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An obstruction to finding algebraic models for smooth manifolds with prescribed algebraic submanifolds

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Abstract

Let $N \subseteq M$ be a pair of closed smooth manifolds and L an algebraic model for the submanifold N. In this paper, we will give an obstruction to finding an algebraic model X of M so that the submanifold N corresponds in X to an algebraic subvariety isomorphic to L.

1. *Introduction and results*

Seifert proved in 1936 that any closed smooth submanifold M of $\mathbb{R}^n$ with trivial normal bundle is isotopic to a nonsingular component of a real algebraic subvariety X of $\mathbb{R}^n$ ([**18**]). In 1952 Nash showed that any closed smooth manifold is diffeomorphic to a component of a nonsingular real algebraic variety ([**13**]). Later, in 1973 Tognoli proved that any closed smooth manifold is diffeomorphic to a nonsingular real algebraic variety ([**22**]) and also observed that the algebraic realization problem is a bordism problem. Later Akbulut and King improved Tognoli's result using this bordism technique. They proved that any closed smooth submanifold M of $\mathbb{R}^n$ is isotopic to a nonsingular real algebraic subvariety X of $\mathbb{R}^{n+1}$ ([**3**, **4**]). Using similar techniques Dovermann and Masuda showed that closed smooth manifolds with certain group actions, such as semifree or odd order finite group actions, can be realized algebraically ([**11**]). Suh has also results in this direction ([**19**]). In 1993 Akbulut and King showed that some submanifolds of $\mathbb{R}^n$ cannot be isotoped to an algebraic subvariety of $\mathbb{R}^n$ with nonsingular complexification ([**5**]).

Given a closed smooth manifold M with a submanifold N, not necessarily connected, there exits a nonsingular real algebraic variety X diffeomorphic to M such that N corresponds to a nonsingular subvariety of X under the diffeomorphism. In this paper we will focus on the following problem: let $N \subseteq M$ be a smooth closed submanifold and L a nonsingular real algebraic variety diffeomorphic to N. Then, is there a nonsingular real algebraic variety X and a diffeomorphism $f: M \to X$ so that $f(N)$ is an algebraic subvariety of X isomorphic to L?

If M is a smooth manifold and $f: M \to X$ a diffeomorphism where X is a nonsingular real algebraic variety then we will call X an algebraic model for the smooth manifold M. Similarly, if $N \subseteq M$ is a smooth submanifold of the closed smooth manifold M and $f: M \to X$ a diffeomorphism so that $L = f(N)$ is a nonsingular real algebraic subvariety of X then the pair (X, L) will be called an algebraic model for the pair (M, N). So the above problem can be restated as follows: given the smooth manifolds $N \subseteq M$ and an algebraic model L of N is there a nonsingular real algebraic variety X so that (X, L) is an algebraic model for (M, N)?

The following theorem, which is a direct consequence of theorem 2·8·4 of [**1**], whose weaker form is originally proved by Benedetti and Tognoli ([**6**]), shows that the algebraic realization question of (M, N) by a pair (X, L), for some X, is indeed an infinitesimal question at L.

Theorem 1·1 ([**1**]). *Let $L \subseteq M \subseteq \mathbb{R}^k$, where L is a nonsingular real algebraic variety and M an embedded closed smooth manifold. Then there is a smooth embedding $g: M \to \mathbb{R}^k \times \mathbb{R}^l$ such that $X = g(M)$ is a nonsingular real algebraic variety with $g(x) = x$, for all $x \in L$, if and only if the normal bundle $N_M(L)$ of L in M has a strongly algebraic structure.*

In general, whether a given topological vector bundle over a compact nonsingular real algebraic variety L has a strongly algebraic structure or not, is a difficult question. If $\dim(L) \leqslant 3$ then the algebraic homology of L, $H_*^A(L, \mathbb{Z}_2)$, determines the answer completely (cf. see section 12·5 of [**7**]).

The next theorem gives a partial answer to the algebraic realization question in one direction, for all dimensions, in terms of the algebraic topology of the pairs $N \subseteq M$ and $L \subseteq L_{\mathbb{C}}$, where $L_{\mathbb{C}}$ is a complexification of L. First some preliminaries.

Let R be any commutative ring with unity. For an R orientable nonsingular compact real algebraic variety X define $KH_*(X, R)$ to be the kernel of the induced map

$$i_*: H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology, where $i: X \to X_{\mathbb{C}}$ is the inclusion map into some nonsingular projective complexification. In [**14**] it is shown that $KH_*(X, R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$ and thus an (entire rational) isomorphism invariant of X (see Section 2 for the definition of complexification we use in this note). Dually, denote the image of the homomorphism

$$i^*: H^*(X_{\mathbb{C}}, R) \to H^*(X, R)$$

by $\operatorname{Im} H^*(X, R)$, which is also an isomorphism invariant.

