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CONSTRUCTIVE APPROXIMATION

Approximation by p-Faber Polynomials in the Weighted Smirnov Class $E^p(G, \omega)$ and the Bieberbach Polynomials

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Abstract. Let $G \subset C$ be a finite domain with a regular Jordan boundary L. In this work, the approximation properties of a p-Faber polynomial series of functions in the weighted Smirnov class $E^p(G, \omega)$ are studied and the rate of polynomial approximation, for $f \in E^p(G, \omega)$ by the weighted integral modulus of continuity, is estimated. Some application of this result to the uniform convergence of the Bieberbach polynomials π_n in a closed domain $\overline{G}$ with a smooth boundary L is given.

1. Introduction

Let G be a finite domain in the complex plane bounded by a rectifiable Jordan curve L, let ω be a weight function on L, and let $1 < p < \infty$. We denote by $L^p(L)$ and $E^p(G)$ the set of all measurable complex valued functions such that $|f|^p$ is Lebesgue integrable with respect to arclength, and the Smirnov class of analytic functions in G, respectively. Each function $f \in E^p(G)$ has a nontangential limit almost everywhere (a.e.) on L, and if we use the same notation for the nontangential limit of f, then $f \in L^p(L)$.

For $p > 1$, $L^p(L)$ and $E^p(G)$ are Banach spaces with respect to the norm

$$\|f\|_{E^p(G)} = \|f\|_{L^p(L)} := \left(\int_L |f(z)|^p |dz| \right)^{1/p}.$$

For further properties, see [7, pp. 168–185] and [14, pp. 438–453].

Theorder of polynomial approximation in $E^p(G)$, $p \geq 1$, has been studied by several authors. In [27], Walsh and Russel give results when L is an analytic curve. For domains with sufficiently smooth boundary, namely when L is a smooth Jordan curve, and $\theta(s)$, the angle between the tangent and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(\theta, s)$ satisfying the Dini-smooth condition

$$\int_0^\delta \frac{\Omega(\theta, s)}{s}\, ds < \infty, \qquad \delta > 0, \tag{1}$$

this problem, for $p > 1$, was studied by S. Y. Alper [1].

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These results were later extended to the domains with regular boundary, which we define in Section 2, for $p > 1$ by V. M. Kokilashvili [21], and for $p \geq 1$ by J. E. Andersson [2]. Similar problems were also investigated in [18]. Let us emphasize that in these works, the Faber operator, Faber polynomials, and p-Faber polynomials were commonly used and the degree of polynomial approximation in $E^p(G)$ has been studied by applying various methods of summation to the Faber series of functions in $E^p(G)$. More extensive knowledge about them can be found in [11, pp. 40–57] and [26, pp. 52–236].

In [19] and [5], for domains with regular boundary we construct the approximants directly as the nth-partial sums of p-Faber polynomial series of $f \in E^p(G)$. In this work, the approximation properties of the p-Faber polynomial series expansions in the ω-weighted Smirnov class $E^p(G, \omega)$ of analytic functions in G, whose boundary is a regular Jordan curve, are studied. Under some restrictive conditions upon weighting functions the approximant polynomials are obtained directly as the nth-partial sums of p-Faber polynomial series of $f \in E^p(G, \omega)$. The degree of this approximation is estimated by a weighted integral modulus of continuity. The results to be obtained in this work are also new in the nonweighted case $\omega = 1$. Finally, applying this result we give a result which improves Mergelyan's estimation about the uniform convergence of the Bieberbach polynomials in the closed domain $\overline{G}$ with a smooth boundary L.

2. Some Definitions, Notations, and Auxiliary Results

Let G be a finite domain in the complex plane bounded by a rectifiable Jordan curve L, let U be the unit disk, $G^- := \operatorname{Ext} L$, $T := \partial U$, $U^- := \operatorname{Ext} T$, $1 < p < \infty$, and let ω be a weight function on L, that is, a nonnegative measurable function on L. We denote by φ the conformal mapping of G^- onto U^- normalized by $\varphi(\infty) = \infty$ and $\lim_{z\to\infty} \varphi(z)/z > 0$. Let $\psi(w)$ be the inverse to $\varphi(z)$. The functions φ and ψ have continuous extensions to L and T, their derivatives $\varphi'(z)$ and $\psi'(w)$ have definite nontangential limit values on L and T a.e., and they are integrable with respect to the Lebesgue measure on L and T, respectively [14, pp. 419, 438].

We shall use $c, c_1, c_2, \ldots$ to denote constants (in general, different in different relations) depending only on numbers that are not important for the questions of interest.

Definition 1. L is called regular if there exists a number $c > 0$ such that for every $r > 0$, $\sup\{|L \cap D(z, r)| : z \in L\} \leq cr$, where $D(z, r)$ is an open disk with radius r and centered at z, and $|L \cap D(z, r)|$ is the length of the set $L \cap D(z, r)$.

We denote by S the set of all regular Jordan curves in the complex plane.

Definition 2. Let ω be a weight function on L. ω is said to satisfy Muckenhoupt's A_p-conditions on L if

$$\sup_{z\in L}\sup_{r>0}\left(\frac{1}{r}\int_{L\cap D(z,r)}\omega(\varsigma)|d\varsigma|\right)\left(\frac{1}{r}\int_{L\cap D(z,r)}[\omega(\varsigma)]^{-1/(p-1)}|d\varsigma|\right)^{p-1} < \infty.$$

Let us denote by $A_p(L)$ the set of all weight functions satisfying Muckenhoupt's A_p-conditions on L.

