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A TAYLOR COLLOCATION METHOD FOR THE SOLUTION OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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In this study, a matrix method called the Taylor collocation method is presented for numerically solving the linear integro-differential equations by a truncated Taylor series. Using the Taylor collocation points, this method transforms the integro-differential equation to a matrix equation which corresponds to a system of linear algebraic equations with unknown Taylor coefficients. Also the method can be used for linear differential and integral equations. To illustrate the method, it is applied to certain linear differential, integral, and integro-differential equations and the results are compared.

Keywords: Taylor polynomials and series; Collocation points; Differential, integral and integro-differential equations

C.R. Categories: G.1.4

1. INTRODUCTION

A Taylor expansion approach to solving integral equations has been presented by Kanwal and Liu [1], and the method extented by Sezer to Volterra integral equations [2], second order linear differential equations [3] and integro-differential equations [4–7].

In this study the basics of the mentioned works, by means of Taylor collocation points, are developed and applied to problems consisting of:

1. *m*th-order linear Fredholm integro-differential equation

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,\mathrm{d}t \tag{1}$$

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and mth-order linear Volterra integro-differential equation

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = f(x) + \lambda \int_a^x K(x,t) y(t)\, \mathrm{d}t \tag{2}$$

where the known functions $P_k(x)$, $f(x)$, $K(x, t)$ are defined on the $a \le x,\ t \le b$; λ is a real parameter, $y(x)$ is the unknown function.

2. The conditions (in the most general)

$$\sum_{j=0}^{m-1} \left[a_{ij} y^{(j)}(a) + b_{ij} y^{(j)}(b) + c_{ij} y^{(j)}(c) \right] = \lambda_i; \quad i = 0, 1, \ldots, m-1 \tag{3}$$

where $a \le c \le b$, provided that the real coefficients, a_{ij}, b_{ij}, c_{ij} and λ_i are appropriate constants, and the approximate solution is expressed in the truncated Taylor series,

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!} (x-c)^n$$

where $y^{(n)}(c)$ are the Taylor coefficients to be determined.

2. METHOD OF SOLUTION FOR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Let us first consider the Fredholm integro-differential equation

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t) y(t)\, \mathrm{d}t \tag{4}$$

We assume that the solution of (4) can be truncated Taylor series

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!} (x-c)^n; \quad a \le x \le b \tag{5}$$

where N is chosen any positive integer such that $N \ge m$. Besides we suppose that the functions $P_k(x)$ in Eq. (4) are defined in $a \le x \le b$ and $K(x, t)$ is defined and if bounded variation in $a \le x,\ t \le b$; that is, $K(x, t)$ can be expanded to Taylor series. Then the solution (5) of Eq. (4) can be expressed in the matrix form

$$[y(x)] = \mathbf{X}\mathbf{M}_0\mathbf{A}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & x-c & (x-c)^2 & \cdots & (x-c)^N \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} y^{(0)}(c) & y^{(1)}(c) & y^{(2)}(c) & \cdots & y^{(N)}(c) \end{bmatrix}^t$$

and

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2!} & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & & \frac{1}{N!} \end{bmatrix}$$

To obtain such a solution, we can use the following matrix method, which is a Taylor Collocation method. This method is based on computing the Taylor coefficients by means of the Taylor collocation points are thereby finding the matrix **A** containing the unknown Taylor coefficients.

Firstly, we substitute the Taylor collocation points defined by

$$x_i = a + i\frac{b-a}{N}; \quad i = 0, 1, \ldots, N; \qquad x_0 = a,\ x_1 = b \tag{6}$$

into Eq. (4) to obtain

$$\sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) = f(x_i) + \lambda \mathbf{I}(x_i) \tag{7}$$

so that

$$I(x_i) = \int_a^b K(x_i, t)y(t)\,\mathrm{d}t$$

then we can write the system (7) in the matrix form

$$\mathbf{P}_0\mathbf{Y}^{(0)} + \mathbf{P}_1\mathbf{Y}^{(1)} + \cdots + \mathbf{P}_m\mathbf{Y}^{(m)} = \sum_{k=0}^{m} \mathbf{P}_k\mathbf{Y}^{(k)} = \mathbf{F} + \lambda\mathbf{I} \tag{8}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_0) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}, \quad \mathbf{Y}^k = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} I(x_0) \\ I(x_1) \\ \vdots \\ I(x_N) \end{bmatrix}$$

