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ON GENERALIZED SPHERICAL SURFACES IN EUCLIDEAN SPACES

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Abstract. In the present study we consider the generalized rotational surfaces in Euclidean spaces. Firstly, we consider generalized spherical curves in Euclidean $(n+1)-$space $\mathbb{E}^{n+1}$. Further, we introduce some kind of generalized spherical surfaces in Euclidean spaces $\mathbb{E}^3$ and $\mathbb{E}^4$ respectively. We have shown that the generalized spherical surfaces of first kind in $\mathbb{E}^4$ are known as rotational surfaces, and the second kind generalized spherical surfaces are known as meridian surfaces in $\mathbb{E}^4$. We have also calculated the Gaussian, normal and mean curvatures of these kind of surfaces. Finally, we give some examples.

1. Introduction

The Gaussian curvature and mean curvature of the surfaces in Euclidean spaces play an important role in differential geometry. Especially, surfaces with constant Gaussian curvature [19], and constant mean curvature conform nice classes of surfaces which are important for surface modelling [5]. Surfaces with constant negative curvature are known as pseudo-spherical surfaces [15].

Rotational surfaces in Euclidean spaces are also important subject of differential geometry. The rotational surfaces in $\mathbb{E}^3$ are called surface of revolution. Recently V. Velickovic classified all rotational surfaces in $\mathbb{E}^3$ with constant Gaussian curvature [18]. Rotational surfaces in $\mathbb{E}^4$ was first introduced by C. Moore in 1919. In the recent years some mathematicians have taken an interest in the rotational surfaces in $\mathbb{E}^4$, for example G. Ganchev and V. Milousheva [13], U. Dursun and N. C. Turgay [12], the second author, et al. [1] and D.W.Yoon [20]. In

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[13], the authors applied invariance theory of surfaces in the four dimensional Euclidean space to the class of general rotational surfaces whose meridians lie in two-dimensional planes in order to find all minimal super-conformal surfaces. These surfaces were further studied in [12], which found all minimal surfaces by solving the differential equation that characterizes minimal surfaces. They then determined all pseudo-umbilical general rotational surfaces in $\mathbb{E}^4$. See, also [3] for Rotational embeddings in E^4 with pointwise 1-type gauss map. The second author et.al in [1] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical, they also shown that each general rotational surface is a Chen surface in $\mathbb{E}^4$ and gave some special classes of generalized rotational surfaces as examples. See also [10] and [4] rotational surfaces with Constant Gaussian Curvature in Four-Space. For higher dimensional case N.H. Kuiper defined rotational embedded submanifolds in Euclidean spaces [16].

The meridian surfaces in $\mathbb{E}^4$ was first introduced by G. Ganchev and V. Milousheva (See, [14] and [2]) which are the special kind of rotational surfaces. Basic source of examples of surfaces in 4-dimensional Euclidean or pseudo-Euclidean space are the standard rotational surfaces and the general rotational surfaces.

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^n$. Section 3 explains some geometric properties of spherical curves $\mathbb{E}^{n+1}$. Section 4 tells about the generalized spherical surfaces in $\mathbb{E}^{n+m}$. Further this section provides some basic properties of generalized spherical surfaces in $\mathbb{E}^4$ and the structure of their curvatures. We also shown that every generalized spherical surfaces in $\mathbb{E}^4$ have constant Gaussian curvature $K = 1/c^2$. Finally, we present some examples of generalized spherical surfaces in $\mathbb{E}^4$.

2. Basic concepts

Let M be a smooth surface in $\mathbb{E}^n$ given with the patch $X(u,v)$: $(u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$g_{11} = \langle X_u, X_u \rangle, g_{12} = \langle X_u, X_v \rangle, g_{22} = \langle X_v, X_v \rangle, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = g_{11}g_{22} - g_{12}^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^n = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^n$.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Given any local vector fields X_1, X_2 tangent to M, consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \le i, j \le 2 \tag{2}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^n$, respectively. This map is well-defined, symmetric and bilinear [7].