It is obvious that if $\omega \in A_p(L)$ then $\omega^{-1/p} \in L^{p/(p-1)}(L)$.

Let $f \in L^1(L)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_L \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma, \qquad z \in G,$$

and

$$f^-(z) = \frac{1}{2\pi i}\int_L \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma, \qquad z \in G^-,$$

are analytic in G and G^-, respectively, and $f^-(\infty) = 0$. When $z_0 \in L$, if the limit of the integral

$$\frac{1}{2\pi i}\int_{L\cap\{\varsigma:|\varsigma - z_0|>\varepsilon\}} \frac{f(\varsigma)}{\varsigma - z_0}\, d\varsigma$$

exists as $\varepsilon \to 0$, this limit is called Cauchy's singular integral of

$$\frac{1}{2\pi i}\int_L \frac{f(\varsigma)}{\varsigma - z}\, d\varsigma$$

at $z_0 \in L$, and it is denoted by $S_L(f)(z_0)$. Namely,

$$S_L(f)(z_0) := (P.V.)\frac{1}{2\pi i}\int_L \frac{f(\varsigma)}{\varsigma - z_0}\, d\varsigma := \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L\cap\{\varsigma:|\varsigma - z_0|>\varepsilon\}} \frac{f(\varsigma)}{\varsigma - z_0}\, d\varsigma.$$

According to the celebrated Privalov theorem [14, p. 431], if one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on L a.e., then $S_L(f)(z)$ exists a.e. on L, and also the other one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on L a.e. Conversely, if $S_L(f)(z)$ exists a.e. on L, then the functions $f^+(z)$ and $f^-(z)$ have nontangential limits a.e. on L. In both cases, the formulas

$$f^+(z) = S_L(f)(z) + \tfrac{1}{2} f(z) \qquad \text{and} \qquad f^-(z) = S_L(f)(z) - \tfrac{1}{2} f(z)$$

hold a.e. on L.

Definition 3. The set $L^p(L,\omega) := \{f \in L^1(L) : |f|^p\omega \in L^1(L)\}$ is called the ω-weighted L^p-space.

Definition 4. The set $E^p(G,\omega) := \{f \in E^1(G) : f \in L^p(L,\omega)\}$ is called the ω-weighted Smirnov class of order p of analytic functions in G.

As was noted in [9, p. 89], the Cauchy singular integrals hold the following result, which is analogously deduced from [6].

Theorem 1. *Let $L \in S$, $1 < p < \infty$, and let ω be a weight function on L. The inequality*

$$\|S_L(f)\|_{L^p(L,\omega)} \le c_1 \|f\|_{L^p(L,\omega)}$$

holds for every $f \in L^p(L,\omega)$ if and only if $\omega \in A_p(L)$.

Lemma 2. *If $f \in L^p(L,\omega)$ and $\omega \in A_p(L)$, then there exists a number $r > 1$ such that $f \in L^r(L)$.*

Proof. Since $\omega \in A_p(L)$, there exists a number $q \in (1, p)$ such that $\omega \in A_q(L)$ [23] (see also [9, p. 49]). Let $r := p/q$. Since $f \in L^p(L,\omega)$, we have $|f|^r \omega^{1/q} \in L^q(L)$. On the other hand, since $\omega^{-(1/q)} \in L^{q/(q-1)}(L)$, Hölder's inequality shows that $f \in L^r(L)$. ■

Lemma 3. *If $L \in S$ and $\omega \in A_p(L)$, then $f^+ \in E^p(G,\omega)$ and $f^- \in E^p(G^-,\omega)$ for each $f \in L^p(L,\omega)$.*

Proof. Let $f \in L^p(L,\omega)$. According to Theorem 1, we have $S_L(f) \in L^p(L,\omega)$. On the other hand, by Lemma 1, there exists a number $r > 1$ such that $f \in L^r(L)$. Since $1 < r < \infty$ and $L \in S$, $S_L : L^r(L) \to L^r(L)$ is a bounded linear operator [6]. Therefore, owing to Havin's work [16] (see also [6, p. 176]), the functions f^+ and f^- belong to $E^r(G)$ and $E^r(G^-)$, respectively. Furthermore, since $f^+(z) = S_L(f)(z) + \frac{1}{2}f(z)$ and $f^-(z) = S_L(f)(z) - \frac{1}{2}f(z)$ hold a.e. on L, it follows that f^+ and f^- are members of $L^p(L,\omega)$. This yields the required result, because $E^r(G) \subset E^1(G)$ and $E^r(G^-) \subset E^1(G^-)$. ■

3. p-Faber Polynomials for $\overline{G}$ and p-Faber Polynomial Series Expansions in $E^p(G,\omega)$

Let k be a nonnegative integer. Then the function $\varphi^k(z)(\varphi'(z))^{1/p}$ has a pole of order k at the point ∞. So there exists a polynomial $F_{k,p}(z)$ of degree k and an analytic function $E_{k,p}(z)$ in G^- such that $E_{k,p}(\infty) = 0$ and $\varphi^k(z)(\varphi'(z))^{1/p} = F_{k,p}(z) + E_{k,p}(z)$ for every $z \in G^-$. The polynomials $F_{k,p}(z)$ $(k = 0, 1, 2, \ldots)$ are called p-Faber polynomials for $\overline{G}$ (see [2]). By means of Cauchy's integral formula, it is easily seen that

$$F_{k,p}(z) = \frac{1}{2\pi i}\int_{L_R} \frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma - z}\, d\varsigma = \frac{1}{2\pi i}\int_{|w|=R} \frac{w^k(\psi'(w))^{1-1/p}}{\psi(w) - z}\, dw,$$

for $R > 1$ and every $z \in \operatorname{Int} L_R$, where $L_R := \{z \in G^- : |\varphi(z)| = R\}$.

