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Normal Subgroups of Hecke Groups $H(\lambda)$

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Abstract Let $\lambda \geq 2$ and let $H(\lambda)$ be the Hecke group associated to λ. Also let $H(\lambda)\backslash U$ be the Riemann surface associated to the Hecke group $H(\lambda)$. In this article, we study the even subgroup $H_e(\lambda)$ and the power subgroups $H^m(\lambda)$ of the Hecke groups $H(\lambda)$. We also study some genus 0 normal subgroups of finite index of $H(\lambda)$. Finally, we discuss free normal subgroups of $H(\lambda)$.

Keywords Hecke group · Even subgroup · Power subgroup · Free normal subgroup

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1 Introduction

The Hecke groups $H(\lambda)$ are the set of linear fractional transformations generated by

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad S(z) = -\frac{1}{z+\lambda},$$

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which take the upper half of the complex plane onto itself. Also we can represent the generators of the Hecke groups $H(\lambda)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix},$$

(throughout this paper, we identify each matrix A with $-A$, so that they each represent the same element of $H(\lambda)$.)

In [7], Hecke proved that H(λ) is discrete only when $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, $q \geq 3$, $q \in \mathbb{N}$, or $\lambda \geq 2$. These groups have come to be known as the *Hecke groups* and we will denote them by $H(\lambda_q)$, $q \geq 3$ or by $H(\lambda)$, $\lambda \geq 2$. The first few Hecke groups are $H(\lambda_3) = PSL(2,\mathbb{Z}) = \Gamma$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H\left(\frac{1+\sqrt{5}}{2}\right)$, and $H(\lambda_6) = H(\sqrt{3})$ for $q = 3, 4, 5$ and 6, respectively.

The Hecke groups $H(\lambda_q)$ and their normal subgroups have been extensively studied from many points of view in the literature, (see, [1, 4, 5] and [16]). The Hecke group $H(\lambda_3)$, the modular group $PSL(2,\mathbb{Z})$, and its normal subgroups have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory and group theory, (see, [9, 10] and [11]).

Here we are interested in the case $\lambda \geq 2$. In this case, the element S is parabolic when $\lambda = 2$, or hyperbolic (boundary) when $\lambda > 2$. When $\lambda > 2$, $H(\lambda)$ is of the second kind; that is, it is a group of infinite volume (see [12]). It is known that when $\lambda \geq 2$, $H(\lambda)$ is a free product of a cyclic group of order 2 and infinity, [13], so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) = < T,\ S \mid T^2 = I > \cong C_2 * \mathbb{Z}. \tag{1.1}$$

Also, the signature of $H(\lambda)\backslash U$ is $(0; 2, \infty; 1)$, i.e., a sphere with one puncture, one elliptic fixed point of order 2, and one hole when $\lambda > 2$, or $(0; 2, \infty, \infty) \cong (0; 2, \infty^{(2)})$, a sphere with two punctures and one elliptic fixed point of order 2 when $\lambda = 2$. Therefore all Hecke groups $H(\lambda)$, $\lambda \geq 2$, can be considered as a triangle group.

As can be understood from its title, this paper is concerned with normal subgroups of Hecke groups $H(\lambda)$, $\lambda \geq 2$. The Reidemeister–Schreier method, the permutation method and the Riemann-Hurwitz formula are used to obtain the abstract group structure, generators and signatures of these normal subgroups. We begin with the even subgroup $H_e(\lambda)$, since $H_e(\lambda)$ is very important amongst the normal subgroups of $H(\lambda)$ and it contains infinitely many normal subgroups of $H(\lambda)$. Also we study the power subgroups $H^m(\lambda)$. Then we find some genus 0 normal subgroups of the finite index of $H(\lambda)$ and finally, we discuss the free normal subgroups of $H(\lambda)$.

2 The Even Subgroup of $H(\lambda)$

We now study the structure of an important normal subgroup of $H(\lambda)$ namely the even subgroup. First we look at the elements of $H(\lambda)$. We need the following definition.

Definition 2.1 Let a, b, c and d be all polynomials of λ^2 with rational integer coefficients. Then

(i) the elements of type $\begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix}$ where $ad - bc\lambda^2 = 1$, are called even elements,

(ii) the elements of type $\begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix}$ where $ad\lambda^2 - bc = 1$, are called odd elements.