For any arbitrary orthonormal frame field $\{N_1, N_2, ..., N_{n-2}\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_k} X_j = -(\widetilde{\nabla}_{X_j} N_k)^T, \quad X_j \in \chi(M). \tag{3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = L_{ij}^k,\ 1 \le i, j \le 2;\ 1 \le k \le n-2 \tag{4}$$

where L_{ij}^k are the coefficients of the second fundamental form. The equation (2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{n-2} L_{ij}^k N_k, \quad 1 \le i, j \le 2 \tag{5}$$

holds. Then the Gaussian curvature K of a regular patch $X(u, v)$ is given by

$$K = \frac{1}{W^2} \sum_{k=1}^{n-2} (L_{11}^k L_{22}^k - (L_{12}^k)^2). \tag{6}$$

Further, the mean curvature vector of a regular patch $X(u, v)$ is given by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{n-2} (L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12}) N_k. \tag{7}$$

We call the functions

$$H_k = \frac{(L_{11}^k g_{22} + L_{22}^k g_{11} - 2L_{12}^k g_{12})}{2W^2}, \tag{8}$$

the k.th mean curvature functions of the given surface. The norm of the mean curvature vector $H = \left\| \overrightarrow{H} \right\|$ is called the mean curvature of M.

Recall that a surface M is said to be *flat* (resp. *minimal*) if its Gauss curvature (resp. mean curvature vector) vanishes identically [8], [9].

The normal curvature K_N of M is defined by (see [11])

$$K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \left\langle R^{\perp}(X_1, X_2)N_\alpha, N_\beta \right\rangle^2 \right\}^{1/2}. \tag{9}$$

where

$$R^{\perp}(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha}X_j) - h(X_j, A_{N_\alpha}X_i), \tag{10}$$

and

$$\left\langle R^{\perp}(X_i, X_j)N_\alpha, N_\beta \right\rangle = \left\langle [A_{N_\alpha}, A_{N_\beta}]X_i, X_j \right\rangle, \tag{11}$$

is called the *equation of Ricci*. We observe that the normal connection D of M is flat if and only if $K_N = 0$ and by a result of Cartan, this equivalent to the diagonalisability of all shape operators A_{N_α} [7].

3. Generalized spherical curves

Let γ be a regular oriented curve in $\mathbb{E}^{n+1}$ that does not lie in any subspace of $\mathbb{E}^{n+1}$. From each point of the curve γ one can draw a segment of unit length along the normal line corresponding to the chosen orientation. The ends of these segments describe a new curve β. The curve $\gamma \in \mathbb{E}^{n+1}$ is called a *generalized spherical curve* if the curve β lies in a certain subspace $\mathbb{E}^n$of $\mathbb{E}^{n+1}$. The curve β is called the trace of γ [15]. Let

$$\gamma(u) = (f_1(u), ..., f_{n+1}(u)), \tag{12}$$

be the radius vector of the curve γ given with arclength parametrization u, i.e., $\|\gamma'(u)\| = 1$. The curve β is defined by the radius vector

$$\beta(u) = (\gamma + c^2\gamma'')(u) = \left((f_1 + c^2 f_1'')(u), ..., (f_{n+1} + c^2 f_{n+1}'')(u)\right), \tag{13}$$

where c is a real constant. If γ is a generalized spherical curve of $\mathbb{E}^{n+1}$ then by definition the curve β lies in the hyperplane $\mathbb{E}^n$ if and only if $f_{n+1}+c^2 f_{n+1}'' = 0$. Consequently, this equation has a non-trivial solution $f_{n+1}(u) = \lambda \cos\left(\frac{u}{c} + c_0\right)$, with some constants λ and c_0. By a suitable choose of arclenght we may assume that

$$f_{n+1}(u) = \lambda \cos\left(\frac{u}{c}\right), \tag{14}$$

with $\lambda > 0$. Thus, the radius vector of the generalized spherical curve γ takes the form

$$\gamma(u) = \left(f_1(u), ..., f_n(u), \lambda \cos\left(\frac{u}{c}\right)\right). \tag{15}$$

Moreover, the condition for the arclength parameter u implies that

$$(f_1')^2 + ... + (f_n')^2 = 1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right). \tag{16}$$

For convenience, we introduce a vector function

$$\phi(u) = (f_1(u), ..., f_n(u); 0).$$

Then the radius vector (15) can be represented in the form

$$\gamma(u) = \phi(u) + \lambda \cos\left(\frac{u}{c}\right) e_{n+1}, \tag{17}$$

where $e_{n+1} = (0, 0, ..., 0, 1)$. Consequently, the condition (16) gives

$$\left\|\phi'(u)\right\|^2 = 1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right). \tag{18}$$

Hence, the radius vector of the trace curve β becomes

$$\beta(u) = \phi(u) + c^2\phi''(u). \tag{19}$$

Consider an arbitrary unit vector function

$$a(u) = (a_1(u), ..., a_n(u); 0), \tag{20}$$

in $\mathbb{E}^{n+1}$ and use this function to construct a new vector function

$$\phi(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} a(u)du, \tag{21}$$

whose last coordinate is equal to zero. Consequently, the vector function $\phi(u)$ satisfies the condition (18) and generates a generalized spherical curve with radius vector (17).

