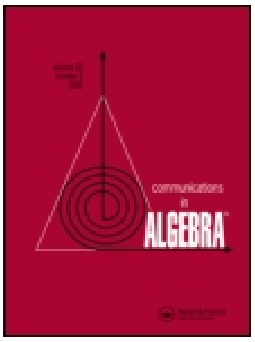

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THE *p*-COCKCROFT PROPERTY OF CENTRAL EXTENSIONS OF GROUPS

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THE p-COCKCROFT PROPERTY OF CENTRAL EXTENSIONS OF GROUPS

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ABSTRACT

A presentation for an arbitrary group extension is well known. A generalization of the work by Conway et al. (Group Tensor **1972**, *25*, 405–418) on central extensions has been given by Baik et al. (J. Group Theor.). As an application of this we discuss necessary and sufficient conditions for the presentation of the central extension to be p-Cockcroft, where p is a prime or 0. Finally, we present some examples of this result.

1991 Mathematical Subject Classification: 20F05; 20F55; 20F32; 57M05; 57M20.

1. INTRODUCTION

Let

$$\mathcal{P} = \langle \mathbf{x}\ ;\ \mathbf{r} \rangle \tag{1}$$

be a group presentation. Let F denote the free group on $\mathbf{x}$, and let N denote the normal closure of $\mathbf{r}$ in F. The quotient $G = F/N$ is the *group defined by* $\mathcal{P}$.

If we regard $\mathcal{P}$ as a 2-complex with one 0-cell, a 1-cell for each $x \in \mathbf{x}$, and a 2-cell for each $R \in \mathbf{r}$ in the standard way, then G is just the fundamental group of $\mathcal{P}$. There is also, of course, the second homotopy group $\pi_2(\mathcal{P})$ of $\mathcal{P}$, which is a left $\mathbb{Z}G$-module. The elements of $\pi_2(\mathcal{P})$ can be represented by geometric configurations called *spherical pictures*. These are described in detail in (1), and we refer the reader these for details. In this paper we need only one basepoint on each disc of our pictures [so we will actually use $*$-pictures, as described in Section 2.4 of (1)]. Also, as described in (1), there are certain operations on spherical pictures.

Suppose $\mathbf{X}$ is a collection of spherical pictures over $\mathbb{P}$. Then, by (1), one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *equivalence* (*rel* $\mathbf{X}$) *of spherical pictures*. Then, by (1), *the elements* $\langle\mathbb{P}\rangle$ $(\mathbb{P} \in \mathbf{X})$ *generate* $\pi_2(\mathcal{P})$ *as a module if and only if every spherical picture is equivalent (rel* $\mathbf{X}$*) to the empty picture*. If the elements $\langle\mathbb{P}\rangle$ $(\mathbb{P} \in \mathbf{X})$ generate $\pi_2(\mathcal{P})$ then we say that $\mathbf{X}$ *generates* $\pi_2(\mathcal{P})$.

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$, is the number of discs of $\mathbb{P}$, labeled by R, minus the number of discs, labeled by R^{-1}. We remark that if pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, then $exp_R(\mathbb{P}_1) = exp_R(\mathbb{P}_2)$ for all $R \in \mathbf{r}$.

Definition 1. Let $\mathcal{P}$ be as in presentation (1), and let n be a nonnegative integer. Then $\mathcal{P}$ is said to be n-Cockcroft if $exp_R(\mathbb{P}) \equiv 0 \pmod{n}$ (where congruence (mod 0) is taken to be equality) for all $R \in \mathbf{r}$ and for all spherical pictures $\mathbb{P}$ over $\mathcal{P}$. A group G is said to be *n-Cockcroft* if it admits an n-Cockcroft presentation.

Remark 2. To verify that the n-Cockcroft property holds, it is enough to check for pictures $\mathbb{P} \in \mathbf{X}$, where $\mathbf{X}$ is a set of generating pictures.

The 0-Cockcroft property is usually just called Cockcroft. In practice, we usually take n to be 0 or a prime p. The Cockcroft property has received considerable attention in (2–6). The p-Cockcroft property has been discussed, for example, in (6).

One can find the definition of *efficiency* for a presentation $\mathcal{P}$, for example, in (7–9). The following result, which is essentially due to Epstein (10), can be found in (6, Theorem 2.1).