Lemma 4. *If $z \in G$ and $w \in U^-$, then*

$$\frac{(\psi'(w))^{1-1/p}}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{F_{k,p}(z)}{w^{k+1}}.$$

Proof. Let us take $z \in G$. Since the function

$$\frac{(\psi'(w))^{1-1/p}}{\psi(w) - z}$$

is analytic in U^- and it is normalized with $\psi(\infty)=\infty$ and $\lim_{w\to\infty}\psi(w)/w>0$, its Laurent series expansion in U^- is of the form

$$\sum_{k=0}^{\infty}\frac{A_{k,p}(z)}{w^{k+1}}$$

and this series converges to

$$\frac{(\psi'(w))^{1-1/p}}{\psi(w)-z}$$

uniformly on compact subsets of U^-. So, for $R>1$ and a nonnegative integer n, we obtain

$$\frac{1}{2\pi i}\int_{|w|=R}\frac{w^n(\psi'(w))^{1-1/p}}{\psi(w)-z}\,dw=\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w|=R}\frac{w^n}{w^{k+1}}\,dw\right)A_{k,p}(z)=A_{n,p}(z).$$

This shows that $F_{n,p}(z)=A_{n,p}(z)$ for $n=0,1,2,\ldots$, and so the proof is completed. ■

Lemma 5. *If $z\in G^-$, then*

$$\lim_{n\to\infty}\int_{L_{1+1/n}}\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma=\int_L\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma,$$

for $k=0,1,2,\ldots$.

Proof. Let

$$\varphi_n(\theta):=\frac{i(1+1/n)^{k+1}e^{i(k+1)\theta}(\psi'((1+1/n)e^{i\theta}))^{1-1/p}}{\psi((1+1/n)e^{i\theta})-z}.$$

It is obvious that the sequence $\{\varphi_n(\theta)\}$ converges a.e. to the function

$$\frac{ie^{i(k+1)\theta}(\psi'(e^{i\theta}))^{1-1/p}}{\psi(e^{i\theta})-z}$$

on the segment $[0,2\pi]$, and

$$\int_{L_{1+1/n}}\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma=\int_0^{2\pi}\varphi_n(\theta)\,d\theta.$$

On the other hand, it is easily proved that the sequence

$$\left\{\int_0^{2\pi}|\varphi_n(\theta)|^{p/(p-1)}\,d\theta\right\}$$

is bounded with respect to n. Thus, by the test for the possibility of taking the limit under the Lebesgue integral sign given in [14, p. 390] we obtain

$$\lim_{n\to\infty}\int_0^{2\pi}\varphi_n(\theta)\,d\theta=\int_0^{2\pi}\frac{ie^{i(k+1)\theta}(\psi'(e^{i\theta}))^{1-1/p}}{\psi(e^{i\theta})-z}\,d\theta.$$

This gives us

$$\lim_{n\to\infty}\int_{L_{1+1/n}}\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma\;=\;\int_L\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma. \qquad \blacksquare$$

Finally, we prove the following lemma for the integral representation of p-Faber polynomials in G^-.

Lemma 6. *If $z\in G^-$, then*

$$F_{k,p}(z)=\varphi^k(z)(\varphi'(z))^{1/p}+\frac{1}{2\pi i}\int_L\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma,$$

for $k=0,1,2,\ldots.$

Proof. The case $z=\infty$ is trivial. Let $z\in G^-\backslash\{\infty\}$. If $R>1$ and the natural numbers n are chosen big enough, z becomes an interior point of the doubly connected domain with the boundary $L_R\cup L_{1+1/n}$. So, by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\int_{L_R}\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma=\varphi^k(z)(\varphi'(z))^{1/p}+\frac{1}{2\pi i}\int_{L_{1+1/n}}\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma$$

and hence by Lemma 4 we obtain

$$F_{k,p}(z)=\varphi^k(z)(\varphi'(z))^{1/p}+\frac{1}{2\pi i}\int_L\frac{\varphi^k(\varsigma)(\varphi'(\varsigma))^{1/p}}{\varsigma-z}\,d\varsigma.$$

The lemma is proved. $\blacksquare$

Let $f\in E^p(G,\omega)$. Since $f\in E^1(G)$, we have for every $z\in G$:

$$f(z)=\frac{1}{2\pi i}\int_L\frac{f(\varsigma)}{\varsigma-z}\,d\varsigma=\frac{1}{2\pi i}\int_T f(\psi(w))(\psi'(w))^{1/p}\frac{(\psi'(w))^{1-1/p}}{\psi(w)-z}\,dw.$$

On the other hand, since

$$\frac{(\psi'(w))^{1-1/p}}{\psi(w)-z}=\sum_{k=0}^{\infty}\frac{F_{k,p}(z)}{w^{k+1}}$$

for $w\in U^-$ and $z\in G$, if we define the coefficients $a_k(f)$ by

$$a_k(f):=\frac{1}{2\pi i}\int_T\frac{f(\psi(w))(\psi'(w))^{1/p}}{w^{k+1}}\,dw,\qquad k=0,1,2,\ldots,$$

we can associate a formal series

$$\sum_{k=0}^{\infty}a_k(f)F_{k,p}(z),$$

in the particular case with the function $f \in E^p(G,\omega)$, i.e.,

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f) F_{k,p}(z).$$

This formal series is called the p-Faber polynomial series expansion of f, and the coefficients $a_k(f)$ are said to be the p-Faber coefficients of f.