Let us assume that the kth derivative of the function (5) with respect to x has the truncated Taylor series expansion defined by Eq. (5).

$$y^{(k)}(x_i) = \sum_{n=k}^{N} \frac{y^{(n)}(c)}{(n-k)!}(x_i - c)^{n-k}; \quad a \leqslant x \leqslant b$$

where $y^{(k)}(x)$ $(k = 0, 1, \ldots, N)$ are Taylor coefficients; clearly $y^{(0)}(x) = y(x)$. Then substituting the Taylor collocation points in this expression, we get the matrix forms

$$[\mathbf{y}^{(k)}(x)_i] = \mathbf{X}_{x_i}\mathbf{M}_k\mathbf{A}, \quad (k = 0, 1, \ldots, N) \tag{9}$$

or the matrix equation

$$\mathbf{Y}^{(k)} = \mathbf{C}\mathbf{M}_k\mathbf{A} \tag{10}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{X}_{x_0} \\ \mathbf{X}_{x_1} \\ \vdots \\ \mathbf{X}_{x_N} \end{bmatrix} \begin{bmatrix} (x_0 - c)^0 & (x_0 - c)^1 & \cdots & (x_0 - c)^N \\ (x_1 - c)^0 & (x_1 - c)^1 & \cdots & (x_1 - c)^N \\ \vdots & \vdots & \ddots & \vdots \\ (x_N - c)^0 & (x_N - c)^1 & \cdots & (x_N - c)^N \end{bmatrix}$$

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 & \cdots & \frac{1}{0!} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{1!} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{(N-k)!} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then we can write the matrix Eq. (8) as

$$\left(\sum_{k=0}^{m} \mathbf{P}_k\mathbf{C}\mathbf{M}_k\right)\mathbf{A} = \mathbf{F} + \lambda\mathbf{I} \tag{11}$$

Let us now find the matrix **I**. The Kernel $K(x, t)$ is expanded to do truncated Taylor series (in the $x = c$ and $t = c$) in the form

$$K(x, t) = \sum_{n=0}^{N}\sum_{m=0}^{N} k_{nm}(x - c)^n(t - c)^m$$

$$k_{nm} = \frac{1}{n!m!}\frac{\partial^{n+m}}{\partial x^n \partial t^m}\bigg|_{(x=c,t=c)}$$

which is given in [2]. Then the matrix representation of $K(x, t)$ can be given by

$$[K(x,t)] = \mathbf{XKT}^t \tag{12}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & x-c & (x-c)^2 & \cdots & (x-c)^N \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & t-c & (t-c)^2 & \cdots & (t-c)^N \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}$$

Besides, the matrix representation of $y(x)$ and $y(t)$ are

$$[y(x)] = \mathbf{XM}_0\mathbf{A}, \quad [y(t)] = \mathbf{TM}_0\mathbf{A} \tag{13}$$

Substituting the expressions (12) and (13) into the integral $I(x_i)$ defined in Eq. (7), we have

$$[I(x)] = \int_a^b \left\{\mathbf{XKT}^t\mathbf{TM}_0\mathbf{A}\right\}\mathrm{d}t = \mathbf{XKHM}_0\mathbf{A} \tag{14}$$

$$\mathbf{H} = [\mathbf{h}_{nm}] = \int_a^b \mathbf{T}^t\mathbf{T}\,\mathrm{d}t,\ \mathbf{h}_{nm} = \left.\frac{(\mathbf{b}-\mathbf{c})^{n+m+1} - (\mathbf{a}-\mathbf{c})^{n+m+1}}{n+m+1}\right|_{n,m=0,1,\ldots,N}$$

which is given in [2].

From (14) we get the matrix $\mathbf{I}$ in the form

$$\mathbf{I} = \mathbf{CKHM}_0\mathbf{A} \tag{15}$$

Finally, substituting (15) in the expression (11), we have the matrix equation

$$\left(\sum_{k=0}^{m} \mathbf{P}_k\mathbf{CM}_k - \lambda\mathbf{CKHM}_0\right)\mathbf{A} = \mathbf{F} \tag{16}$$

which is the fundamental relation for solving of Fredholm integro-differential equation defined in the range $a \le x \le b$.