Theorem 1·2. *Let M be a closed smooth manifold, $N \subseteq M$ a smooth closed n-dimensional submanifold and L an algebraic model for N. Suppose that one of the following conditions hold*:

(i) *N is oriented and there exists a cohomology class $u \in H^n(M, \mathbb{Q})$, which belongs to the subalgebra generated by the Pontrjagin classes of (the tangent bundle of) M, with $u([L]) \neq 0$ and $[L] \in KH_n(L, \mathbb{Q})$.*

(ii) *There exists a cohomology class $u \in H^n(M, \mathbb{Z}_2)$, which belongs to the subalgebra generated by the squares of the Stiefel–Whitney classes of (the tangent bundle of) M, with $u([L]) \neq 0$ and $[L] \in KH_n(L, \mathbb{Z}_2)$.*

Then the pair (M, N) *has no algebraic model of the form* (X, L).

If L is as in the above theorem, then the vector bundle over L, obtained by pulling back the normal bundle of N in M, has no strongly algebraic structure.

Example 1·3. Consider the smooth manifolds $N' = \mathbb{R}P^2 \subseteq \mathbb{R}P^4 = M$ so that the fundamental class of N' is not zero in $H_2(M, \mathbb{Z}_2)$. Inside a small four ball centred at a point p of N' connect sum another copy of $\mathbb{R}P^2$ to N' (note that $\mathbb{R}P^2 \subseteq \mathbb{R}^4$). So we have obtained an embedded Klein bottle, $N = KB \subseteq M$ realizing the same homology class as N'. Hence, if ω_1 is the first Stiefel–Whitney class of M, then $\omega_1^2([N]) = \omega_1^2([N']) \neq 0$.

Proposition 1·4. *There exists an algebraic model L of the Klein bottle with* $[L] \in KH_2(L_{\mathbb{C}}, \mathbb{Z}_2)$.

Hence, if L is as in the above proposition then by the above theorem the smooth pair (M, N) has no algebraic model of the form (X, L).

Example 1·5. Consider two copies of the smooth manifold $\mathbb{C}P^2$ one containing an embedded oriented closed surface F and the other an embedded torus T^2 both realizing nonzero homology classes. For example, let $F \subseteq CP^2$ be any smooth algebraic curve and T^2 an elliptic curve in $\mathbb{C}P^2$. Now embed $\mathbb{C}P^2 \times \mathbb{C}P^2$ into $\mathbb{C}P^8$ using the Segre embedding

$$([z_0, z_1, z_2], [w_0, w_1, w_2]) \longmapsto [z_0 w_0, \ldots, z_i w_j, \ldots, z_2 w_2].$$

Then $F \times T^2$ realizes a nonzero homology class, say $\alpha \in H_4(\mathbb{C}P^8, \mathbb{Q})$. In particular, $p_1([F \times T^2]) \neq 0$, where p_1 is the first Pontrjagin class of $\mathbb{C}P^8$. Embed smoothly $\mathbb{C}P^8$ into some Euclidean space $\mathbb{R}^n$ so that the submanifold $F \times T^2$ maps diffeomorphically onto $F' \times S^1 \times S^1 \subseteq \mathbb{R}(n-4k) \times \mathbb{R}^2 \times \mathbb{R}^2$, where $F' \subseteq \mathbb{R}(n-4k)$ is an algebraic model for F and S^1 is the standard unit circle. Call this algebraic variety L. Since S^1 bounds in its complexification $S^1_{\mathbb{C}} = \mathbb{C}P^1 = S^2$ so does L and hence by the above theorem the pair $(\mathbb{C}P^8, F \times T^2)$ has no real algebraic model of the form (X, L).

Indeed, it is apparent from the above argument that the same works if we replace $S^1 \times S^1$ by $X_1 \times X_2$, where both are nonsingular compact connected real algebraic curves one of which is separating (homologously trivial in its complexification).

Remark 1·6. In Example 1·5 let $F = S^2 = \mathbb{C}P^1 \subseteq \mathbb{C}P^2$. By the example below any topological vector bundle S^2 is strongly algebraic. We also know that any topological real vector bundle over $S^1 \times S^1$ is strongly algebraic because the homology of $S^1 \times S^1$ is algebraic (cf. corollary 12·5·4 and remark 12·6·8 of [**7**]). Hence, by Theorem 1·1 we conclude that, not every topological real vector bundle over $S^2 \times S^1 \times S^1$ has a strongly algebraic structure, even though any topological real vector bundle over S^2 or $S^1 \times S^1$ has a strongly algebraic structure.