4. Main Results

Let $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$. Since $L^p(T,\omega)$ is noninvariant with respect to the usual shift, we consider the following mean value function as a shift for $g \in L^p(T,\omega)$:

$$g_h(w) := \frac{1}{2h}\int_{-h}^{h} g(we^{it})dt, \qquad 0 < h < \pi, \qquad w \in T.$$

Using the relation (see, e.g., [9, p. 110]):

$$\|g_h\|_{L^p(T,\omega)} \le c_p \|g\|_{L^p(T,\omega)}, \qquad 1 < p < \infty,$$

we get that $g_h \in L^p(T,\omega)$.

Definition 5. If $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$, then the function $\Omega_{p,\omega}(g,\cdot) : [0,\infty) \to [0,\infty)$, defined by

$$\Omega_{p,\omega}(g,\delta) := \sup\{\|g - g_h\|_{L^p(T,\omega)}, h \le \delta\}, \qquad 1 < p < \infty,$$

is called the ω-weighted integral modulus of continuity of order p for g.

Note that the idea of defining such a modulus of continuity originates from [29]. It can be shown easily that $\Omega_{p,\omega}(g,\cdot)$ is a continuous nonnegative nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p,\omega}(g,\delta) = 0, \qquad \Omega_{p,\omega}(g_1 + g_2,\cdot) \le \Omega_{p,\omega}(g_1,\cdot) + \Omega_{p,\omega}(g_2,\cdot).$$

Lemma 7. *If $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$, then*

$$\Omega_{p,\omega}(S_T(g),\cdot) \le c_2 \Omega_{p,\omega}(g,\cdot).$$

Proof. Let $\delta \in (0,\pi)$, $h < \delta$, and $w \in T$. Applying the Fubini theorem we have

$$\begin{aligned}
[S_T(g)]_h(w) &= \frac{1}{2h}\int_{-h}^{h} S_T(g(we^{i\theta}))\,d\theta \\
&= \frac{1}{2h}\int_{-h}^{h} \frac{1}{2\pi i}\left((P.V.)\int_T \frac{g(\tau)\,d\tau}{\tau - we^{i\theta}}\right)\,d\theta \\
&= \frac{1}{2h}\int_{-h}^{h} \frac{1}{2\pi i}\left((P.V.)\int_T \frac{g(\tau e^{i\theta})e^{i\theta}\,d\tau}{\tau e^{i\theta} - we^{i\theta}}\right)\,d\theta
\end{aligned}$$

$$= \frac{1}{2h}\int_{-h}^{h}\frac{1}{2\pi i}\left((P.V.)\int_T \frac{g(\tau e^{i\theta})\,d\tau}{\tau-w}\right)d\theta$$
$$= \frac{1}{2\pi i}(P.V.)\int_T \frac{(1/2h)\int_{-h}^{h} g(\tau e^{i\theta})\,d\theta}{\tau-w}\,d\tau$$
$$= \frac{1}{2\pi i}(P.V.)\int_T \frac{g_h(\tau)}{\tau-w}\,d\tau = [S_T(g_h)](w).$$

Therefore,

$$[S_T g)](w) - [S_T(g)]_h(w) = [S_T(g-g_h)](w),$$

and by virtue of Theorem 1 we obtain

$$\|S_T(g) - [S_T(g)]_h\|_{L^p(T,\omega)} = \|S_T(g-g_h)\|_{L^p(T,\omega)} \le c_2\|g-g_h\|_{L^p(T,\omega)}.$$

The last inequality shows that

$$\Omega_{p,\omega}(S_T(g),\cdot) \le c_2\Omega_{p,\omega}(g,\cdot),$$

and the proof is completed. ∎

Lemma 8. *If $g \in L^p(T,\omega)$ and $\omega \in A_p(T)$, then*

$$\Omega_{p,\omega}(g^+,\cdot) \le (c_2 + \tfrac{1}{2})\Omega_{p,\omega}(g,\cdot).$$

Proof. Since $g^+ = \frac{1}{2}g + S_T(g)$ a.e. on T, by means of Lemma 6 we obtain

$$\Omega_{p,\omega}(g^+,\cdot) \le (c_2 + \tfrac{1}{2})\Omega_{p,\omega}(g,\cdot).$$ ∎

Lemma 9. *Let $g \in E^p(U,\omega)$ and $\omega \in A_p(T)$. If*

$$\sum_{k=0}^{n}\alpha_k(g)w^k$$

is the nth partial sum of the Taylor series of g at the origin, then there exists a constant $c_3 > 0$, such that

$$\left\|g(w) - \sum_{k=0}^{n}\alpha_k(g)w^k\right\|_{L^p(T,\omega)} \le c_3\Omega_{p,\omega}\left(g,\frac{1}{n}\right),$$

for every natural number n.

