}\right)\int_0^{S(u_{n-1},u_{n-1},u_n)}\varsigma(t)\mathrm{d}t. \tag{10}$$

If we put $h=\frac{h_1+h_3+h_4}{1-2h_3-h_4}$ then we find $h<1$ since $h_1+3h_3+2h_4<1$. Using the inequality (10) we have

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le h^n \int_0^{S(u_0,u_0,u_1)} \varsigma(t)\mathrm{d}t. \tag{11}$$

If we take limit for $n \to \infty$, using the inequality (11) we get

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t = 0,$$

since $h \in (0,1)$. The condition (1) implies $\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0$.

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence $\{u_n\}$ is Cauchy. Then there exists $w \in X$ such that

$$\lim_{n\to\infty} T^n u_0 = w,$$

since (X, S) is a complete S-metric space. From the inequality (9) we find

$$\begin{aligned}\int_0^{S(u_n,u_n,Tw)} \varsigma(t)\mathrm{d}t = \int_0^{S(Tu_{n-1},Tu_{n-1},Tw)} \varsigma(t)\mathrm{d}t &\le h_1 \int_0^{S(u_{n-1},u_{n-1},w)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(u_n,u_n,w)} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(Tw,Tw,u_{n-1})} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{\max\{S(u_n,u_n,u_{n-1}),S(Tw,Tw,w)\}} \varsigma(t)\mathrm{d}t.\end{aligned}$$

Taking limit for $n \to \infty$ and using Lemma 2.1 we get

$$\int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t \le (h_3 + h_4) \int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t,$$

which implies $Tw = w$ since $h_3 + h_4 < 1$.

Now we show the uniqueness of the fixed point. Let w_1 be another fixed point of T. Using the inequality (9) and Lemma 2.1, we get

$$\begin{aligned}\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = \int_0^{S(Tw,Tw,Tw_1)} \varsigma(t)\mathrm{d}t &\le h_1 \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(w_1,w_1,w)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{\max\{S(w,w,w),S(w_1,w_1,w_1)\}} \varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \le (h_1 + h_2 + h_3) \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t.$$

Then we obtain

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = 0,$$

that is, $w = w_1$ since $h_1 + h_2 + h_3 < 1$. Consequently, T has a unique fixed point $w \in X$. □

Remark 2.8

(1) If we set the function $\varsigma : [0,\infty) \to [0,\infty)$ in Theorem 2.7 as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$, then we obtain Theorem 3 in [6].

(2) Theorem 2.7 is a generalization of Theorem 2.4 on a complete S-metric space. Indeed, if we take $h_1 = h$ and $h_2 = h_3 = h_4 = 0$ in Theorem 2.7, then we get Theorem 2.4.

(3) Since Theorem 2.7 is a generalization of Theorem 2.4, Theorem 2.7 generalizes the classical Banach's fixed-point theorem.

(4) If we set the S-metric as $S : X \times X \times X \to \mathbb{C}$ and take the function $\varsigma : [0,\infty) \to [0,\infty)$ as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$ in Theorem 2.7, then we get Theorem 3.1 in [7].

Now we give the second generalization of Theorem 2.4.

Theorem 2.9 *Let (X, S) be a complete S-metric space, the function $\varsigma : [0,\infty) \to [0,\infty)$ be defined as in (1) and $T : X \to X$ be a self-mapping of X such that*

$$\begin{aligned}\int_0^{S(Tu,Tu,Tv)} \varsigma(t)\mathrm{d}t &\le h_1 \int_0^{S(u,u,v)} \varsigma(t)\mathrm{d}t + h_2 \int_0^{S(Tu,Tu,u)} \varsigma(t)\mathrm{d}t \\ &+ h_3 \int_0^{S(Tu,Tu,v)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{S(Tv,Tv,u)} \varsigma(t)\mathrm{d}t + h_5 \int_0^{S(Tv,Tv,v)} \varsigma(t)\mathrm{d}t \\ &+ h_6 \int_0^{\max\{S(u,u,v),S(Tu,Tu,u),S(Tu,Tu,v),S(Tv,Tv,u),S(Tv,Tv,v)\}} \varsigma(t)\mathrm{d}t,\end{aligned} \tag{12}$$

for all $u, v \in X$ *with nonnegative real numbers* h_i $(i \in \{1,2,3,4,5,6\})$ *satisfying* $\max\{h_1 + h_2 + 3h_4 + h_5 + 3h_6, h_1 + h_3 + h_4 + h_6\} < 1$. *Then* T *has a unique fixed point* $w \in X$ *and we have*

$$\lim_{n\to\infty} T^n u = w,$$

for each $u \in X$.