Example 3.1. *The ordinary circular curve in $\mathbb{E}^2$ is given with the radius vector*

$$\gamma(u) = \left(\int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2\left(\frac{u}{c}\right)} du, \lambda \cos\left(\frac{u}{c}\right)\right). \tag{22}$$

Example 3.2. *Consider the unit vector* $a(u) = (\cos\alpha(u), \sin\alpha(u); 0)$ *in* $\mathbb{E}^2$. *Then using (21), the corresponding generalized spherical curve in* $\mathbb{E}^3$ *is defined by the radius vector*

$$
\begin{aligned}
f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} \cos\alpha(u)du, \\
f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} \sin\alpha(u)du, \qquad (23)\\
f_3(u) &= \lambda\cos\left(\frac{u}{c}\right).
\end{aligned}
$$

Example 3.3. *Consider the unit vector* $a(u) = \big(\cos\alpha(u), \cos\alpha(u)\sin\alpha(u), \sin^2\alpha(u); 0\big)$ *in* $\mathbb{E}^3$. *Then using (21), the corresponding generalized spherical curve in* $\mathbb{E}^4$ *is defined by the radius vector*

$$
\begin{aligned}
f_1(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} \cos\alpha(u)du, \\
f_2(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} \cos\alpha(u)\sin\alpha(u)du, \qquad (24)\\
f_3(u) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} \sin^2\alpha(u)du; \\
f_4(u) &= \lambda\cos\left(\frac{u}{c}\right).
\end{aligned}
$$

4. Generalized spherical surfaces

Consider the space $\mathbb{E}^{n+1} = \mathbb{E}^n \oplus \mathbb{E}^1$ as a subspace of $\mathbb{E}^{n+m} = \mathbb{E}^n \oplus \mathbb{E}^m$, $m \geq 2$ and Cartesian coordinates $x_1, x_2, ..., x_{n+m}$ and orthonormal basis $e_1, ..., e_{n+m}$ in $\mathbb{E}^{n+m}$. Let M^2 be a local surface given with the regular patch (radius vector) $\mathbb{E}^n \subset \mathbb{E}^{n+1}$

$$X(u,v) = \phi(u) + \lambda\cos\left(\frac{u}{c}\right)\rho(v), \qquad (25)$$

where the vector function $\phi(u) = (f_1(u), ..., f_n(u), 0, ..., 0)$, satisfies (18) and generates a generalized spherical curve with radius vector

$$\gamma(u) = \phi(u) + \lambda\cos\left(\frac{u}{c}\right)e_{n+1}, \qquad (26)$$

and the vector function $\rho(v) = (0, ..., 0, g_1(v), ..., g_m(v))$, satisfying the conditions $\|\rho(v)\| = 1, \|\rho'(v)\| = 1$ and specifies a curve $\rho = \rho(v)$

parametrized by a natural parameter on the unit sphere $Sm-1 \subset \mathbb{E}^m$. Consequently, the surface M^2 is obtained as a result of the rotation of the generalized spherical curve γ along the spherical curve ρ, which is called generalized spherical surface in $\mathbb{E}^{n+m}$.

In the sequel, we will consider some type of generalized spherical surface;

CASE I. For $n = 1$ and $m = 2$, the radius vector (25) satisfying the indicated properties describes the spherical surface in $\mathbb{E}^3$ with the radius vector

$$X(u,v) = (\phi(u), \lambda\cos\left(\frac{u}{c}\right)\cos v, \lambda\cos\left(\frac{u}{c}\right)\sin v), \tag{27}$$

where the function $\phi(u)$ is found from the relation $|\phi'(u)| = \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}$. The surface given with the parametrization (27) is a kind of surface of revolution which is called ordinary sphere.