Any strongly algebraic complex line bundle over the standard torus $S^1 \times S^1$ is trivial, because any entire rational map from $S^1 \times S^1$ to the Grassmann variety $\mathbb{C}P^n$ is null homotopic (see theorems 2·4 and 4·2 of [**8**]). However, we cannot use this fact to get examples as above. Indeed, since any topological real vector bundle over $S^1 \times S^1$ is strongly algebraic we can even find an algebraic model $(S^1 \times S^1, X)$ for the pair $(E, \mathbb{C}P^2)$, where E is any given smooth elliptic curve in $\mathbb{C}P^2$. In other words, the (strongly algebraic) normal bundle of $S^1 \times S^1$ in X has topologically the structure of a complex vector bundle, even though this complex structure cannot be made complex algebraic.

Example 1·7. It is well known that any continuous vector bundle over the standard k-sphere $S^k \subseteq \mathbb{R}^{k+1}$ has a strongly algebraic structure ([**20**, **21**]) and therefore if M is a closed smooth manifold with an embedded k-sphere as a submanifold then M has an algebraic model where this submanifold is replaced with a subvariety isomorphic to the standard sphere S^k.

Remark 1·8. In their work [**10**] Bos, Levenberg, Milman and Taylor prove the following nice result: let $M \subseteq R^n$ be a smooth compact submanifold. Then M is algebraic (a union of components of an algebraic variety) if and only if M satisfies a tangential Markov inequality with exponent one, i.e. there exists $C = C(M) > 0$ such that

$$|D_T p(x)| \leqslant C\ (\deg p)\ \|p\|_M, \quad x \in M$$

for all polynomials p, where D_T denotes any tangential derivative and $\|p\|_M$ the supremum norm of p on M. Combining this with Example 1·5 (Example 1·3) we arrive at the following interesting conclusion: the Markov inequality, mentioned above, will never hold on the embedded manifold $M = \mathbb{C}P^8 \subseteq R^n$ ($M = \mathbb{R}P^4 \subseteq R^n$) no matter how we isotop it, even in some larger space $\mathbb{R}^{n+k}$, provided that the isomorphism type of L is kept fixed. On the other hand, by the Akbulut–King result mentioned in the introduction we can isotop M to an algebraic variety in some larger space $\mathbb{R}^{n+k}$, on which the Markov inequality is trivially satisfied, if we are willing to replace L with some other algebraic model of the smooth manifold $F \times T^2$ (Klein bottle).

2. *Proofs*

All real algebraic varieties under consideration in this report are compact and nonsingular. It is well known that real projective varieties are affine (proposition 2·4·1 of [**1**] or theorem 3·4·4 of [**7**]). Moreover, compact affine real algebraic varieties are projective (corollary 2·5·14 of [**1**]) and therefore we will not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$ a map $F: X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say X and Y are isomorphic to each other if there are entire rational maps $F: X \to Y$ and $G: Y \to X$ such that $F \circ G = \mathrm{id}_Y$ and $G \circ F = \mathrm{id}_X$. Isomorphic algebraic varieties will be regarded the same. A complexification $X_{\mathbb{C}} \subseteq \mathbb{C}P^N$ of X will mean that X is embedded into some projective space $\mathbb{R}P^N$ and $X_{\mathbb{C}} \subseteq \mathbb{C}P^N$ is the complexification of the pair $X \subseteq \mathbb{R}P^N$. We also require the complexification to be nonsingular (blow up $X_{\mathbb{C}}$ along smooth centres away from X defined over reals if necessary, [**9**, **12**]). We refer the reader to [**1**, **7**] for the basic definitions and facts about real algebraic geometry.

For a compact nonsingular real algebraic variety X, let $H_k^A(X, \mathbb{Z}_2) \subseteq H_k(X, \mathbb{Z}_2)$ be the subgroup of classes represented by algebraic subvarieties of X and let $H_A^k(X, \mathbb{Z}_2)$ be the Poincaré dual of $H_{n-k}^A(X, \mathbb{Z}_2)$. These are well known and very useful in the study of real algebraic varieties. Also we define $H_A^k(X, \mathbb{Z}_2)^2$ to be the subgroup

$$\{\alpha^2 \mid \alpha \in H_A^k(X, \mathbb{Z}_2)\} \subseteq H_A^{2k}(X, \mathbb{Z}_2)$$

(cup product preserves algebraic cycles [**2**]).

It is well known that Grassmann varieties together with their canonical bundles have canonical real algebraic structures. Pullbacks of these canonical bundles via entire rational maps, from X into the Grassmannians, are called strongly algebraic vector bundles over X. A continuous vector bundle $E \to X$ is said to have a strongly algebraic structure if it is continuously isomorphic to a strongly algebraic vector bundle, or equivalently, if the continuous map classifying E is homotopic to an entire rational map.