Proof Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined as

$$T^n u_0 = u_n.$$

Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (12), the condition $(S2)$ and Lemma 2.1 we get

$$\begin{aligned}\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t &= \int_0^{S(Tu_{n-1},Tu_{n-1},Tu_n)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(u_n,u_n,u_{n-1})} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(u_n,u_n,u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{S(u_{n+1},u_{n+1},u_{n-1})} \varsigma(t)\mathrm{d}t + h_5 \int_0^{S(u_{n+1},u_{n+1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_6 \int_0^{\max\{S(u_{n-1},u_{n-1},u_n),S(u_n,u_n,u_{n-1}),S(u_n,u_n,u_n),S(u_{n+1},u_{n+1},u_{n-1}),S(u_{n+1},u_{n+1},u_n)\}} \varsigma(t)\mathrm{d}t \\ &\le (h_1 + h_2 + h_4 + h_6) \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t + (2h_4 + h_5 + 2h_6) \int_0^{S(u_{n+1},u_{n+1},u_n)} \varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le \left(\frac{h_1 + h_2 + h_4 + h_6}{1 - 2h_4 - h_5 - 2h_6}\right) \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t. \tag{13}$$

If we put $h = \frac{h_1+h_2+h_4+h_6}{1-2h_4-h_5-2h_6}$ then we find $h < 1$ since $h_1 + h_2 + 3h_4 + h_5 + 3h_6 < 1$. Using the inequality (13) we have

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le h^n \int_0^{S(u_0,u_0,u_1)} \varsigma(t)\mathrm{d}t. \tag{14}$$

If we take limit for $n \to \infty$, using the inequality (14) we get

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t = 0,$$

since $h \in (0,1)$. The condition (1) implies

$$\lim_{n\to\infty} S(u_n, u_n, u_{n+1}) = 0.$$

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence $\{u_n\}$ is Cauchy. Then there exists $w \in X$ such that

$$\lim_{n\to\infty} T^n u_0 = w,$$

since (X, S) is a complete S-metric space. From the inequality (12) we find

$$\begin{aligned}\int_0^{S(u_n,u_n,Tw)} \varsigma(t)\mathrm{d}t &= \int_0^{S(Tu_{n-1},Tu_{n-1},Tw)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(u_{n-1},u_{n-1},w)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(u_n,u_n,u_{n-1})} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(u_n,u_n,w)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{S(Tw,Tw,u_{n-1})} \varsigma(t)\mathrm{d}t + h_5 \int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t \\ &+ h_6 \int_0^{\max\{S(u_{n-1},u_{n-1},w),S(u_n,u_n,u_{n-1}),S(u_n,u_n,w),S(Tw,Tw,u_{n-1}),S(Tw,Tw,w)\}} \varsigma(t)\mathrm{d}t.\end{aligned}$$

If we take limit for $n \to \infty$, using Lemma 2.1 we get

$$\int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t \le (h_4+h_5+h_6) \int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t,$$

which implies $Tw = w$ since $h_4 + h_5 + h_6 < 1$.

Now we show the uniqueness of the fixed point. Let w_1 be another fixed point of T. Using the inequality (12) and Lemma 2.1, we get

$$\begin{aligned}\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = \int_0^{S(Tw,Tw,Tw_1)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \\ + h_2 \int_0^{S(w,w,w)} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \\ + h_4 \int_0^{S(w_1,w_1,w)} \varsigma(t)\mathrm{d}t + h_5 \int_0^{S(w_1,w_1,w_1)} \varsigma(t)\mathrm{d}t \\ + h_6 \int_0^{\max\{S(w,w,w_1),S(w,w,w),S(w,w,w_1),S(w_1,w_1,w),S(w_1,w_1,w_1)\}} \varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \le (h_1+h_3+h_4+h_6) \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t.$$

Then we obtain

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = 0,$$

that is, $w = w_1$ since $h_1 + h_3 + h_4 + h_6 < 1$. Consequently, T has a unique fixed point $w \in X$. □

Remark 2.10

(1) In Theorem 2.9, if we set the function $\varsigma : [0,\infty) \to [0,\infty)$ as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$, then we obtain Theorem 4 in [6].