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u(u,v) &= (\phi'(u), \frac{-\lambda}{c}\sin\left(\frac{u}{c}\right)\cos v, \frac{-\lambda}{c}\sin\left(\frac{u}{c}\right)\sin v), \\ X_v(u,v) &= (0, -\lambda\cos\left(\frac{u}{c}\right)\sin v, \lambda\cos\left(\frac{u}{c}\right)\cos(v)). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u(u,v), X_u(u,v)\rangle = 1 \\ g_{12} &= \langle X_u(u,v), X_v(u,v)\rangle = 0 \\ g_{22} &= \langle X_v(u,v), X_v(u,v)\rangle = \lambda^2\cos^2\left(\frac{u}{c}\right), \end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^3$.

For a regular patch $X(u,v)$ the unit normal vector field or surface normal N is defined by

$$\begin{aligned} N(u,v) &= \frac{X_u \times X_v}{\| X_u \times X_v \|}(u,v) \\ &= \left(-\frac{\lambda}{c}\sin\left(\frac{u}{c}\right), -\phi'(u)\cos v, -\phi'(u)\sin v\right), \end{aligned}$$

where

$$\|X_u \times X_v\| = \sqrt{g_{11}g_{22} - g_{12}^2} = \lambda\cos\left(\frac{u}{c}\right) \neq 0.$$

The second partial derivatives of $X(u,v)$ are expressed as follows

$$\begin{aligned}
X_{uu}(u,v) &= (\phi''(u), \frac{-\lambda}{c^2}\cos\left(\frac{u}{c}\right)\cos v, \frac{-\lambda}{c^2}\cos\left(\frac{u}{c}\right)\sin v),\\
X_{uv}(u,v) &= (0, \frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\sin v, -\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\cos(v)),\\
X_{vv}(u,v) &= (0, -\lambda\cos\left(\frac{u}{c}\right)\cos v, -\lambda\cos\left(\frac{u}{c}\right)\sin(v)).
\end{aligned}$$

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{aligned}
L_{11} &= \langle X_{uu}(u,v), N(u,v)\rangle = -\kappa_\gamma(u),\\
L_{12} &= \langle X_{uv}(u,v), N(u,v)\rangle = 0,\\
L_{22} &= \langle X_{vv}(u,v), N(u,v)\rangle = \phi'(u)\lambda\cos\left(\frac{u}{c}\right)
\end{aligned} \tag{28}$$

where

$$\kappa_\gamma(u) = -\frac{\lambda}{c^2}\phi'(u)\cos\left(\frac{u}{c}\right) + \frac{\lambda}{c}\phi''(u)\sin\left(\frac{u}{c}\right), \tag{29}$$

is the curvature function of the profile curve γ. Furthermore, substituting (28) into (6)-(7) we obtain the following result.

Proposition 4.1. *Let M be a spherical surface in $\mathbb{E}^3$ given with the parametrization (27). Then the Gaussian and mean curvature of M become*

$$K = 1/c^2,$$

and

$$H = \frac{\frac{2\lambda^2}{c^2}\cos^2\left(\frac{u}{c}\right) - \frac{\lambda^2}{c^2} + 1}{2\lambda\cos\left(\frac{u}{c}\right)\sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}},$$

respectively.

Corollary 4.2. [18] *Let M be a spherical surface in $\mathbb{E}^3$ given with the parametrization (27). Then we have the following assertions*

i) If $\lambda = c$ then the corresponding surface is a sphere with radius c and centered at the origin,

ii) If $\lambda > c$ then the corresponding surface is a hyperbolic spherical surface,

iii) If $\lambda < c$ then the corresponding surface is an elliptic spherical surface.

CASE II. For $n = 2$ and $m = 2$, the radius vector (25) satisfying the indicated properties describes the generalized spherical surface given with the radius vector

$$X(u, v) = (f_1(u), f_2(u), \lambda \cos\left(\frac{u}{c}\right) \cos v, \lambda \cos\left(\frac{u}{c}\right) \sin v), \tag{30}$$

where

$$\begin{aligned} f_1(u) &= \int \sqrt{1 - \tfrac{\lambda^2}{c^2} \sin^2\left(\tfrac{u}{c}\right)} \cos \alpha(u) du, \\ f_2(u) &= \int \sqrt{1 - \tfrac{\lambda^2}{c^2} \sin^2\left(\tfrac{u}{c}\right)} \sin \alpha(u) du. \end{aligned} \tag{31}$$

are differentiable functions.