Briefly, we can also write the Eq. (16) in the form

$$\mathbf{WA} = \mathbf{F} \tag{17}$$

which corresponds to a system of $(N+1)$ algebraic equations with the unknown Taylor coefficients so that

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{C} \mathbf{M}_k - \lambda \mathbf{CKHM}_0, \quad i,j = 0,1,\ldots,N$$

Then we can find the unknown coefficients by means of the augmented matrix of Eq. (17)

$$[\mathbf{W};\mathbf{F}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N};\ f(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N};\ f(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN};\ f(x_N) \end{bmatrix} \tag{18}$$

In Eq. (17) if $|\mathbf{W}| \neq 0$ we get

$$\mathbf{A} = \mathbf{W}^{-1}\mathbf{F} \tag{19}$$

Thus, the unknown coefficients are uniquely determined by Eq. (19) and thereby we find any particular solution of Fredholm integro-differential equation in the truncated Taylor series.

Note that, if we take $P_0(x) = 1, P_1(x) = P_2(x) = \cdots = P_N(x) = 0$, Eq. (4) is reduce to Fredholm integral equation and the augmented matrix (18) can be used for the approximate solution of Fredholm integral equations.

Now let us form the matrix representation of the conditions. For the interval $a \leq x \leq b$, the condition (3) reduces to

$$\sum_{j=0}^{m-1} \left[a_{ij} y^{(j)}(a) + b_{ij} y^{(j)}(b) + c_{ij} y^{(j)}(c) \right] = \lambda_i; \qquad i = 0,1,\ldots,m-1, \quad a \leq c \leq b \tag{20}$$

Using the relation (9), we find the matrix representations of the functions at the points a, b and c in the forms

$$[y^{(j)}(a)] = \mathbf{PM}_j\mathbf{A} \tag{21}$$

$$[y^{(j)}(b)] = \mathbf{QM}_j\mathbf{A} \tag{22}$$

$$[y^{(j)}(c)] = \mathbf{RM}_j\mathbf{A} \tag{23}$$

where

$$\begin{aligned} \mathbf{P} &= \begin{bmatrix} 1 & (a-c) & (a-c)^2 & \cdots & (a-c)^N \end{bmatrix} \\ \mathbf{Q} &= \begin{bmatrix} 1 & (b-c) & (b-c)^2 & \cdots & (b-c)^N \end{bmatrix} \\ \mathbf{R} &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

Substituting the matrix representations (21), (22) and (23) into the Eq. (20), we obtain

$$\sum_{j=0}^{m-1}\{a_{ij}\mathbf{P}+b_{ij}\mathbf{Q}+c_{ij}\mathbf{R}\}\mathbf{M}_j\mathbf{A}=[\lambda_i]$$

Let us define $\mathbf{U}_i$ as

$$\mathbf{U}_i=\sum_{j=0}^{m-1}\{a_{ij}\mathbf{P}+b_{ij}\mathbf{Q}+c_{ij}\mathbf{R}\}\mathbf{M}_j\equiv[\,u_{i0}\quad u_{i1}\quad\cdots\quad u_{iN}\,],$$
$$i=0,1,\ldots,m-1$$

Thus, the matrix form conditions (2) become

$$\mathbf{U}_i\mathbf{A}=[\lambda_i] \tag{24}$$

and the augmented matrices of them are

$$[\mathbf{U}_i;\,\lambda_i]=[\,u_{i0}\quad u_{i1}\quad\cdots\quad u_{iN};\;\lambda_i\,] \tag{25}$$

Consequently, replacing the m row matrices (25) by the last m rows of the augmented matrix (18), we have the required augmented matrix

$$[\widetilde{\mathbf{W}};\,\widetilde{\mathbf{F}}]$$

where

$$\widetilde{\mathbf{W}}=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}\\ w_{10} & w_{11} & \cdots & w_{1N}\\ \vdots & \vdots & \cdots & \vdots\\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N}\\ u_{00} & u_{01} & \cdots & u_{0N}\\ \vdots & \vdots & \cdots & \vdots\\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N}\end{bmatrix},\quad \widetilde{\mathbf{F}}=\begin{bmatrix} f(x_0)\\ f(x_1)\\ \vdots\\ f(x_{N-m})\\ \lambda_0\\ \vdots\\ \lambda_{m-1}\end{bmatrix}$$

If, $|\widetilde{\mathbf{W}}|\neq 0$ we can write

$$\mathbf{A}=\widetilde{\mathbf{W}}^{-1}\widetilde{\mathbf{F}}$$

and thus the matrix $\mathbf{A}$ is uniquely determined. Then we can say that the integro-differential Eq. (4) with conditions (20) has a unique solution in the form (5).