(2) Theorem 2.9 is a generalization of Theorem 2.4 on a complete S-metric space. Indeed, if we take $h_1 = h$ and $h_2 = h_3 = h_4 = h_5 = h_6 = 0$ in Theorem 2.9, then we get Theorem 2.4.

(3) Since Theorem 2.9 is another generalization of Theorem 2.4, Theorem 2.9 generalizes the classical Banach's fixed-point theorem.

(4) If we set the S-metric as $S : X \times X \times X \to \mathbb{C}$ and take the function $\varsigma : [0,\infty) \to [0,\infty)$ as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$ in Theorem 2.9, then we get Theorem 3.4 in [7].

In the following example we give a self-mapping satisfying the conditions of Theorems 2.7 and 2.9, respectively, but does not satisfy the condition of Theorem 2.4.

Example 2.11 Let $\mathbb{R}$ be the complete S-metric space with the S-metric defined in Example 1 given in [9]. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$Tu = \begin{cases} u+80 & \text{if} \quad u \in \{0,2\} \\ 75 & \text{if} \quad \text{otherwise} \end{cases},$$

for all $u \in \mathbb{R}$ and the function $\varsigma : [0,\infty) \to [0,\infty)$ as

$$\varsigma(t) = 2t,$$

for all $t \in [0,\infty)$. Then we get

$$\int_0^{\varepsilon} \varsigma(t)\mathrm{d}t = \int_0^{\varepsilon} 2t\mathrm{d}t = \varepsilon^2 > 0,$$

for each $\varepsilon > 0$. Therefore T satisfies the inequality (9) in Theorem 2.7 for $h_1 = h_2 = h_3 = 0$, $h_4 = \frac{1}{2}$ and the inequality (12) in Theorem 2.9 for $h_1 = h_3 = h_4 = h_5 = 0$, $h_2 = h_6 = \frac{1}{3}$. Hence T has a unique fixed point $u = 75$. But T does not satisfy the inequality (2) in Theorem 2.4. Indeed, if we take $u = 0$ and $v = 1$, then we obtain

$$\int_0^{10} 2t\mathrm{d}t = 100 \le h \int_0^{2} 2t\mathrm{d}t = 4h,$$

which is a contradiction since $h \in (0,1)$.

Finally, we give another generalization of Theorem 2.4.

Theorem 2.12 Let (X, S) be a complete S-metric space, the function $\varsigma : [0,\infty) \to [0,\infty)$ be defined as in (1) and $T : X \to X$ be a self-mapping of X such that

$$\begin{aligned}\int_0^{S(Tu,Tu,Tv)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(u,u,v)} \varsigma(t)\mathrm{d}t + h_2 \int_0^{S(Tu,Tu,u)} \varsigma(t)\mathrm{d}t \\ + h_3 \int_0^{S(Tv,Tv,v)} \varsigma(t)\mathrm{d}t \\ + h_4 \int_0^{\max\{S(Tu,Tu,v),S(Tv,Tv,u)\}} \varsigma(t)\mathrm{d}t,\end{aligned} \tag{15}$$

for all $u, v \in X$ with nonnegative real numbers h_i $(i \in \{1,2,3,4\})$ satisfying $h_1 + h_2 + h_3 + 3h_4 < 1$. Then T has a unique fixed point $w \in X$ and we have

$$\lim_{n\to\infty} T^n u = w,$$

for each $u \in X$.

Proof Let $u_0 \in X$ and the sequence $\{u_n\}$ be defined as $T^n u_0 = u_n$.