Theorem 3. *Let $\mathcal{P}$ be as in* (1). *Then $\mathcal{P}$ is efficient if and only if it is p-Cockcroft for some prime p.*

1.1. Central Extensions

Let Q be a group with the presentation $\mathcal{P}_Q = \langle \mathbf{a}\ ;\ \mathbf{r}\rangle$, and let K be a cyclic group of order m generated by x ($m = 0$ if x has infinite order). Any *central extension* of K by Q will have a presentation of the form

$$\mathcal{P}_c = \langle \mathbf{a},\ x\ ;\ Rx^{-k_R}\ (R \in \mathbf{r}),\ x^m,\ [a, x]\ (a \in \mathbf{a})\rangle, \tag{2}$$

where $0 \leq k_R < m$, ($k_R \in \mathbb{Z}$ if $m = 0$).

However, not every presentation of this form defines an extension of K by Q, because the order of x may not be m in $G \cong G(\mathcal{P}_c)$. But, by (11) [see also (12) Corollary 7.2], if we know a generating set, say $\mathbf{Y}$, of $\pi_2(\mathcal{P}_Q)$ then we can give necessary and sufficient conditions for x to have order m (Theorem 4 later).

Let $\mathbb{Q}$ ($\mathbb{Q} \in \mathbf{Y}$) have discs $\Delta_1,\ \Delta_2, \ldots,\ \Delta_t$ labeled $R_1^{\varepsilon_1},\ R_2^{\varepsilon_2}, \ldots,\ R_t^{\varepsilon_t}$, respectively ($R_i \in \mathbf{r},\ \varepsilon_i = \pm 1,\ 1 \leq i \leq t$). Then let us choose a spray

$$\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_t \tag{3}$$

for $\mathbb{Q}$, and suppose the label on γ_i is W_{Δ_i} which is a word on $\mathbf{a}$ ($1 \leq i \leq t$). [The reader can be found the details of spray in (1).] Let $\beta(\mathbb{Q}) = \sum_{i=1}^{t} \varepsilon_i k_{R_i}$.

Theorem 4. (11, 12)**.** *Let G be the group defined by the presentation* (2). *Then the order of x is m in G if and only if*

$$\beta(\mathbb{Q}) \equiv 0\ (mod\ m) \quad (\mathbb{Q} \in \mathbf{Y}). \tag{4}$$

For $\mathbb{Q} \in \mathbf{Y}$ as above and $a \in \mathbf{a}$, we let $\alpha_a(\mathbb{Q}) = \sum_{i=1}^{t} \varepsilon_i\ \exp_a(W_{\Delta_i}) k_{R_i}$.

1.2. The General Theorem

Theorem 5. *Let p be a prime or* 0, *and let* $\mathcal{P}_c$ *be a presentation as in Equation* (2) *such that the condition* (4) *holds. Then* $\mathcal{P}_c$ *is p-Cockcroft if and only if*

(i) $m \equiv 0\ (mod\ p)$,
(ii) $\exp_a(R) \equiv 0\ (mod\ p)$, *for all* $a \in \mathbf{a}$, $R \in \mathbf{r}$,
(iii) $\mathcal{P}_Q$ *is p-Cockcroft*,
(iv) $\alpha_a(\mathbb{Q}) \equiv 0\ (mod\ p)$, *for all* $a \in \mathbf{a}$, $\mathbb{Q} \in \mathbf{Y}$,
(v) $\beta(\mathbb{Q}) \equiv 0\ (mod\ m \cdot p)$, *for all* $\mathbb{Q} \in \mathbf{Y}$.

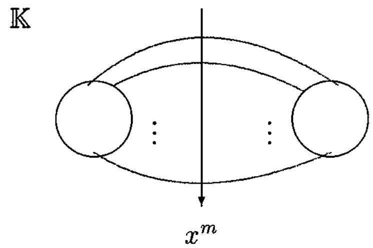


Figure 1.