Note that if we consider the multiplication of these elements, the situation is similar to the multiplication of negative and positive numbers. Here, we have

$$odd.odd = even.even = even,$$

$$even.odd = odd.even = odd.$$

Theorem 2.1 *The Hecke groups $H(\lambda)$ consist exactly of odd and even elements.*

Proof Since the generators of $H(\lambda)$ have form $T = \begin{pmatrix} 0.\lambda & -1 \\ 1 & 0.\lambda \end{pmatrix}$ and $S = \begin{pmatrix} 0.\lambda & -1 \\ 1 & 1.\lambda \end{pmatrix}$, they are odd elements. If we take an element V of $H(\lambda)$, we can write this element as a cyclically reduced word of the form $V = TS^{\varepsilon_1}TS^{\varepsilon_2}...TS^{\varepsilon_n}$ where $1 \leq \varepsilon_i$. Thus if the sum of the T's and S's in the element V is even, then V is even, otherwise V is odd. □

Definition 2.2 The even elements form a subgroup of $H(\lambda)$ of index 2 called the *even subgroup*, denoted by $H_e(\lambda)$:

$$H_e(\lambda) = \left\{ M = \begin{pmatrix} a & b\lambda \\ c\lambda & d \end{pmatrix} : M \in H(\lambda) \right\}.$$

The set of odd elements

$$H_o(\lambda) = \left\{ N = \begin{pmatrix} a\lambda & b \\ c & d\lambda \end{pmatrix} : N \in H(\lambda) \right\},$$

forms the other coset of $H_e(\lambda)$ in $H(\lambda)$.

Theorem 2.2 *The even subgroup $H_e(\lambda)$ of $H(\lambda)$ is a normal subgroup of index two of $H(\lambda)$. Also*

$$H(\lambda) = H_e(\lambda) \cup T.H_e(\lambda),$$

$$H_e(\lambda) \cong \langle ST \rangle * \langle TS \rangle, \tag{2.1}$$

and therefore $H_e(\lambda)$ is isomorphic to the free product of two infinite cyclic groups.

Proof First, we consider the case $\lambda > 2$. Having index two $H_e(\lambda)$ is a normal subgroup of $H(\lambda)$. Let us now choose $\{I, T\}$ as a Schreier transversal for the even subgroup. According to the Reidemeister–Schreier method, we can form all possible products:

$$I.T.(T)^{-1} = I,$$

$$T.T.(I)^{-1} = I,$$

$$I.S.(T)^{-1} = ST,$$

$$T.S.(I)^{-1} = TS.$$

The generators of $H_e(\lambda)$ are ST and TS. It is easily seen that both generators are parabolic elements of infinite order. We have

$$H_e(\lambda) \cong \langle ST \rangle * \langle TS \rangle \cong \mathbb{Z} * \mathbb{Z}.$$

Finally as $T \notin H_e(\lambda)$, (2.1) follows.

Now we consider the homomorphism

$$H(\lambda) \to H(\lambda)/H_e(\lambda) \cong C_2.$$

Here T is mapped to an element of order two and S is mapped to an element of order two. Hence TS is mapped to the identity. Then the permutation method and the Riemann-Hurwitz formula together give the signature of $H_e(\lambda)\backslash U$ as $(0; \infty^{(2)}; 1)$, i.e., a sphere with two punctures, and one hole.

If we consider the case $\lambda = 2$ then similarly to the case $\lambda > 2$, we obtain the signature of $H_e(2)\backslash U$ as $(0; \infty^{(3)})$, i.e., a sphere with three punctures. □

3 Power Subgroups of $H(\lambda)$

Let m be a positive integer. Let us define $H^m(\lambda)$ to be the subgroup generated by the m^{th} powers of all elements of $H(\lambda)$. The subgroup $H^m(\lambda)$ is called the $m^{\text{th}} - $ *power subgroup* of $H(\lambda)$. As fully invariant subgroups, they are normal in $H(\lambda_q)$.

From the definition one can easily deduce that

$$H^m(\lambda) > H^{mk}(\lambda)$$

and that

$$(H^m(\lambda))^k > H^{mk}(\lambda).$$

The power subgroups of the Hecke groups $H(\sqrt{n})$, n square-free integer, was studied in [15]. Here we show that this nicely generalizes to Hecke groups $H(\lambda)$ with $\lambda \geq 2$.