We call this surface the *generalized spherical surface of first kind.* Actually, these surfaces are the special type of rotational surfaces [13], see also [4].

The tangent space is spanned by the vector fields

$$\begin{aligned} X_u(u, v) &= (f_1{}'(u), f_2{}'(u), \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v), \\ X_v(u, v) &= (0, 0, -\lambda \cos\left(\frac{u}{c}\right) \sin v, \lambda \cos\left(\frac{u}{c}\right) \cos(v)). \end{aligned}$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{aligned} g_{11} &= \langle X_u(u, v), X_u(u, v) \rangle = 1 \\ g_{12} &= \langle X_u(u, v), X_v(u, v) \rangle = 0 \\ g_{22} &= \langle X_v(u, v), X_v(u, v) \rangle = \lambda^2 \cos^2\left(\frac{u}{c}\right), \end{aligned}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{E}^4$.

The second partial derivatives of $X(u, v)$ are expressed as follows

$$\begin{aligned} X_{uu}(u, v) &= (f_1{}''(u), f_2{}''(u), \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \sin v), \\ X_{uv}(u, v) &= (0, 0, \frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \sin v, -\frac{\lambda}{c} \sin\left(\frac{u}{c}\right) \cos(v)), \\ X_{vv}(u, v) &= (0, 0, -\lambda \cos\left(\frac{u}{c}\right) \cos v, -\lambda \cos\left(\frac{u}{c}\right) \sin(v)). \end{aligned}$$

The normal space is spanned by the vector fields

$$N_1 = \frac{1}{\kappa_\gamma}(f_1{}''(u), f_2{}''(u), \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \cos v, \frac{-\lambda}{c^2} \cos\left(\frac{u}{c}\right) \sin v)$$

$$\begin{aligned} N_2 = {} & \tfrac{1}{\kappa_\gamma}(-\tfrac{\lambda f_2{}'(u)}{c^2} \cos\left(\tfrac{u}{c}\right) + \tfrac{\lambda f_2{}''(u)}{c} \sin\left(\tfrac{u}{c}\right), -\tfrac{\lambda f_1{}''(u)}{c} \sin\left(\tfrac{u}{c}\right) + \tfrac{\lambda f_1{}'(u)}{c^2} \cos\left(\tfrac{u}{c}\right), \\ & (f_1{}'(u) f_2{}''(u) - f_1{}''(u) f_2{}'(u)) \cos v, (f_1{}'(u) f_2{}''(u) - f_1{}''(u) f_2{}'(u)) \sin v) \end{aligned}$$

where

$$\kappa_\gamma = \sqrt{(f_1'')^2 + (f_2'')^2 + \frac{\lambda^2}{c^4}\cos^2\left(\frac{u}{c}\right)}, \tag{32}$$

is the curvature of the profile curve γ. Hence, the coefficients of the second fundamental form of the surface are

$$\begin{aligned}
L_{11}^1 &= \langle X_{uu}(u,v), N_1(u,v)\rangle = \kappa_\gamma(u),\\
L_{12}^1 &= \langle X_{uv}(u,v), N_1(u,v)\rangle = 0,\\
L_{22}^1 &= \langle X_{vv}(u,v), N_1(u,v)\rangle = \frac{\lambda^2\cos^2\left(\frac{u}{c}\right)}{c^2\kappa_\gamma(u)},\\
L_{11}^2 &= \langle X_{uu}(u,v), N_2(u,v)\rangle = 0,\\
L_{12}^2 &= \langle X_{uv}(u,v), N_2(u,v)\rangle = 0,\\
L_{22}^2 &= \langle X_{vv}(u,v), N_2(u,v)\rangle = -\frac{\lambda\cos\left(\frac{u}{c}\right)\kappa_1(u)}{\kappa_\gamma(u)}.
\end{aligned} \tag{33}$$

where

$$\kappa_1(u) = f_1{}'(u)f_2{}''(u) - f_1{}''(u)f_2{}'(u), \tag{34}$$

is the curvature of the projection of the curve γ on the Oe_1e_2- plane.

Furthermore, by the use of (33) with (6)-(7) we obtain the following results.