Suppose that $u_n \neq u_{n+1}$ for all n. Using the inequality (15), the condition ($S2$) and Lemma 2.1 we get

$$\begin{aligned}\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t &= \int_0^{S(Tu_{n-1},Tu_{n-1},Tu_n)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(u_n,u_n,u_{n-1})} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(u_{n+1},u_{n+1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{\max\{S(u_n,u_n,u_n),S(u_{n+1},u_{n+1},u_{n-1})\}} \varsigma(t)\mathrm{d}t \\ &\le h_1 \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t + h_2 \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ h_3 \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{2S(u_n,u_n,u_{n+1})+S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \\ &\le (h_1+h_2+h_4) \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t \\ &+ (h_3+2h_4) \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le \left(\frac{h_1+h_2+h_4}{1-h_3-2h_4}\right) \int_0^{S(u_{n-1},u_{n-1},u_n)} \varsigma(t)\mathrm{d}t. \tag{16}$$

If we put $h = \frac{h_1+h_2+h_4}{1-h_3-2h_4}$ then we find $h<1$ since $h_1+h_2+h_3+3h_4<1$. Using the inequality (16) and mathematical induction, we have

$$\int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t \le h^n \int_0^{S(u_0,u_0,u_1)} \varsigma(t)\mathrm{d}t. \tag{17}$$

Taking limit for $n \to \infty$ and using the inequality (17) we find

$$\lim_{n\to\infty} \int_0^{S(u_n,u_n,u_{n+1})} \varsigma(t)\mathrm{d}t = 0,$$

since $h \in (0,1)$. The condition (1) implies

$$\lim_{n\to\infty} S(u_n,u_n,u_{n+1}) = 0.$$

By the similar arguments used in the proof of Theorem 2.4, we see that the sequence $\{u_n\}$ is Cauchy. Then there exists $w \in X$ such that

$$\lim_{n\to\infty} T^n u_0 = w,$$

since (X, S) is a complete S-metric space. From the inequality (15) we find

$$\begin{aligned}\int_0^{S(u_n,u_n,Tw)} \varsigma(t)\mathrm{d}t &= \int_0^{S(Tu_{n-1},Tu_{n-1},Tw)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(u_{n-1},u_{n-1},w)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(u_n,u_n,u_{n-1})} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{\max\{S(u_n,u_n,w),S(Tw,Tw,u_{n-1})\}} \varsigma(t)\mathrm{d}t.\end{aligned}$$

If we take limit for $n \to \infty$, using Lemma 2.1 we get

$$\int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t \le (h_3+h_4) \int_0^{S(Tw,Tw,w)} \varsigma(t)\mathrm{d}t,$$

which implies $Tw = w$ since $h_3+h_4<1$.

Now we show the uniqueness of the fixed point. Let w_1 be another fixed point of T. Using the inequality (15) and Lemma 2.1, we get

$$\begin{aligned}\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t &= \int_0^{S(Tw,Tw,Tw_1)} \varsigma(t)\mathrm{d}t \le h_1 \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \\ &+ h_2 \int_0^{S(w,w,w)} \varsigma(t)\mathrm{d}t + h_3 \int_0^{S(w_1,w_1,w_1)} \varsigma(t)\mathrm{d}t \\ &+ h_4 \int_0^{\max\{S(w,w,w_1),S(w_1,w_1,w)\}} \varsigma(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t \le (h_1+h_4) \int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t.$$

Then we obtain

$$\int_0^{S(w,w,w_1)} \varsigma(t)\mathrm{d}t = 0,$$

that is, $w = w_1$ since $h_1 + h_4 < 1$. Consequently, T has a unique fixed point $w \in X$. □

Remark 2.13

(1) If we set the function $\varsigma : [0,\infty) \to [0,\infty)$ in Theorem 2.12 as

$$\varsigma(t) = 1,$$

for all $t \in [0,\infty)$, then we obtain Theorem 2 in [6].

(2) Theorem 2.12 is another generalization of Theorem 2.4 on a complete S-metric space. Indeed, if we take $h_1 = h$ and $h_2 = h_3 = h_4 = 0$ in Theorem 2.12, then we get Theorem 2.4.

(3) Since Theorem 2.12 is another generalization of Theorem 2.4, Theorem 2.12 generalizes the classical Banach's fixed-point theorem.

Let us consider the self-mapping $T : \mathbb{R} \to \mathbb{R}$ and the function $\varsigma : [0,\infty) \to [0,\infty)$ defined in Example 2.11. Then T satisfy the contractive condition (15) in Theorem 2.12 and so $u = 75$ is a unique fixed point of T. Notice that T does not satisfy the inequality (2) in Theorem 2.4.