Proof. Let

$$\sum_{k=-\infty}^{\infty}\beta_k e^{ik\theta}$$

be the Fourier series of $g \in E^p(U,\omega)$ and

$$S_n(g,\theta) := \sum_{k=-n}^{n} \beta_k e^{ik\theta}$$

be its nth-partial sum. Since $g \in E^1(U)$, we have $\beta_k = 0$ for $k < 0$, and $\beta_k = \alpha_k(g)$ for $k \geq 0$ [7, p. 38]. Hence

$$\left\| g(w) - \sum_{k=0}^{n} \alpha_k(g) w^k \right\|_{L^p(T,\omega)} = \|g(e^{i\theta}) - S_n(g,\theta)\|_{L^p([0,2\pi],\omega)}. \tag{2}$$

Now, let $T_n^*(\theta)$ be the best approximate trigonometric polynomial for $g(e^{i\theta})$ in $L^p([0,2\pi],\omega)$. That is,

$$\|g(e^{i\theta}) - T_n^*(\theta)\|_{L^p([0,2\pi],\omega)} = E_{n,p}(g,\omega), \tag{3}$$

where $E_{n,p}(g,\omega) := \inf\{\|g(e^{i\theta}) - T(\theta)\|_{L^p([0,2\pi],\omega)} : T \in \Pi_n\}$ denotes the minimal error in approximating g by trigonometric polynomials of degree at most n. Then from (2) we get

$$\left\| g(w) - \sum_{k=0}^{n} \alpha_k(g) w^k \right\|_{L^p(T,\omega)} \leq \|g(e^{i\theta}) - T_n^*(\theta)\|_{L^p([0,2\pi],\omega)} + \|S_n(g - T_n^*,\theta)\|_{L^p([0,2\pi],\omega)}. \tag{4}$$

On the other hand, under the condition $\omega \in A_p(T)$ the result [17] (see also [9, p. 108]) states that, for every $g \in L^p([0,2\pi],\omega)$:

$$\left\| \sup_{n\geq 0} |S_n(g,\theta)| \right\|_{L^p([0,2\pi],\omega)} \leq c_4 \|g\|_{L^p([0,2\pi],\omega)}.$$

By applying this inequality to the function $g - T_n^*$ and taking into account the relation (3), from (4) we get

$$\left\| g(w) - \sum_{k=0}^{n} \alpha_k(g) w^k \right\|_{L^p(T,\omega)} \leq (c_4 + 1) E_{n,p}(g,\omega). \tag{5}$$

Further, using the estimation

$$E_{n,p}(g,\omega) \leq c_5 \Omega_{p,\omega}\left(g, \frac{1}{n}\right),$$

which was proved in [15, Theorem 1.4], from (5) we obtain

$$\left\| g(w) - \sum_{k=0}^{n} \alpha_k(g) w^k \right\|_{L^p(T,\omega)} \leq c_3 \Omega_{p,\omega}\left(g, \frac{1}{n}\right).$$

The lemma is proved. ∎

Now, for $w \in T$, we set

$$\omega_0(w) := \omega(\psi(w)), \qquad f_0(w) := f(\psi(w))(\psi'(w))^{1/p},$$

and state the main theorem in our work.

Theorem 10. *Let $f \in E^p(G, \omega)$ and let*

$$S_n(f, z) := \sum_{k=0}^{n} a_k(f) F_{k,p}(z)$$

be the nth partial sums of its p-Faber polynomial series expansion. If $L \in S$, $\omega \in A_p(L)$, and $\omega_0 \in A_p(T)$, then there exists a constant $c_6 > 0$ such that

$$\| f - S_n(f, \cdot)\|_{L^p(L,\omega)} \leq c_6 \Omega_{p,\omega_0}\left(f_0, \frac{1}{n}\right)$$

for every natural number n.

Proof. It is obvious that $f_0 \in L^p(T, \omega_0)$. Let us consider the functions f_0^+ and f_0^- defined by

$$f_0^+(w) := \frac{1}{2\pi i}\int_T \frac{f_0(\tau)}{\tau - w}\, d\tau, \qquad w \in U,$$

and

$$f_0^-(w) := \frac{1}{2\pi i}\int_T \frac{f_0(\tau)}{\tau - w}\, d\tau, \qquad w \in U^-.$$

Let $a_k(f)$ be the kth p-Faber coefficient of $f \in E^p(G, \omega)$. Since by Lemma 2, $f_0^+ \in E^p(U, \omega_0)$ and $f_0^- \in E^p(U^-, \omega_0)$, moreover, $f_0^-(\infty) = 0$ and $f_0 = f_0^+ - f_0^-$ a.e. on T, and

$$a_k(f) := \frac{1}{2\pi i}\int_T \frac{f_0(\tau)}{\tau^{k+1}}\, d\tau,$$

we obtain

$$a_k(f) = \frac{1}{2\pi i}\int_T \frac{f_0^+(\tau)}{\tau^{k+1}}\, d\tau.$$

It is seen that the kth p-Faber coefficient of $f \in E^p(G, \omega)$ is the kth Taylor coefficient of $f_0^+ \in E^p(U, \omega_0)$ at the origin. On the other hand, the assumption $f \in E^p(G, \omega)$ implies

$$\int_L \frac{f(\varsigma)}{\varsigma - z'}\, d\varsigma = 0, \qquad z' \in G^-,$$

and considering $f_0 = f_0^+ - f_0^-$ a.e. on T:

(6) $$f(\varsigma) = (f_0^+(\varphi(\varsigma)) - f_0^-(\varphi(\varsigma)))(\varphi'(\varsigma))^{1/p}$$

holds a.e. on L.