Proposition 4.3. *The generalized spherical surface of first kind has constant Gaussian curvature $K = 1/c^2$.*

Proposition 4.4. *Let M be a generalized spherical surface of first kind given with the surface patch (30). Then the mean curvature vector of M becomes*

$$\vec{H} = \frac{1}{2}\left\{\left(\frac{\kappa_\gamma^2c^2+1}{c^2\kappa_\gamma}\right)N_1 - \frac{\kappa_1}{\kappa_\gamma\lambda\cos\left(\frac{u}{c}\right)}N_2\right\}. \tag{35}$$

where

$$\kappa_\gamma = \sqrt{(\varphi\prime)^2 + \varphi^2\left((\alpha')^2 + \frac{1}{c^2}\right) + \frac{\lambda^2}{c^4}\left(1 - \frac{c^2}{\lambda^2}\right)},\ \kappa_1 = \varphi^2\alpha', \tag{36}$$

and

$$\varphi = \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}. \tag{37}$$

Corollary 4.5. *Let M be a generalized spherical surface of first kind given with the surface patch (30). If the second mean curvature H_2 vanishes identically then the angle function $\alpha(u)$ is a real constant.*

For any local surface $M \subset \mathbb{E}^4$ given with the regular surface patch $X(u,v)$ the normal curvature K_N is given with the following result.

Proposition 4.6. [6] *Let $M \subset \mathbb{E}^4$ be a local surface given with a regular patch $X(u,v)$ then the normal curvature K_N of the surface becomes*

(38)
$$K_N = \frac{g_{11}(L_{12}^1 L_{22}^2 - L_{12}^2 L_{22}^1) - g_{12}(L_{11}^1 L_{22}^2 - L_{11}^2 L_{22}^1) + g_{22}(L_{11}^1 L_{12}^2 - L_{11}^2 L_{12}^1)}{W^3}.$$

As a consequence of (33) with (38) we get the following result.

Corollary 4.7. *Any generalized spherical surface of first kind has flat normal connection, i.e., $K_N = 0$.*

Example 4.8. *In 1966, T. Otsuki considered the following special cases*

$$a) f_1(u) = \frac{4}{3}\cos^3(\frac{u}{2}), \quad f_2(u) = \frac{4}{3}\sin^3(\frac{u}{2}), \quad f_3(u) = \sin u,$$
$$b) f_1(u) = \frac{1}{2}\sin^2 u\cos(2u), \quad f_2(u) = \frac{1}{2}\sin^2 u\sin(2u), \quad f_3(u) = \sin u.$$

For the case a) the surface is called Otsuki (non-round) sphere in $\mathbb{E}^4$ which does not lie in a 3-dimensional subspace of $\mathbb{E}^4$. It has been shown that these surfaces have constant Gaussian curvature [17].

CASE III. For $n = 1$ and $m = 3$, the radius vector (25) satisfying the indicated properties describes the generalized spherical surface given with the radius vector

$$X(u,v) = \phi(u)\overrightarrow{e_1} + \lambda\cos\left(\frac{u}{c}\right)\rho(v), \tag{39}$$

where

$$\phi(u) = \int\sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}du, \tag{40}$$

and $\rho = \rho(v)$ parametrized by

$$\rho(v) = (g_1(v), g_2(v), g_3(v)),$$
$$\|\rho(v)\| = 1, \left\|\rho'(v)\right\| = 1,$$

which lies on the unit sphere $S^2 \subset \mathbb{E}^4$. The spherical curve ρ has the following Frenet Frames;

$$\begin{aligned} \rho'(v) &= T(v) \\ T'(v) &= \kappa_\rho(v)N(v) - \rho(v) \\ N'(v) &= -\kappa_\rho(v)T(v). \end{aligned}$$

We call this surface a *generalized spherical surface of second kind.* Actually, these surfaces are the special type of meridian surface defined in [14], see also [2].

Proposition 4.9. *Let M be a meridian surface in $\mathbb{E}^4$ given with the parametrization (39). Then M has the Gaussian curvature*

$$K = -\frac{\kappa_\gamma \phi'(u)}{\lambda \cos\left(\frac{u}{c}\right)}, \tag{41}$$

where

$$\kappa_\gamma(u) = -\frac{\lambda}{c^2}\phi'(u)\cos\left(\frac{u}{c}\right) + \frac{\lambda}{c}\phi''(u)\sin\left(\frac{u}{c}\right)$$

is the curvature of the profile curve γ.