3. THE METHOD OF SOLUTION FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Let us consider the Volterra integro-differential equation in the range $a \leq x \leq b$. Then Eq. (2) becomes

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = f(x) + \lambda \int_a^x K(x, t) y(t) \, \mathrm{d}t \tag{26}$$

Let us now approximate to solution $y(x)$ by means of a finite Taylor series. We apply the same method presented before to Volterra integro-differential equation for finding the solution in the form (5).

In order to determine the $(N + 1)$ coefficients, firstly, we replace Eq. (26) by the $(N + 1)$ equations

$$\sum_{k=0}^{m} P_k(x_i) y^{(k)}(x_i) = f(x_i) + \lambda \int_a^{x_i} K(x_i, t) y(t) \, \mathrm{d}t \tag{27}$$

for the $(N + 1)$ points x_i, defined by Eq. (6). Then defining

$$I(x_i) = \lambda \int_a^{x_i} K(x_i, t) y(t) \, \mathrm{d}t \tag{28}$$

we can write (27) in the matrix form

$$\sum_{k=0}^{m} \mathbf{P}_k \mathbf{Y}^{(k)} = \mathbf{F} + \lambda \mathbf{I} \tag{29}$$

where $\mathbf{P}_k$ are matrices of order $(N + 1)$, and $\mathbf{Y}^{(k)}$, $\mathbf{F}$ and $\mathbf{I}$ are $(N + 1)$-by-1 matrices defined in previous section.

Substituting matrix Eqs. (12) and (13) into (28) yields the matrix equation

$$[I(x_i)] = \mathbf{X}_{x_i} \mathbf{K} \mathbf{H}_{x_i} \mathbf{M}_0 \mathbf{A} \tag{30}$$

where

$$\mathbf{X}_{x_i} = \begin{bmatrix} 1 & x_i - c & (x_i - c)^2 & \cdots & (x_i - c)^N \end{bmatrix}$$

and

$$\mathbf{H}_{x_i} = \int_a^{x_i} \mathbf{T}^t\mathbf{T}\,\mathrm{d}t$$

$$\mathbf{H}_{x_i} = \begin{bmatrix} h_{00}(x_i) & h_{01}(x_i) & \cdots & h_{0N}(x_i) \\ h_{10}(x_i) & h_{11}(x_i) & \cdots & h_{1N}(x_i) \\ \vdots & \vdots & \vdots & \vdots \\ h_{N0}(x_i) & h_{N1}(x_i) & \cdots & h_{NN}(x_i) \end{bmatrix}$$

hence, we obtain the matrix **I** as

$$\mathbf{I} = \overline{\mathbf{X}}\,\overline{\mathbf{K}}\,\overline{\mathbf{H}}\,\overline{\mathbf{M}}_0\mathbf{A} \tag{31}$$

where $\overline{\mathbf{X}}$, $\overline{\mathbf{K}}$, $\overline{\mathbf{H}}$ and $\overline{\mathbf{M}}_0$, respectively, $(N+1)$-by-$(N+1)^2$, $(N+1)^2$-by-$(N+1)^2$, $(N+1)^2$ -by-$(N+1)^2$ and $(N+1)^2$-by-$(N+1)$ matrices and can be written by the blocked matrices as follows:

$$\overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}_{x_0} & 0 & \cdots & 0 \\ 0 & \mathbf{X}_{x_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}_{x_N} \end{bmatrix}, \quad \overline{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & 0 & \cdots & 0 \\ 0 & \mathbf{K} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{K} \end{bmatrix}$$

$$\overline{\mathbf{H}} = \begin{bmatrix} \mathbf{H}_{x_0} & 0 & \cdots & 0 \\ 0 & \mathbf{H}_{x_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{x_N} \end{bmatrix}, \quad \overline{\mathbf{M}}_0 = \begin{bmatrix} \mathbf{M}_0 \\ \mathbf{M}_0 \\ \vdots \\ \mathbf{M}_0 \end{bmatrix}$$

where $\mathbf{X}_{x_i}$ are 1-by-$(N+1)$ and $\mathbf{H}_{x_i}$, $\mathbf{K}$ and $\mathbf{M}_0$ are square matrices with the order $(N+1)$. Inserting the Eqs. (10) and (31) into (29), we get

$$\left(\sum_{k=0}^{m} \mathbf{P}_k\mathbf{C}\mathbf{M}_k - \lambda\overline{\mathbf{X}}\,\overline{\mathbf{K}}\,\overline{\mathbf{H}}\,\overline{\mathbf{M}}_0\right)\mathbf{A} = \mathbf{F} \tag{32}$$