Let us take a $z' \in G^-$. By means of Lemma 5 we obtain

$$\begin{aligned}
\sum_{k=0}^{n} a_k(f) F_{k,p}(z') &= (\varphi'(z'))^{1/p} \sum_{k=0}^{n} a_k(f)\varphi^k(z') \\
&\quad + \frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p}\sum_{k=0}^{n} a_k(f)\varphi^k(\varsigma)}{\varsigma - z'}\, d\varsigma, \\
-\frac{1}{2\pi i}\int_L \frac{f(\varsigma)}{\varsigma - z'}\, d\varsigma &= (\varphi'(z'))^{1/p} \sum_{k=0}^{n} a_k(f)\varphi^k(z') \\
&\quad + \frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p}\sum_{k=0}^{n} a_k(f)\varphi^k(\varsigma)}{\varsigma - z'}\, d\varsigma \\
&\quad - \frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p} f_0^+(\varphi(\varsigma))}{\varsigma - z'}\, d\varsigma \\
&\quad + \frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p} f_0^-(\varphi(\varsigma))}{\varsigma - z'}\, d\varsigma.
\end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p} f_0^-(\varphi(\varsigma))}{\varsigma - z'}\, d\varsigma = -(\varphi'(z'))^{1/p} f_0^-(\varphi(z'))$$

we get

$$\begin{aligned}
\sum_{k=0}^{n} a_k(f) F_{k,p}(z') &= (\varphi'(z'))^{1/p} \sum_{k=0}^{n} a_k(f)\varphi^k(z') \\
&\quad + \frac{1}{2\pi i}\int_L \frac{(\varphi'(\varsigma))^{1/p}[\sum_{k=0}^{n} a_k(f)\varphi^k(\varsigma) - f_0^+(\varphi(\varsigma))]}{\varsigma - z'}\, d\varsigma \\
&\quad - (\varphi'(z'))^{1/p} f_0^-(\varphi(z')).
\end{aligned}$$

Taking the limit as $z' \to z$ along all nontangential paths outside L, it appears that

$$\begin{aligned}
\sum_{k=0}^{n} a_k(f) F_{k,p}(z) &= \tfrac{1}{2}(\varphi'(z))^{1/p}\left[\sum_{k=0}^{n} a_k(f)\varphi^k(z) - f_0^+(\varphi(z))\right] \\
&\quad + [f_0^+(\varphi(z)) - f_0^-(\varphi(z))](\varphi'(z))^{1/p} \\
&\quad + S_L\left[(\varphi')^{1/p}\left(\sum_{k=0}^{n} a_k(f)\varphi^k - f_0^+\circ\varphi\right)\right](z)
\end{aligned}$$

holds on L a.e. Further, taking relation (6) into account, and applying Minkowski's inequality and Theorem 1, from the last equality we obtain

$$\|f - S_n(f,\cdot)\|_{L^p(L,\omega)} \le (c_1 + \tfrac{1}{2})\left\| f_0^+(w) - \sum_{k=0}^{n} \alpha_k(f) w^k \right\|_{L^p(T,\omega_0)}.$$

Now, the proof follows from Lemmas 8 and 7. ■

Note that if L is a sufficiently smooth curve then one out of the conditions $\omega \in A_p(L)$ and $\omega_0 \in A_p(T)$ may be omitted in the above Theorem 2. In particular, the following theorem holds.

Theorem 11. *Let L be the smooth boundary satisfying condition* (1). *If* $f \in E^p(G, \omega)$, *and one out of the conditions* $\omega \in A_p(L)$ *and* $\omega_0 \in A_p(T)$ *holds, then there exists a constant* $c_7 > 0$ *such that*

$$\| f - S_n(f, \cdot)\|_{L^p(L,\omega)} \le c_7 \Omega_{p,\omega_0}\left(f_0, \frac{1}{n}\right).$$

Proof. According to Theorem 2 it is sufficient to prove the equivalence of the conditions $\omega \in A_p(L)$ and $\omega_0 \in A_p(T)$. Since the boundary L is smooth, it can be shown easily that the condition $\omega \in A_p(L)$ is equivalent to the inequality

$$\left(\frac{1}{|I|}\int_I \omega(\varsigma)|d\varsigma|\right) \Big/ \left(\frac{1}{|I|}\int_I [\omega(\varsigma)]^{-1/(p-1)}|d\varsigma|\right)^{p-1} \le c < \infty \quad \text{for every arc} \quad I \subset L, \tag{7}$$

On the other hand, under the restrictive conditions upon L, by the result [28]:

$$0 < c_8 \le |\psi'(w)| \le c_9 < \infty \qquad \text{for every} \quad |w| \ge 1,$$

and from this we have

$$|\psi(I)| = \int_I |\psi'(w)||d_w| \le c_9 |I|,$$
$$|I| = \int_{\psi(I)} |\varphi'(z)||d_z| \le \frac{|\psi(I)|}{c_8},$$

for every arc $I \subset T$.

Substituting $\varsigma = \psi(w)$ in (7) and using the last three relations, as result of simple computations we obtain the desired equivalence. ■

5. Application to the Uniform Convergence of the Bieberbach Polynomials in Closed Domains with Smooth Boundary

Let G be a finite simply connected domain of the complex plane C and let $z_0 \in G$. By the Riemann mapping theorem, there exists a unique conformal mapping $w = \varphi_0(z)$ of G onto $D(0, r_0) := \{w : |w| < r_0\}$ with the normalization $\varphi_0(z_0) = 0$, $\varphi_0'(z_0) = 1$. The radius r_0 of this disk is called the conformal radius of G with respect to z_0. Let $\psi_0(w)$ be the inverse to $\varphi_0(z)$.