We now discuss the group theoretical structure of these subgroups. Let us consider the presentation of the Hecke group $H(\lambda)$ given in (1.1):

$$H(\lambda) =< T, S \mid T^2 = I > .$$

We find a presentation for the quotient $H(\lambda)/H^m(\lambda)$ by adding the relation $X^m = I$ to the presentation of $H(\lambda)$. The order of $H(\lambda)/H^m(\lambda)$ gives us the index. We have

$$H(\lambda)/H^m(\lambda) \cong< T, S \mid T^2 = T^m = S^m = (TS)^m = ... = I > . \tag{3.1}$$

Thus we use the Reidemeister-Schreier process to find the presentation of the power subgroups $H^m(\lambda)$. First we have

Theorem 3.1 *The normal subgroup $H^2(\lambda)$ is the free product of three infinite cyclic groups. Also*

$$H(\lambda)/H^2(\lambda) = C_2 \times C_2,$$

$$H^2(\lambda) =< S^2 > * < TS^2T > * < TSTS^{-1} >,$$

and

$$H(\lambda) = H^2(\lambda) \cup TH^2(\lambda) \cup SH^2(\lambda) \cup TSH^2(\lambda).$$

The elements of $H^2(\lambda)$ can be characterized by the requirement that are the sums of the exponents of T and S are both even.

Proof If $\lambda > 2$, then by (3.1), we get

$$H(\lambda)/H^2(\lambda) \cong < T, S \mid T^2 = S^\infty = (TS)^\infty = T^2 = S^2 = (TS)^2 = ... = I > .$$

Since $S^\infty = S^2 = I$ and $(TS)^\infty = (TS)^2 = I$, we obtain

$$H(\lambda)/H^2(\lambda) \cong < T, S \mid T^2 = S^2 = (TS)^2 = I >$$

and therefore we have

$$H(\lambda)/H^2(\lambda) \cong C_2 \times C_2 \cong D_2,$$

$$\left|H(\lambda) : H^2(\lambda)\right| = 4.$$

Now we choose $\{I, T, S, TS\}$ as a Schreier transversal. Hence, all possible products are

$$I.T.(T)^{-1} = I, \qquad I.S.(S)^{-1} = I,$$

$$T.T.(I)^{-1} = I, \qquad T.S.(TS)^{-1} = I,$$

$$S.T.(TS)^{-1} = STS^{-1}T, \quad S.S.(I)^{-1} = S^2,$$

$$TS.T.(S)^{-1} = TSTS^{-1}, \quad TS.S.(T)^{-1} = TS^2T.$$

Since $(STS^{-1}T)^{-1} = TSTS^{-1}$, the generators of $H^2(\lambda)$ are S^2, TS^2T, $TSTS^{-1}$. Thus $H^2(\lambda)$ has a presentation

$$H^2(\lambda) =< S^2 > * < TS^2T > * < TSTS^{-1} >$$

and we get

$$H(\lambda) = H^2(\lambda) \cup TH^2(\lambda) \cup SH^2(\lambda) \cup TSH^2(\lambda).$$

Now we consider the homomorphism

$$H(\lambda) \to H(\lambda)/H^2(\lambda) \cong D_2.$$

Here T is mapped to an element of order two, S is mapped to an element of order two, and TS is mapped to an element of order two. Then the permutation method and the Riemann-Hurwitz formula together give the signature of $H^2(\lambda)\backslash U$ as $(0; \infty^{(2)}; 2)$, i.e., a sphere with two punctures and two holes.

If $\lambda = 2$, then the signature of $H^2(2)\backslash U$ is $(0; \infty^{(4)})$, i.e., a sphere with four punctures. □

Now we can write the following result from the Theorem 2.2 and the Theorem 3.1.

Corollary 3.2 *The group $H^2(\lambda)$ is a subgroup of index two of the even subgroup $H_e(\lambda)$.*

Theorem 3.3 *Let m be a positive odd integer. The normal subgroup $H^m(\lambda)$ is isomorphic to the free product of m cyclic groups of order two and an infinite cyclic group. Also*

$$H(\lambda)/H^m(\lambda) \cong C_m,$$

$$H^m(\lambda) =< S^m > * < T > * < STS^{-1} > * < S^2TS^{-2} > \\ * \ldots * < S^{m-1}TS^{-(m-1)} >,$$

and

$$H(\lambda) = H^m(\lambda) \cup SH^m(\lambda) \cup S^2H^m(\lambda) \cup \ldots \cup S^{m-1}H^m(\lambda).$$

Proof If $\lambda > 2$, then by (3.1), we have

$$H(\lambda)/H^m(\lambda) \cong < T, S \mid T^2 = S^\infty = (TS)^\infty = I, \\ T^m = S^m = (TS)^m = ... = I > .$$