An application to the Fredholm integral equation

In this section, we give an application of the contraction condition (2) to the Fredholm integral equation

$$y(u) = l(u) + \lambda \int_a^b k(u,t)y(t)\mathrm{d}t, \tag{18}$$

where $y : [a,b] \to \mathbb{R}$ with $-\infty < a < b < \infty$, $k(u, t)$ which is called the kernel of the integral equation (18) is continuous on the squared region $[a,b] \times [a,b]$ with $|k(u,t)| \leq M$ $(M > 0)$ and $l(u)$ is continuous on $[a,b]$.

Let $C[a,b] = \{f \mid f : [a,b] \to \mathbb{R} \text{isacontinuousfunction}\}$. Now we define the function $S : C[a,b] \times C[a,b] \times C[a,b] \to [0,\infty)$ by

$$S(f,g,h) = \sup_{u\in[a,b]} |f(u) - h(u)| + \sup_{u\in[a,b]} |f(u) + h(u) - 2g(u)|, \tag{19}$$

for all $f,g,h \in C[a,b]$. Then the function S is an S-metric. Now we show that this S-metric can not be generated by any metric ρ. We assume that this S-metric is generated by any metric ρ, that is, there exists a metric ρ such that

$$S(f,g,h) = \rho(f,h) + \rho(g,h), \tag{20}$$

for all $f,g,h \in C[a,b]$. Then we get

$$S(f,f,h) = 2\rho(f,h) = 2\sup_{u\in[a,b]} |f(u) - h(u)|$$

and

$$\rho(f,h) = \sup_{u\in[a,b]} |f(u) - h(u)|. \tag{21}$$

Similarly, we obtain

$$S(g,g,h) = 2\rho(g,h) = 2\sup_{u\in[a,b]} |g(u) - h(u)|$$

and

$$\rho(g,h) = \sup_{u\in[a,b]} |g(u) - h(u)|. \tag{22}$$

Using the equalities (20), (21) and (22), we find

$$\begin{aligned} &\sup_{u\in[a,b]} |f(u) - h(u)| + \sup_{u\in[a,b]} |f(u) + h(u) - 2g(u)| \\ &= \sup_{u\in[a,b]} |f(u) - h(u)| + \sup_{u\in[a,b]} |g(u) - h(u)|, \end{aligned}$$

which is a contradiction. Hence this S-metric is not generated by any metric ρ. Consequently, $(C[a,b], S)$ is a complete S-metric space.

Proposition 3.1 *Let $(C[a,b], S)$ be a complete S-metric space with the S-metric defined in (19) and λ be a real number with*

$$|\lambda| < \frac{1}{M(b-a)}.$$

Then the Fredholm integral equation (18) has a unique solution $y : [a,b] \to \mathbb{R}$.

Proof Let us define the function $T : C[a,b] \to C[a,b]$ as

$$Ty(u) = l(u) + \lambda \int_a^b k(u,t)y(t)\mathrm{d}t.$$

Now we show that T satisfies the contractive condition (2). We get

$$\begin{aligned} S(Ty_1, Ty_1, Ty_2) &= 2\sup_{u\in[a,b]} |Ty_1(u) - Ty_2(u)| \\ &= 2\sup_{u\in[a,b]} \left| \lambda \int_a^b k(u,t)(y_1(u) - y_2(u))\mathrm{d}t \right| \\ &\leq 2|\lambda| M \sup_{u\in[a,b]} \left| \int_a^b (y_1(u) - y_2(u))\mathrm{d}t \right| \\ &\leq 2|\lambda| M \sup_{u\in[a,b]} \int_a^b |y_1(u) - y_2(u)|\mathrm{d}t \\ &\leq 2|\lambda| M \sup_{u\in[a,b]} |y_1(u) - y_2(u)| \left| \int_a^b \mathrm{d}t \right| \\ &\leq |\lambda| M(b-a) S(y_1, y_1, y_2) \\ &< S(y_1, y_1, y_2), \end{aligned}$$

which implies

$$\int_0^{S(Ty_1,Ty_1,Ty_2)} \varsigma(t)\mathrm{d}t < \int_0^{S(y_1,y_1,y_2)} \varsigma(t)\mathrm{d}t.$$

Consequently, the contractive condition (2) is satisfied and the Fredholm integral equation (18) has a unique solution y. □

Now we give an example of Proposition 3.1.