For an arbitrary function f given on G and $p > 0$ we set

$$\| f\|_{\overline{G}} := \sup\{|f(z)|, z \in \overline{G}\}, \qquad \| f\|^2_{L_2(G)} := \iint_G |f(z)|^2\, d\sigma_z,$$
$$\| f\|^2_{L_2^1(G)} := \iint_G |f'(z)|^2\, d\sigma_z, \qquad d\sigma_z = dx\, dy.$$

It is well known that the function $\varphi_0(z)$ minimizes the integral $\|f\|^2_{L^1_2(G)}$ in the class of all functions analytic in G with the normalization $f(z_0)=0$, $f'(z_0)=1$. On the other hand, let Π_n be the class of all polynomials p_n of degree at most n satisfying the conditions $p_n(z_0)=0$, $p_n'(z_0)=1$. Then the integral $\|p_n\|^2_{L^1_2(G)}$ is minimized in Π_n by a unique polynomial π_n which is called the nth Bieberbach polynomial for the pair (G,z_0).

If G is a Carathéodory domain, then $\|\varphi_0-\pi_n\|_{L^1_2(G)}\to 0$ $(n\to\infty)$ and from this it follows that $\pi_n(z)\to\varphi_0(z)$ $(n\to\infty)$ for $z\in G$, uniformly on compact subsets of G.

First of all, the uniform convergence of the sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\overline{G}$ was investigated by M. V. Keldych. He showed [20] that if the boundary L of G is a smooth Jordan curve with bounded curvature then the following estimate holds for every $\varepsilon>0$:

$$\|\varphi_0-\pi_n\|_{\overline{G}}\le\frac{c_{10}}{n^{1-\varepsilon}}.$$

In [20] the author also gives an example of domains G with a Jordan rectifiable boundary L for which the appropriate sequence of the Bieberbach polynomials diverges on a set which is everywhere dense in L.

Furthermore, S. N. Mergelyan [22] has shown that the Bieberbach polynomials satisfy

$$\|\varphi_0-\pi_n\|_{\overline{G}}\le\frac{c_{11}}{n^{1/2-\varepsilon}} \tag{8}$$

for every $\varepsilon>0$, whenever L is a smooth Jordan curve.

In addition to this the author [22] noted it is possible to replace the exponent $\frac{1}{2}-\varepsilon$ in (8) by $1-\varepsilon$.

Therefore the uniform convergence of the sequence $\{\pi_n\}_{n=1}^{\infty}$ in $\overline{G}$ and the estimate of the error $\|\varphi_0-\pi_n\|_{\overline{G}}$ depend on the geometric properties of boundary L. If L has a certain degree of smoothness, this error tends to zero with a certain speed. In several papers (see, e.g., [25], [24], [3], [4], [12], [13]) various estimates of the error $\|\varphi_0-\pi_n\|_{\overline{G}}$ and sufficient conditions on the geometry of the boundary L are given to guarantee the uniform convergence of the Bieberbach polynomials on $\overline{G}$. More extensive knowledge about them can be found in [4], [12].

To the best of the author's knowledge in the literature there are no results improving the above cited Mergelyan's result yet. In this section, applying Theorem 2, we give a result which improves estimate (8).

For the mapping φ_0 and a weight function ω we set

$$\varepsilon_n(\varphi_0')_2:=\inf_{p_n}\|\varphi_0'-p_n\|_{L_2(G)},\qquad E_n^{\circ}(\varphi_0',\omega)_2:=\inf_{p_n}\|\varphi_0'-p_n\|_{L^2(L,\omega)},$$

where inf is taken over all polynomials p_n of degree at most n.

At first we prove the following result, about the A_p-properties of the conformal maps φ_0 and φ.

Lemma 12. *Let G be a finite domain with a smooth boundary L. Then the functions $1/|\varphi_0'|$ and $1/|\varphi'|$ belong to $A_p(L)$ for every $p\in(1,\infty)$.*

Proof. We prove only the relation $1/|\varphi_0'| \in A_p(L)$. The other relation is proved similarly. Moreover, taking into account the property $A_{p_1}(L) \subset A_{p_2}(L)$ for $p_1 < p_2$, it is sufficient to consider the case $1 < p < 2$.

Since L is smooth, Theorem 3 of [10] states that, for every $p > 1$:

$$|\varphi'|, |\varphi_0'| \in A_p(L) \qquad \text{and} \qquad |\psi_0'| \in A_p(\partial D(0, r_0)). \tag{9}$$

It is easy to verify that the relation $|\psi_0'| \in A_p(\partial D(0, r_0))$ is equivalent to the inequality

$$\left(\frac{1}{|I|}\int_I |\varphi_0'|^q |dz|\right)^{1/q} \Bigg/ \left(\frac{1}{|I|}\int_I |\varphi_0'||dz|\right) \le c < \infty, \quad \text{for every arc } I \subset L, \tag{10}$$

where $q := p/(p-1)$. If we write

$$\begin{aligned}\left(\frac{1}{|I|}\int_I |\varphi_0'|^{-1}|dz|\right)&\left(\frac{1}{|I|}\int_I |\varphi_0'|^{1/(p-1)}|dz|\right)^{p-1} \\ &= \left[\left(\frac{1}{|I|}\int_I |\varphi_0'||dz|\right)\left(\frac{1}{|I|}\int_I |\varphi_0'|^{-1}|dz|\right)\right] \\ &\quad \times \left[\left(\frac{1}{|I|}\int_I |\varphi_0'|^{1/(p-1)}|dz|\right)^{p-1} \Bigg/ \left(\frac{1}{|I|}\int_I |\varphi_0'||dz|\right)\right],\end{aligned}$$

then the first factor is bounded because $|\varphi_0'| \in A_2(L)$. Further, applying inequality (10) for $q = 1/(p-1)$ we obtain the boundedness of the second factor. This completes the proof. ■

Now we can formulate the main result of this section.