2. THE PROOF OF THEOREM 5

Throught this section Q, K will be finite groups with the presentations $\mathcal{P}_Q = \langle \mathbf{a}\ ;\ \mathbf{r}\rangle$ and $\mathcal{P}_K = \langle x\ ;\ x^m\rangle$, respectively, and let $\mathcal{P}_c$ be a presentation of the central extension of K by Q as in Equation (2) such that the condition (4) holds. By (12), we can give a set of generating pictures over $\mathcal{P}_c$ as follows:

(I) The generating picture of the presentation $\mathcal{P}_K$ can be illustrated as in Figure 1.Note that if $m = 0$ then the picture $\mathbb{K}$ simply becomes the empty picture.
(II) For each $a \in \mathbf{a}$, we have a spherical picture $\mathbb{K}_a$ as in Figure 2. Also note that if $m = 0$ then the picture $\mathbb{K}_a$ becomes the empty picture.
(III) For each $R \in \mathbf{r}$, we have a spherical picture as in Figure 3a (or Fig. 3b if $k_R = 0$).
(IV) For each $\mathbb{Q} \in \mathbf{Y}$, a picture $\hat{\mathbb{Q}}$ defined as follows.

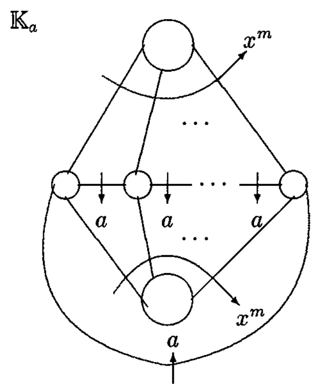


Figure 2.

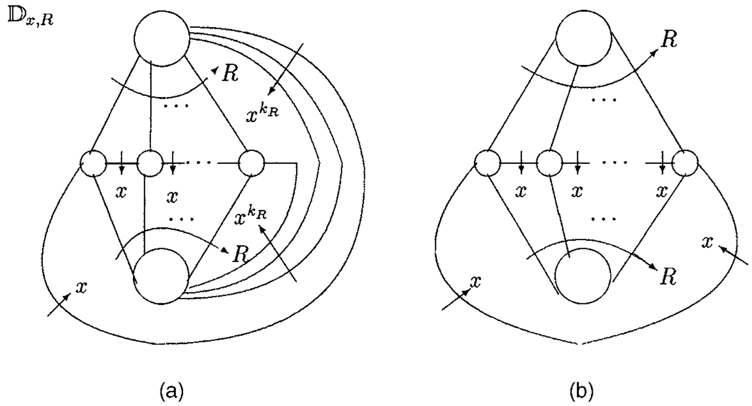


Figure 3.

For the picture $\mathbb{Q}$, we have the spray as defined in Equation (3). Then, for each disc Δ_i labeled $R_i^{\varepsilon_i}$ $(1 \leq i \leq t)$, we replace the spray line (transverse path) γ_i by a picture consisting of discs labeled $[a, x]$ $(a \in \mathbf{a})$ and with boundary label $W_{\Delta_i} x^{\varepsilon_i k_{R_i}} W_{\Delta_i}^{-1} x^{-\varepsilon_i k_{R_i}}$. This can be illustrated as in Figure 4. This gives a picture $\mathbb{Q}^*$ with the boundary label

$$\begin{aligned} W(\mathbb{Q}) &= (x^{\varepsilon_1 k_{R_1}} x^{\varepsilon_2 k_{R_2}} \cdots x^{\varepsilon_t k_{R_t}})^{-1} \\ &= x^{-\beta(\mathbb{Q})} \text{ by the definition of } \beta(\mathbb{Q}). \end{aligned}$$