Therefore we find $S^m = T = I$ from the relations

$$T^2 = T^m = I, \; S^\infty = S^m = I,$$

and as $(m, 2) = 1$. Thus we obtain

$$H(\lambda)/H^m(\lambda) \cong< S \mid S^m = I >$$

and therefore we get

$$H(\lambda)/H^m(\lambda) \cong C_m,$$

$$\left|H(\lambda) : H^m(\lambda)\right| = m.$$

Now we choose $\{I, S, S^2, \dots S^{m-1}\}$ as a Schreier transversal. Hence, all possible products are

$$\begin{array}{ll} I.T.(T)^{-1} = T, & I.S.(S)^{-1} = I, \\ S.T.(S)^{-1} = STS^{-1}, & S.S.(S^2)^{-1} = I, \\ S^2.T.(S^2)^{-1} = S^2TS^{-2}, & S^2.S.(S^3)^{-1} = I, \\ \vdots & \vdots \\ S^{m-1}.T.(S^{m-1})^{-1} = S^{m-1}TS^{-(m-1)}, & S^{m-1}.S.(I)^{-1} = S^m. \end{array}$$

The generators of $H^m(\lambda)$ are S^m, T, STS^{-1}, $S^2TS^{-2}, \dots, S^{m-1}TS^{-(m-1)}$. Thus $H^m(\lambda)$ has a presentation

$$H^m(\lambda) = < S^m > * < T > * < STS^{-1} > * < S^2TS^{-2} > * \dots * < S^{m-1}TS^{-(m-1)} >,$$

and we get

$$H(\lambda) = H^m(\lambda) \cup SH^m(\lambda) \cup S^2H^m(\lambda) \cup \dots \cup S^{m-1}H^m(\lambda).$$

Now we consider the homomorphism

$$H(\lambda) \to H(\lambda)/H^m(\lambda) \cong C_m.$$

Here T is mapped to the identity and S is mapped to an element of order m. Hence TS is mapped to an element of order m. Using the permutation method and the Riemann–Hurwitz formula we have the signature of $H^m(\lambda)\backslash U$ as $(0; 2^{(m)}, \infty; 1)$, i.e., a sphere with one puncture, m elliptic fixed points of order 2, and one hole.

If $\lambda = 2$, we find the signature of $H^m(2)\backslash U$ as $(0; 2^{(m)}, \infty^{(2)})$, i.e., a sphere with two punctures, m elliptic fixed points of order 2. □

Finally, if $m > 2$ is even, then there are the relations $T^2 = S^m = (TS)^m = I$ and the other relations in the quotient group $H(\lambda)/H^m(\lambda)$. Then the order of factor group is unknown. In this case the above techniques do not say much about $H^m(\lambda)$. But we can write $m = 2k$, $k \geq 2$ integer. Then because $H^2(\lambda)$ is a free group and $H^2(\lambda) \supset (H^2(\lambda))^k \supset H^{2k}(\lambda)$, we have by Schreier's Theorem the following :

Theorem 3.4 *Let $m > 2$ be a positive even integer. The groups $H^m(\lambda)$ are free groups.*

4 Normal Subgroups of $H(\lambda)$

Normal subgroups of Hecke groups $H(\lambda_q)$ has been studied by Cangül, Bizim and Singerman and classification theorems are given in [1 – 4]. Also normal subgroups of the Hecke group $H(\sqrt{5})$ have been investigated by Özgür and Cangül in [14] (note that $\lambda = \sqrt{5} > 2$). Our aim in this section is to generalise these results to Hecke groups $H(\lambda)$ for all $\lambda \geq 2$ and find some normal subgroups of them. One way of doing this is to use the regular map theory.

The study of maps is closely related to the study of subgroups of certain triangle groups. In [8], Jones and Singerman showed that there is a natural correspondence between maps and Schreier coset graphs for the subgroups of the triangle groups $(2, m, n)$. Regularity is an important property of maps. Jones and Singerman showed the existence of a $1:1$ correspondence between regular maps and normal subgroups of certain triangle groups including Hecke groups $H(\lambda_q)$ (see [8]). We can generalise this correspondence to all Hecke groups $H(\lambda)$. By means of this correspondence we can find normal subgroups of $H(\lambda)$ and prove many important results related to them if we know the corresponding regular maps.

We use this correspondence between regular maps and normal subgroups to obtain normal subgroups of Hecke groups from regular maps in the following way: Firstly, there is a homomorphism θ from $H(\lambda) \cong (2, \infty; 1)$ (or $H(\lambda) \cong (2, \infty, \infty)$) to the triangle group $(2, m, n)$. Let now $\mathcal{M}$ be a regular map of type $\{m, n\}$. By Jones and Singerman's result, associated to $\mathcal{M}$ there is a normal subgroup N of the triangle group $(2, m, n)$. If we consider the inverse image $\theta^{-1}(N)$ of N, it is a normal subgroup of $H(\lambda)$. We shall say that N is a normal subgroup of $H(\lambda)$ corresponding to the regular map $\mathcal{M}$ of type $\{m, n\}$. The number n corresponds to the level of the normal subgroup $\theta^{-1}(N)$.