Theorem 13. *Let G be a finite domain with a smooth Jordan boundary L. Then the Bieberbach polynomials π_n, for the pair (G, z_0), satisfy*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le c_{12}\left(\frac{\ln n}{n}\right)^{1/2} \Omega_{2,\omega_0}\left(\varphi_0'\left[\psi(w)\right]\left(\psi'(w)\right)^{1/2}, \frac{1}{n}\right), \qquad n \ge 2, \tag{11}$$

where $\omega := 1/|\varphi'|$, $\omega_0 := |\psi'|$, and $\Omega_{2,\omega_0}(\cdot, 1/n)$ is the ω_0-weighted integral modulus of continuity of order 2 for $\varphi_0'[\psi(w)](\psi'(w))^{1/2}$.

Proof. Since G is a finite domain with a smooth boundary, the functions $|\varphi'|$ and $1/|\varphi_0'|$ belong to $A_p(L)$ for every $p > 1$, by (9) and Lemma 9, respectively. Then by means of Hölder's inequality we get $\varphi_0' \in L^2(L, 1/|\varphi'|)$. On the other hand $\varphi_0' \in E^1(G)$. Hence, by definition, we have $\varphi_0' \in E^2(G, 1/|\varphi'|)$. Then the result [8, (Theorem 11, Remark (ii))] states that, for φ_0', $\omega := 1/|\varphi'|$ and $p = 2$:

$$\varepsilon_n(\varphi_0')_2 \le c_{13} n^{-1/2} E_n^\circ\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2. \tag{12}$$

For the polynomials $q_n(z)$, best approximating φ_0' in the norm $\|\cdot\|_{L_2(G)}$, we set

$$Q_n(z) := \int_{z_0}^{z} q_n(t)\,dt, \qquad t_n(z) := Q_n(z) + [1 - q_n(z_0)](z - z_0).$$

Then $t_n\ (z_0) = 0$, $t_n'\ (z_0) = 1$ and from (12) we obtain

$$\begin{aligned} \|\varphi_0' - t_n'\|_{L_2(G)} &= \|\varphi_0' - q_n - 1 + q_n(z_0)\|_{L_2(G)} \\ &\le \|\varphi_0' - q_n\|_{L_2(G)} + \|1 - q_n(z_0)\|_{L_2(G)} \\ &\le c_{13} n^{-1/2} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2 + \|\varphi_0'(z_0) - q_n(z_0)\|_{L_2(G)}. \end{aligned} \tag{13}$$

On the other hand, by the inequality

$$|f(z_0)| \le \frac{\|f\|_{L_2(G)}}{\operatorname{dist}(z_0, L)},$$

which holds for every analytic function f with $\|f\|_{L_2(G)} < \infty$, from (13) and (12) we get

$$\|\varphi_0' - t_n'\|_{L_2(G)} \le c_{13} n^{-1/2} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2 + \frac{\varepsilon_n(\varphi_0')_2}{\operatorname{dist}(z_0, L)} \le c_{14} n^{-1/2} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2.$$

So, according to the extremal property of the polynomials π_n, we have

$$\|\varphi_0 - \pi_n\|_{L_2^1(G)} \le c_{14} n^{-1/2} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2. \tag{14}$$

Further applying Andrievskii's [3] polynomial lemma

$$\|p_n\|_{\overline{G}} \le c(\ln n)^{1/2} \|p_n\|_{L_2^1(G)},$$

which holds for every polynomial p_n of degree $\le n$ with $p_n(z_0) = 0$, and using the familiar method of Simonenko [24] and Andrievskii [4], from (14) we get

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le c_{15} \left(\frac{\ln n}{n}\right)^{1/2} E_n^{\circ}\left(\varphi_0', \frac{1}{|\varphi'|}\right)_2, \qquad n \ge 2. \tag{15}$$

On the other hand, as is shown above, $\varphi_0' \in E^2(G, 1/|\varphi'|)$, and by Lemma 9 the function $\omega = 1/|\varphi'|$ belongs to $A_2(L)$. In addition, by [8, Lemma 3] $\omega_0 = |\psi'| \in A_2(T)$. Since every smooth Jordan boundary L belongs to S, finally we see that, the conditions of Theorem 2 are satisfied. Then relation (15) and Theorem 2 complete the proof. ■

The following improvement of Mergelyan's estimation (8) immediately follows from Theorem 4.

Corollary 14. *Let G be a finite domain with a smooth Jordan boundary L. Then the Bieberbach polynomials π_n, for the pair (G, z_0), satisfy*

$$\|\varphi_0 - \pi_n\|_{\overline{G}} \le c_{16} \left(\frac{\ln n}{n}\right)^{1/2}, \qquad n \ge 2. \tag{16}$$

In fact, estimation (11) is better than (16), because it contains the factor $\Omega_{2,\omega_0}(\varphi_0'[\psi(w)] (\psi'(w))^{1/2}, 1/n)$ which also tends to zero with a certain speed.

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