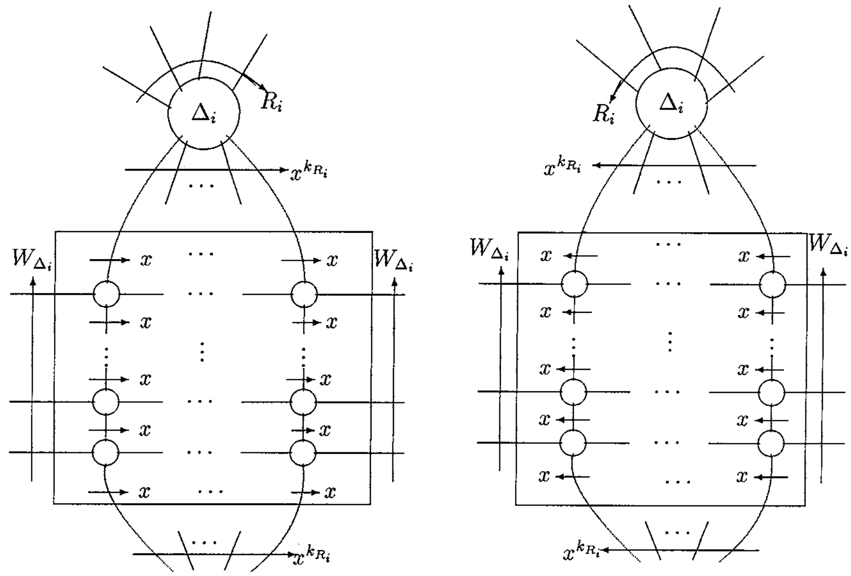
We then cap off $\mathbb{Q}^*$ with a picture consisting of $\frac{\beta(\mathbb{Q})}{m}$ times x^m-discs (where $\frac{\beta(\mathbb{Q})}{m}$ is taken to be 0 if $m = 0$), positively oriented if $\beta(\mathbb{Q}) > 0$, negatively oriented if $\beta(\mathbb{Q}) < 0$, to obtain a spherical picture $\hat{\mathbb{Q}}$. In doing this it may be necessary to join up loose oppositely oriented x-arcs.

In (12, Theorem 6.4), Baik–Harlander–Pride proved that if the presentation $\mathcal{P}_c$ be as in Equation (2) such that the condition (4) holds then $\pi_2(\mathcal{P}_c)$ is generated by the pictures

$$\mathbb{K}, \quad \mathbb{K}_a \, (a \in \mathbf{a}), \quad \mathbb{D}_{x,R} \, (R \in \mathbf{r}), \quad \text{and} \quad \hat{\mathbb{Q}} \, (\mathbb{Q} \in \mathbf{Y}).$$

At this part of the proof we must check the conditions of Theorem 5 by using these above generating pictures. Let C_R, C_a denote the relators Rx^{-k_R} $(R \in \mathbf{r})$, $[a, x]$ $(a \in \mathbf{a})$, respectively, in presentation $\mathcal{P}_c$.

First assume that $m \neq 0$. Let us consider the picture $\mathbb{K}$. It is clear that $\exp_{x^m}(\mathbb{K}) = 1 - 1 = 0$. Also, let us consider a picture $\mathbb{K}_a$ $(a \in \mathbf{a})$. Clearly $\exp_{x^m}(\mathbb{K}_a) = 1 - 1 = 0$, and it is easy to see that $\exp_{C_a}(\mathbb{K}_a) = \exp_x(x^m) = m$, so we must have $m \equiv 0 \pmod{p}$. Hence the condition (i) must hold. Consider a picture $\mathbb{D}_{x,R}$ $(R \in \mathbf{r})$. We have $\exp_{C_R}(\mathbb{D}_{x,R}) = 1 - 1 = 0$, and we get $\exp_{C_a}(\mathbb{D}_{x,R}) = \exp_a(R)$, for all $a \in \mathbf{a}$. Thus, the condition (ii) must hold. Now consider a picture $\hat{\mathbb{Q}}$ $(\mathbb{Q} \in \mathbf{Y})$. We must have $\exp_{C_R}(\hat{\mathbb{Q}}) \equiv 0 \pmod{p}$. But $\exp_{C_R}(\hat{\mathbb{Q}}) = \exp_R(\mathbb{Q})$, so



$(\varepsilon_i = 1)$ $(\varepsilon_i = -1)$

Figure 4.

we must have $\exp_R(\mathbb{Q}) \equiv 0 \pmod{p}$, that is, $\mathcal{P}_Q$ must be p-Cockcroft. This gives the condition (iii). Also, for a fixed $a \in \mathbf{a}$, we have $\exp_{C_a}(\hat{\mathbb{Q}}) = \alpha_a(\mathbb{Q})$. Then we must have $\alpha_a(\mathbb{Q}) \equiv 0 \pmod{p}$, which gives the condition (iv). Finally, we have that $\exp_{x^m}(\hat{\mathbb{Q}}) = \frac{\beta(\mathbb{Q})}{m}$. Then we must have $\beta(\mathbb{Q}) \equiv 0 \pmod{m.p}$. So the condition (v) must hold.