Example 3.2 Let us consider the Fredholm integral equation defined as

$$y(u) = e + \lambda \int_1^e \frac{\ln u}{t} y(t)\mathrm{d}t. \tag{23}$$

Now we find a solution of the Fredholm integral equation (23) with the initial condition $y_0(u) = 0$. We solve this equation for $|\lambda| < \frac{1}{e-1}$ since $\left|\frac{\ln u}{t}\right| < 1$ for all $1 \le u, t \le e$. We obtain

$$y_1(u) = e,$$

$$y_2(u) = e + \lambda \int_1^e \frac{\ln u}{t} e\mathrm{d}t = e + \lambda e \ln u,$$

$$\begin{aligned} y_3(u) &= e + \lambda \int_1^e \frac{\ln u}{t}(e + \lambda e \ln t)\mathrm{d}t \\ &= e + \lambda e \ln u + \frac{\lambda^2}{2} e \ln u, \end{aligned}$$

$$\begin{aligned} y_4(u) &= e + \lambda \int_1^e \frac{\ln u}{t}\left(e + \lambda e \ln t + \frac{\lambda^2}{2} e \ln t\right)\mathrm{d}t \\ &= e + \lambda e \ln u + \frac{\lambda^2}{2} e \ln u + \frac{\lambda^3}{2} e \ln u, \end{aligned}$$

...

$$\begin{aligned} y_n(u) &= e + \lambda e \ln u \left[1 + \frac{\lambda}{2} + \frac{\lambda^2}{4} + \cdots + \frac{\lambda^n}{2^n}\right] \\ &\to e + \frac{2\lambda}{2-\lambda} e \ln u. \end{aligned}$$

Consequently, this is a solution of the Fredholm integral equation (18) for $|\lambda| < \frac{1}{e-1} < 1$.

References

1. Branciari, A.: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. **29**(9), 531–536 (2002)
2. Dey, D., Ganguly, A., Saha, M.: Fixed point theorems for mappings under general contractive condition of integral type. Bull. Math. Anal. Appl. **3**(1), 27–34 (2011)
3. Gu, F., Ye, H.: Common fixed point theorems of Altman integral type mappings in *G*-metric spaces, *Abstr. Appl. Anal.* vol. 2012, Article ID 630457, 13 pages. doi:10.1155/2012/630457
4. Hieu, N.T., Ly, N.T., Dung, N.V.: A generalization of Ciric quasi-contractions for maps on *S*-metric spaces. Thai J. Math. **13**(2), 369–380 (2015)
5. Mlaiki, N.M.: Common fixed points in complex *S*-metric space. Adv. Fixed Point Theory **4**(4), 509–524 (2014)
6. Özgür, N.Y., Taş, N.: Some Generalizations of Fixed Point Theorems on *S*-Metric Spaces, Essays in Mathematics and Its Applications in Honor of Vladimir Arnold. Springer, New York (2016)
7. Özgür, N.Y., Taş, N.: Some generalizations of the Banach's contraction principle on a complex valued *S*-metric space. J. New Theory **2**(14), 26–36 (2016)
8. Özgür, N.Y., Taş, N.: Some fixed point theorems on *S*-metric spaces. Mat. Vesnik **69**(1), 39–52 (2017)
9. Özgür, N.Y., Taş, N.: Some new contractive mappings on *S*-metric spaces and their relationships with the mapping **(S25)**. Math. Sci. **11**(7), 7–16 (2017). doi:10.1007/s40096-016-0199-4
10. Özgür, N.Y., Taş, N.: Common fixed point results on complex valued *S*-metric spaces. (submitted for publication)
11. Özgür, N.Y., Taş, N.: The Picard theorem on *S*-metric spaces. (submitted for publication)
12. Rahman, M.U., Sarwar, M., Rahman, M.U.: Fixed point results of Altman integral type mappings in *S*-metric spaces. Int. J. Anal. Appl. **10**(1), 58–63 (2016)
13. Sedghi, S., Shobe, N., Aliouche, A.: A generalization of fixed point theorems in *S*-metric spaces. Mat. Vesn. **64**(3), 258–266 (2012)
14. Sedghi, S., Dung, N.V.: Fixed point theorems on *S*-metric spaces. Mat. Vesn. **66**(1), 113–124 (2014)