This is a system of $(N+1)$ equations for the coefficients and also the fundamental matrix for Volterra integro-differential equation. Therefore, we can write Eq. (32) in the form

$$\mathbf{WA} = \mathbf{F}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k\mathbf{C}\mathbf{M}_k - \lambda\overline{\mathbf{X}}\,\overline{\mathbf{K}}\,\overline{\mathbf{H}}\,\overline{\mathbf{M}}_0$$

Note that if $\mathbf{P}_k = 0,\ k = 1, 2, \ldots, N$ and $|\mathbf{W}| \neq 0$, the Volterra integral equation has one and only solution; if $|\mathbf{W}| = 0$ then the integral equation either is insoluble or has an infinite number of solution.

On the other hand, in order to solve integro-differential or differential equation with conditions, we find the matrix forms of the conditions (20) as given in (24). Then we obtain a new matrix equation by writing values that belong to the conditions with replacing the rows as number of conditions that are erased from the last matrix equation. Hence, the Taylor coefficients can be simply computed and the solution of Eq. (26) under the mixed conditions is obtained.

4. ACCURACY OF SOLUTION

We can easily check the accuracy of the solutions obtained in the form (5) as follows.

The solution (5) or the corresponding polynomial expansion must satisfy approximately the Eq. (4) or Eq. (26) for Volterra integro-differential equation when $y(x)$ and its derivatives $y^{(k)}(x)$ are substituted in this equation since the finite Taylor series (5) is an approximate solution of Fredholm integro-differential equation. That is, for any points $x = x_i$, $a \leq x_i \leq b,\ i = 0, 1, \ldots, N$.

$$D(x_i) = \sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) - f(x_i) - \lambda I(x_i) \cong 0$$

or

$$|D(x_i)| \cong 10^{-k_i}$$

where k_i are positive integers.

If max $10^{-k_i} = 10^{-k}$ (k any positive integer) is prescribed, then the truncation limit N is increased until the difference $|D(x_i)|$ becomes smaller than the prescribed 10^{-k} at each of the points x_i. Thus, we can get better the solution (5) by choosing k appropriately so that 10^{-k} is very close to zero.

5. ILLUSTRATIONS

We now give some examples to illustrate the use of the method.

Example 1 Let us first consider the linear Fredholm integro-differential equation

$$y'' + xy' - xy = \mathrm{e}^x - 2\sin x + \int_{-1}^{1} \sin x\ \mathrm{e}^{-t} y(t)\,\mathrm{d}t$$

with $y(0) = 1$ and $y'(0) = 1,\ -1 \leq x, t \leq 1$ and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{y^{(n)}(0)}{n!} x^n$$

where $a = -1,\ b = 1,\ c = 0,\ \lambda = 1,\ P_0 = 1,\ P_1 = x,\ P_2 = -x,\ f(x) = \exp x - 2\sin x$, $K(x, t) = \sin x \exp(-t)$.

Then, for $N = 5$, the matrix Eq. (16)

$$(\mathbf{P}_2\mathbf{C}\mathbf{M}_2 + \mathbf{P}_1\mathbf{C}\mathbf{M}_1 - \mathbf{P}_0\mathbf{C}\mathbf{M}_0 - \mathbf{C}\mathbf{K}\mathbf{H}\mathbf{M}_0)\mathbf{A} = \mathbf{F}$$

where $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{H}, \mathbf{C}, \mathbf{K}, \mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2$ are matrices of order (6×6) defined by