Suppose that $m = 0$. Then the five conditions (i)–(v) reduce to the three conditions

(ii) $\exp_a(R) \equiv 0 \pmod{p}$, for all $a \in \mathbf{a}$, $R \in \mathbf{r}$,
(iii) $\mathcal{P}_Q$ is p-Cockcroft,
(iv) $\alpha_a(\mathbb{Q}) \equiv 0 \pmod{p}$, for all $a \in \mathbf{a}$, $\mathbb{Q} \in \mathbf{Y}$,

since the conditions (i) and (v) automatically hold. Because the pictures $\mathbb{K}$ and $\mathbb{K}_a$ are trivial, so impose no restrictions, and there are no x^m discs, then the above process on pictures will carry over.

3. SOME EXAMPLES

Example 6. Let Q be the (k, l, n)-triangle group with the presentation $\mathcal{P}_Q = \langle a, b\,;\, a^k, b^l, (ab)^n \rangle$, where $k, l, n \in \mathbb{Z}^+$ and $\frac{1}{k} + \frac{1}{l} + \frac{1}{n} \leq 1$, and let K be a

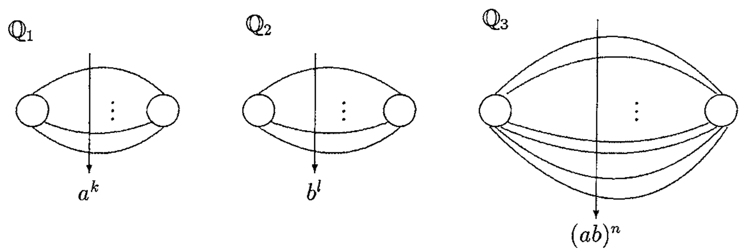


Figure 5.

cyclic group of order m generated by x (m is taken to be 0 if x has infinite order). Consider the presentation

$$\mathcal{P}_c = \langle a,\ b,\ x\ ;\ a^k x^{-r},\ b^l x^{-s},\ (ab)^n x^{-t},\ x^m,\ C_a,\ C_b\rangle, \tag{5}$$

where $0 \le r,\ s,\ t < m$ (or $r,\ s,\ t \in \mathbb{Z}$, if $m = 0$) and let $C_a := [a, x]$ and $C_b := [b, x]$. By the weight test [see (13, 14)], $\mathcal{P}_Q$ is Combinatorial Aspherical [see (8)] and then Cockcroft. We can give a set of generating pictures of $\pi_2(\mathcal{P}_Q)$ as in Figure 5, so $\beta(\mathbb{Q}_1) = 0$, $\beta(\mathbb{Q}_2) = 0$, $\beta(\mathbb{Q}_3) = 0$ and then condition (4) holds. Hence, by Theorem 4, the group G defined by $\mathcal{P}_c$ is a central extension of K by Q. We have $\exp_a(a^k) = k$, $\exp_b(b^l) = l$, $\exp_a((ab)^n) = n$, $\exp_b((ab)^n) = n$. Moreover, we get $\alpha_a(\mathbb{Q}_1) = r$, $\alpha_b(\mathbb{Q}_1) = 0$, $\alpha_a(\mathbb{Q}_2) = 0$, $\alpha_b(\mathbb{Q}_2) = s$, $\alpha_a(\mathbb{Q}_3) = t$, $\alpha_b(\mathbb{Q}_3) = t$. Also, for any prime p, we always have $\beta(\mathbb{Q}_i) \equiv 0 \,(\mathrm{mod}\ m.p)(i = 1, 2, 3)$.

Thus, we get the following result for Example 6, as a consequence of Theorems 3 and 5.