Akbulut and King showed that $H^k_A(X, \mathbb{Z}_2)^2$ and Pontrjagin classes of X are pullbacks of some classes of $X_{\mathbb{C}}$ ([**5**]). Indeed, the same works for any strongly algebraic vector bundle $E \to X$ over X, not just for the tangent bundle, because the complexification (as a vector bundle) of any strongly algebraic vector bundle over X extends over some complexification $X_{\mathbb{C}}$ of X. The reason is that the real Grassmann variety, $G_{\mathbb{R}}(n, k)$, of the real k-planes in $\mathbb{R}^n$ has the complex Grassmann variety, $G_{\mathbb{C}}(n, k)$, of the complex k-planes in $\mathbb{C}^n$ as its natural complexification and therefore any entire rational map from X into $G_{\mathbb{R}}(n, k)$ gives rise to a regular map, maybe after some blowing-ups of the domain along centres away from the real part X ([**9**, **12**]), from $X_{\mathbb{C}}$ into $G_{\mathbb{C}}(n, k)$. We can summarize this as follows:

Theorem 2·1 ([**16**]). *Let X be a nonsingular compact connected real algebraic variety and*

$$P = \{e^2(E),\ p_i(E) \mid E \to X \text{ is a strongly algebraic vector bundle}\}$$

and

$$W^2 = \{w_i^2(E) \mid E \to X \text{ is a strongly algebraic vector bundle}\}$$

which are subsets of $H^*(X, \mathbb{Q})$ and $H^*(X, \mathbb{Z}_2)$ respectively, where $e(E)$, $p_i(E)$ and $w_i(E)$ are the Euler, the Pontrjagin and the Stiefel–Whitney classes of E. Then, $\operatorname{Im} H^*(X, \mathbb{Q})$ *and* $\operatorname{Im} H^*(X, \mathbb{Z}_2)$ *contain the subalgebras generated by P and W^2 respectively.*

Proof of Theorem 1·2. Suppose there exists an algebraic model of the form (X, L). Then, by Theorem 2·1 we have $u = i^*(v)$ for some $v \in H^n(X_{\mathbb{C}}, R)$, where $i\colon X \to X_{\mathbb{C}}$ is the inclusion map and R is either $\mathbb{Q}$ or $\mathbb{Z}_2$. By the hypothesis $0 \neq u([L]) = i^*(v)([L]) = v(i_*([L])) = v(0) = 0$, which is a contradiction. Hence we are done.

Proof of Proposition 1·4. Consider the 2-torus

$$T^2 = S^1 \times S^1 = \{(x_1, x_2, y_1, y_2) \subseteq \mathbb{R}^4 \mid x_1^2 + x_2^2 = 1,\ y_1^2 + y_2^2 = 1\}$$

with the algebraic $\mathbb{Z}_2$-action given by

$$(x_1, x_2, y_1, y_2) \mapsto (-x_1, -x_2, -y_1, y_2).$$

The quotient is the smooth Klein bottle. Indeed, it is a nonsingular real algebraic variety. To see this first consider the affine complexification of $S^1 \times S^1$ in $\mathbb{C}^4$ given by the same equations. The $\mathbb{Z}_2$-action extends over the complexification so that the subset of the complexification on which the $\mathbb{Z}_2$-action agrees with the complex conjugation is the empty set. Now, Theorem 2·2(a) of [**15**] (or [**17**]) proves that the quotient is a nonsingular real algebraic variety, say L.

Let us now show that $[L] \in KH_2(L_{\mathbb{C}}, \mathbb{Z}_2)$. Choose an orientation for the first factor of $S^1 \times S^1$ and let D denote the closure of one of two the disk components of $S^1_{\mathbb{C}} = S^2 - S^1$, whose complex orientation agrees with this orientation of S^1 on the boundary. Note also that the action extends over the complexification $S^1_{\mathbb{C}} \times S^1_{\mathbb{C}}$:

$$([x_0, x_1, x_2], [y_0, y_1, y_2]) \mapsto ([x_0, -x_1, -x_2], [y_0, -y_1, y_2]),$$

where we identify S^1 with

$$\{[x_0, x_1, x_2] \in \mathbb{R}P^2 \mid x_1^2 + x_2^2 = x_0^2\}$$

with projective complexification

$$S^1_{\mathbb{C}} = \{[x_0, x_1, x_2] \in \mathbb{C}P^2 \mid x_1^2 + x_2^2 = x_0^2\}$$

which is isomorphic to $\mathbb{C}P^1$.

Since the $\mathbb{Z}_2$-action on the first factor of $S^1 \times S^1$ is orientation preserving (180° rotation) its leaves invariant the solid torus $D \times S^1$, which bounds $S^1 \times S^1$ in its projective complexification $S^1_{\mathbb{C}} \times S^1_{\mathbb{C}}$. In the quotient, the two ends of the half of this solid torus,

$$D \times \{(y_1, y_2) \in S^1 \mid y_2 \geqslant 0\},$$

identifies and gives the solid Klein bottle bounding L. This finishes the proof.

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