$$\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \mathbf{P}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{7} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{9} \\ \frac{2}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{9} & 0 \\ 0 & \frac{2}{7} & 0 & \frac{2}{9} & 0 & \frac{2}{11} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{3}{5} & \left(-\frac{3}{5}\right)^2 & \left(-\frac{3}{5}\right)^3 & \left(-\frac{3}{5}\right)^4 & \left(-\frac{3}{5}\right)^5 \\ 1 & -\frac{1}{5} & \left(-\frac{1}{5}\right)^2 & \left(-\frac{1}{5}\right)^3 & \left(-\frac{1}{5}\right)^4 & \left(-\frac{1}{5}\right)^5 \\ 1 & \frac{1}{5} & \left(\frac{1}{5}\right)^2 & \left(\frac{1}{5}\right)^3 & \left(\frac{1}{5}\right)^4 & \left(\frac{1}{5}\right)^5 \\ 1 & \frac{3}{5} & \left(\frac{3}{5}\right)^2 & \left(\frac{3}{5}\right)^3 & \left(\frac{3}{5}\right)^4 & \left(\frac{3}{5}\right)^5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & \frac{1}{2!} & -\frac{1}{3!} & \frac{1}{4!} & -\frac{1}{5!} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{3!} & \frac{1}{3!} & -\frac{1}{3!2!} & \frac{1}{3!3!} & -\frac{1}{3!4!} & \frac{1}{3!5!} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5!} & -\frac{1}{5!} & \frac{1}{5!2!} & -\frac{1}{5!3!} & \frac{1}{5!4!} & -\frac{1}{5!5!} \end{bmatrix}$$

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2!} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3!} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4!} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5!} \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 0 & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1!} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2!} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3!} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4!} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 & \frac{1}{0!} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1!} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2!} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3!} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix forms of the conditions for $N = 5$ are

$$[1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0; \quad 1]$$
$$[0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0; \quad 1]$$

Taking $N = 5$, we obtain the approximate solution. The solution is

$$y(x) = 1 + x + 0.500343x^2 + 0.166886x^3 + 0.0403378x^4 + 0.00577493x^5$$

Taking $N = 5$, the solutions obtained are compared with the results given by Akyüz and Sezer [7] and by Nas, Yalzinbas and Sezer. [4] and the exact solution $y = (x)$ in Table I.

Example 2 Let us consider the boundary-value problem

$$(1 + 2x)y'''(x) + 4xy''(x) + (2x - 1)y'(x) = e^{-x}, \quad 0 \leq x \leq 1$$

with $y(0) = 1,\ y'(0) = 1/2$ and $y''(0) = -1$

TABLE I Numerical Results for $N=5,6,10$

r	x_r	*Present method* $y(x_r)$ $N=5$	$N=6$	$N=10$	*Exact solution* $y=\exp(x_r)$	*Chebyshev collocation method* ($N=10$)	*Taylor matrix method* ($N=5$)
0	1	2.71334	2.71766	2.71828	2.718282	2.718282	2.71653
1	$\cos(\pi/10)$	2.58467	2.58800	2.58844	2.588443	2.588443	2.58714
2	$\cos(\pi/5)$	2.24414	2.24556	2.24569	2.245699	2.245699	2.24518
3	$\cos(3\pi/10)$	1.79975	1.79999	1.8	1.799997	1.799997	1.79990
4	$\cos(2\pi/5)$	1.36210	1.36208	1.36208	1.362085	1.362085	1.36207
5	$\cos(\pi/2)$	1	1	1	1	0.999999	1
6	$\cos(3\pi/5)$	0.734188	0.734166	0.734168	0.7341683	0.7341683	0.734171
7	$\cos(7\pi/10)$	0.555598	0.55553	0.555555	0.5555564	0.5555565	0.555556
8	$\cos(4\pi/5)$	0.445373	0.445291	0.445295	0.4452955	0.4452958	0.445021
9	$\cos(9\pi/10)$	0.386454	0.386323	0.386332	0.3863326	0.386333	0.385544
10	$\cos(\pi)$	0.368019	0.367867	0.367879	0.3678795	0.3678799	0.366796

TABLE II

x_i	*Present method* $N=5$	*Exact solution* $y=[x/2\exp(-x)]+1$
0	1	1
0.2	1.08188	1.081873
0.4	1.13416	1.134064
0.6	1.16469	1.164643
0.8	1.17859	1.179731
1	1.17820	1.183939

and approximate the solution $y(x)$ by the truncated Taylor series in the form

$$y(x)=\sum_{n=0}^{4}\frac{y^{(n)}(0)}{n!}x^n$$

so that $a=-1,\ b=1,\ c=0,\ \lambda=1,\ P_0=0,\ P_1=2x-1,\ P_2=4x,\ P_3=1+2x$ and $f(x)=\exp(-x)$.