Proof. Let M be a meridian surface in $\mathbb{E}^4$ defined by (39). Differentiating (39) with respect to u and v and we obtain

$$\begin{aligned} X_u &= \phi'(u)\overrightarrow{e}_1 - \frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\rho(v), \\ X_v &= \lambda\cos\left(\frac{u}{c}\right)\rho'(v), \\ X_{uu} &= \phi''(u)\overrightarrow{e}_1 - \frac{\lambda}{c^2}\cos\left(\frac{u}{c}\right)\rho(v), \\ X_{uv} &= -\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\rho'(v), \\ X_{vv} &= \lambda\cos\left(\frac{u}{c}\right)\rho''(v). \end{aligned} \tag{42}$$

The normal space of M is spanned by

$$\begin{aligned} N_1 &= N(v), \\ N_2 &= -\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\overrightarrow{e}_1 - \phi'(u)\rho(v), \end{aligned} \tag{43}$$

where $N(v)$ is the normal vector of the spherical curve ρ.

Hence, the coefficients of first and second fundamental forms are becomes

$$\begin{aligned} g_{11} &= \langle X_u(u,u), X_u(u,u)\rangle = 1, \\ g_{12} &= \langle X_u(u,v), X_v(u,v)\rangle = 0, \\ g_{22} &= \langle X_v(v,v), X_v(v,v)\rangle = \lambda^2\cos^2\left(\frac{u}{c}\right), \end{aligned} \tag{44}$$

and

$$\begin{aligned} L_{11}^1 &= L_{12}^1 = L_{12}^2 = 0, \\ L_{22}^1 &= \kappa_\rho(v)\lambda\cos\left(\frac{u}{c}\right), \\ L_{11}^2 &= -\kappa_\gamma(u), \\ L_{11}^2 &= \phi'(u)\lambda\cos\left(\frac{u}{c}\right). \end{aligned} \tag{45}$$

respectively, where

$$\begin{aligned} \kappa_\gamma(u) &= f_1'(u)f_2''(u) - f_1''(u)f_2'(u) \\ &= -\frac{\lambda}{c^2}\phi'(u)\cos\left(\frac{u}{c}\right) + \frac{\lambda}{c}\phi''(u)\sin\left(\frac{u}{c}\right). \end{aligned}$$

Consequently, substituting (44)-(45) into (6) we obtain the result. □

As a consequence of (45) with (38) we get the following result.

Proposition 4.10. *Any generalized spherical surface of second kind has flat normal connection, i.e.,* $K_N = 0$.

Corollary 4.11. *Every generalized spherical surface of second kind is a meridian surface given with the parametrization*

$$\begin{aligned} f_1(u) = \int \sqrt{1 - \tfrac{\lambda^2}{c^2}\sin^2\left(\tfrac{u}{c}\right)}du \\ f_2(u) = \lambda\cos\left(\tfrac{u}{c}\right) \end{aligned} \tag{46}$$

By the use of (40)-(41) with (46) we get the following result.

Corollary 4.12. *The generalized spherical surface of second kind has constant Gaussian curvature* $K = 1/c^2$.

As consequence of (7) we obtain the following results.

Proposition 4.13. *Let* M *be a generalized spherical surface of second kind given with the parametrization (39). Then the mean curvature vector of* M *becomes*

$$\overrightarrow{H} = \frac{1}{2f_2(u)}\left\{\kappa_\rho(v)N_1 + \left(-\kappa_\gamma f_2(u) + f_1'(u)\right)N_2\right\}. \tag{47}$$

where

$$\kappa_\rho(v) = \sqrt{g_1''(v)^2 + g_2''(v)^2 + g_3''(v)^2}.$$

Corollary 4.14. *Let M be a generalized spherical surface of second kind given with the parametrization (39). If $\kappa_\gamma(u) = \frac{f_1'(u)}{f_2(u)}$ then M has vanishing second mean curvature, i.e., $H_2 = 0$.*

Example 4.15. *Consider the curve $\rho(v) = \left(\cos v, \cos v \sin v, \sin^2 v\right)$ in $S^2 \subset \mathbb{E}^3$. The corresponding generalized spherical surface*

$$\begin{aligned} x_1(u,v) &= \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)} du \\ x_2(u,v) &= \lambda \cos\left(\frac{u}{c}\right)\cos v \\ x_3(u,v) &= \lambda \cos\left(\frac{u}{c}\right)\cos v \sin v \\ x_4(u,v) &= \lambda \cos\left(\frac{u}{c}\right)\sin^2 v. \end{aligned} \tag{48}$$

is of second kind.

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