Corollary 7. *Let p be a prime. Then the presentation $\mathcal{P}_c$ as in Equation* (5), *is p-Cockcroft if and only if*

$$\begin{array}{lll} & m \equiv 0\,(mod\ p), & \\ k \equiv 0\,(mod\ p), & l \equiv 0\,(mod\ p), & n \equiv 0\,(mod\ p), \\ r \equiv 0\,(mod\ p), & s \equiv 0\,(mod\ p), & t \equiv 0\,(mod\ p). \end{array}$$

Hence $\mathcal{P}_c$ is efficient if and only if

$$hcf(m,\ k,\ l,\ n,\ r,\ s,\ t) \neq 1.$$

Example 8. Let Q be the group $\mathbb{Z}_k \oplus \mathbb{Z}_l$ ($k, l \in \mathbb{Z}^+$) with the presentation

$$\mathcal{P}_Q = \langle a,\ b\ ;\ a^k,\ b^l,\ [a, b]\rangle,$$

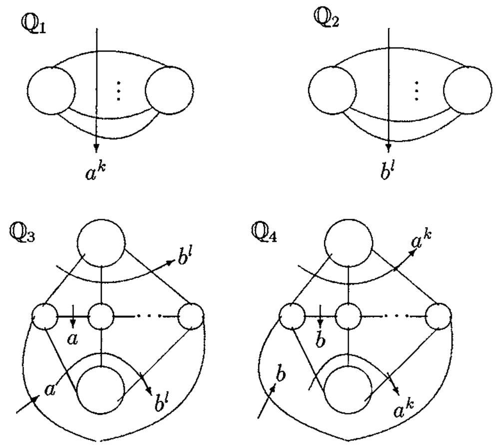


Figure 6.

and let K be a finite cyclic group of order m generated by x. Let us consider the presentation

$$\mathcal{P}_c = \langle a,\ b,\ x\ ;\ a^k x^{-r},\ b^l x^{-s},\ [a,b]x^{-t},\ x^m,\ C_a,\ C_b \rangle, \tag{6}$$

where $0 \leq r, s, t < m$ and let $C_a := [a, x]$ and $C_b := [b, x]$. We can give a set of generating pictures of $\pi_2(\mathcal{P}_Q)$, as in Figure 6. We have $\beta(\mathbb{Q}_1) = 0$, $\beta(\mathbb{Q}_2) = 0$, $\beta(\mathbb{Q}_3) = lt$ and $\beta(\mathbb{Q}_4) = kt$. Suppose that $lt \equiv 0\,(\text{mod}\ m)$ and $kt \equiv 0\,(\text{mod}\ m)$. So condition (4) holds. Then, by Theorem 4, the group G defined by $\mathcal{P}_c$ is a central extension of K by Q. It is clear that $\exp_a(a^k) = k$, $\exp_b(b^l) = l$, $\exp_a([a,b]) = 1 - 1 = 0$, $\exp_b([a,b]) = 1 - 1 = 0$. Also, by the definition, we get $\alpha_a(\mathbb{Q}_1) = r$, $\alpha_b(\mathbb{Q}_1) = 0$, $\alpha_a(\mathbb{Q}_2) = 0$, $\alpha_b(\mathbb{Q}_2) = s$, $\alpha_a(\mathbb{Q}_3) = s$, $\alpha_b(\mathbb{Q}_3) = -\frac{1}{2}l(l-1)t, \alpha_a(\mathbb{Q}_4) = \frac{1}{2}k(k-1)t$, $\alpha_b(\mathbb{Q}_3) = r$.

Therefore, we get the following result for Example 8, as a consequence of Theorems 3 and 5.

Corollary 9. *Let p be a prime. Then the presentation $\mathcal{P}_c$, as in Equation* (6), *is p-Cockcroft if and only if*

$$\begin{gathered} m \equiv 0\,(mod\ p), \\ k \equiv 0\,(mod\ p), \quad r \equiv 0\,(mod\ p), \quad kt \equiv 0\,(mod\ m.p), \\ l \equiv 0\,(mod\ p), \quad s \equiv 0\,(mod\ p), \quad lt \equiv 0\,(mod\ m.p), \\ \tfrac{1}{2}k(k-1)t \equiv 0\,(mod\ p) \\ -\tfrac{1}{2}l(l-1)t \equiv 0\,(mod\ p). \end{gathered}$$

Thus, $\mathcal{P}_c$ *is efficient if and only if*

$$hcf\left(m, k, l, r, s, \frac{1}{2}k(k-1)t, \frac{1}{2}l(l-1)t, \frac{1}{m}kt, \frac{1}{m}lt\right) \neq 1.$$

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