For $N=4$, the collocation points

$$x_0=0,\quad x_1=\frac{1}{4},\quad x_2=\frac{1}{2},\quad x_3=\frac{3}{4},\quad x_4=1$$

and the matrix form of the problems defined by

$$(\mathbf{P}_3\mathbf{C}\mathbf{M}_3+\mathbf{P}_2\mathbf{C}\mathbf{M}_2+\mathbf{P}_1\mathbf{C}\mathbf{M}_1)\mathbf{A}=\mathbf{F}$$

After the augmented matrşces of the system and conditions are computed, we obtain the new augmented matrix in the form

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]=\begin{bmatrix}0 & -1 & 0 & 1 & 0;\ \exp(0)\\ 2 & -\frac{1}{2} & \frac{7}{8} & \frac{111}{64} & \frac{311}{768};\ \exp(-\frac{1}{4})\\ 1 & 0 & 0 & 0 & 0;\ 1\\ 0 & 1 & 0 & 0 & 0;\ \frac{1}{2}\\ 0 & 0 & 1 & 0 & 0;\ -1\end{bmatrix}$$

This system has the solution

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 & -1 & 1.5 & -1.72308 \end{bmatrix}^t$$

Therefore, we find the solution

$$y(x) = 1 + 0.5x - 0.5x^2 + 0.25x^3 + 0.0717954x^4$$

The values of this solution at taking $i = 0.2$ decimal in [0, 1] points are compare with the exact solution in Table II.

Example 3 Let us consider the linear Volterra integro-differential equation

$$y'(x) + y(x) = 1 + 2x + \int_0^x x(1+2x)e^{t(x-t)}y(t)\,dt, \quad 0 \le x \le 1$$

with $y(0) = 1$ and $0 \le x,\ t \le 1$. The analytical solutions $y(x) = e^{x^2}$.

For $N = 3$, the collocation points

$$x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1$$

and the matrix form of the problem is defined by

$$(\mathbf{P}_1\mathbf{C}\mathbf{M}_1 + \mathbf{P}_0\mathbf{C}\mathbf{M}_0 - \overline{\mathbf{X}}\,\overline{\mathbf{K}}\,\overline{\mathbf{H}}\,\overline{\mathbf{M}}_0)\mathbf{A} = \mathbf{F}$$

After the augmented matrices of the system and conditions are computed, we obtain this solution

$$y(x) = 1 + 0.64754\,x^2 + 0.966173\,x^3$$

The values of this solution at taking $i = 0.2$ decimal in [0, 1] points are compare with the exact solution [8] in Table III.

Example 4 Our last example is the linear Fredholm integro-differential equation

$$xy'(x) + y(x) = 3x^2 + \frac{14}{3}x + 2 - \int_{-1}^{1}(x+t)y(t)\,dt$$

with $y(0) = 2$ and approximate the solution $y(x)$ by the truncated Taylor series ($N = 4$).

TABLE III

x_i	*Present method* $N=3$	*Exact solution* $y=exp(x^2)$
0	1	1
0.2	1.03361	1.04081
0.4	1.16536	1.17351
0.6	1.44163	1.43332
0.8	1.90879	1.89648
1	2.61322	2.7182

The system has the solution

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 & 0 & 0 \end{bmatrix}^t$$

Therefore, we find the exact solution

$$y(x) = x^2 + 2$$

6. CONCLUSIONS

High order integro-differential equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed.

The method presented in this study is a method for computing the coefficients in the Taylor expansion of the solution of a linear integro-differential equation, and is valid when the functions $P_k(x)$ are defined in $[a, b]$ and the kernel function $K(x, t)$ has a Taylor series expansion in this range. Moreover, it would appear that the method shows the best advantage when the functions $K(x, t)$, $f(x)$ and $P_k(x)$ can be expanded to the Taylor series which converges rapidly.

To obtain the best approximating solution of the equation, we take more terms from the Taylor expansion of functions; that is, the truncation limit N must be chosen to be large enough. For computational efficiency, some estimate for N, the degree of the approximating polynomial (the truncation limit of the Taylor series) to $y(x)$, should be available.

The Taylor collocation method can also be applied to the differential and integral equations. This is demonstrated by the examples in the last section. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a polynomial of degree N or less than N.

The method can also be extended to the partial integro-differential equations and to the system of ordinary differential equations with variable coefficients.

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