Now we discuss some genus 0 normal subgroups of finite index of $H(\lambda)$. Let N be a normal subgroup of genus 0 in $H(\lambda)$. Then $H(\lambda)/N$ acts on the Riemann surface $\hat{U}/N$ where $\hat{U}$ is the upper half plane (see [6]). This gives a regular map on the sphere, so that $H(\lambda)/N$ is isomorphic to a finite subgroup of $SO(3)$, and therefore, is isomorphic to one of the finite triangle groups. These are known to be isomorphic to A_4, S_4, A_5, C_n, and D_n, for $n \in \mathbb{N}$. Now considering each of these groups as a quotient group of $H(\lambda)$, whenever possible, we can find all genus 0 normal subgroups of Hecke groups.

First we consider the case $\lambda > 2$:

Let us begin with the cyclic group $C_n \cong (1, n, n)$ of order n. By mapping T to the identity and S to the generator α of the cyclic group of order n, we obtain a homomorphism of $H(\lambda)$ to C_n. For each such n, we get a normal subgroup N of genus 0. By the permutation method, $N\backslash U$ has signature $(0; 2^{(n)}, \infty; 1)$, i.e., a sphere with one puncture, n elliptic fixed points of order 2, and one hole. We denote this class of normal subgroups of $H(\lambda)$ by $Y_n(\lambda)$. They are isomorphic to the free product $\mathbb{Z}$ with n cyclic groups of order two. These subgroups have the property that each $Y_n(\lambda)$ contains infinitely many normal subgroups $Y_n(\lambda)$ of genus 0, since we have $Y_n(\lambda) \triangleright Y_{nk}(\lambda)$, $k \in \mathbb{N}$. The corresponding regular maps are called star maps. They consist of a vertex surrounded by a number of edges.

There is another homomorphism of $H(\lambda)$ to a cyclic group C_2 of order two with signature $(2, 1, 2)$ (that is, C_2 can be thought of as finite triangle group with a presentation $< x, y \mid x^2 = y^1 = (xy)^2 = I >$). But as this quotient is a member of the class $D_n \cong (2, n, 2)$ of dihedral groups of order $2n$, it will be considered in the following paragraph:

Let us now map $H(\lambda)$ to a dihedral group $D_n \cong (2, n, 2) \cong < x, y \mid x^2 = y^n = (xy)^2 = I >$ of order $2n$, by taking T to x, and S to y. This is a homomorphism and similarly we obtain a normal subgroup denoted by $S_n(\lambda)\backslash U$ with signature $(0; \infty^{(2)}; n)$, i.e., a sphere with two punctures and n holes. It is isomorphic to the free-product of $(n+1)$ infinite cyclic groups. Corresponding regular maps are regular polygons on the sphere.

Now we map $H(\lambda)$ to $A_4 \cong (2, 3, 3) \cong < x, y \mid x^2 = y^3 = (xy)^3 = I >$. By mapping T to x and S to y we obtain a homomorphism of $H(\lambda)$ onto A_4 and this gives a normal subgroup denoted by $T_1(\lambda)\backslash U$ with signature $(0; \infty^{(4)}; 4)$, i.e., a sphere with four punctures and four holes. Corresponding regular map is a tetrahedron.

If we map $H(\lambda)$ to $S_4 \cong (2, 3, 4)$ by taking T to the generator of order 2 and S to the generator of order 3 of S_4, then we get a normal subgroup $T_2(\lambda)\backslash U$ with signature $(0; \infty^{(8)}; 6)$, i.e., a sphere with 8 punctures and 6 holes. Corresponding regular map is an octahedron. Also we can map $H(\lambda)$ to $S_4 \cong (2, 4, 3)$ by taking T to the generator of order 2 and S to the generator of order 4 of S_4, then we have a normal subgroup $T_3(\lambda)\backslash U$ with signature $(0; \infty^{(6)}; 8)$, i.e., a sphere with 6 punctures, and 8 holes. Corresponding regular map is a cube.

If we map $H(\lambda)$ to $A_5 \cong (2, 3, 5)$ such that T is taken to the generator of order 2 and S is taken to the generator of order 3, we obtain a normal subgroup $T_4(\lambda)\backslash U$ with signature $(0; \infty^{(20)}; 12)$−a sphere with 20 punctures and 12 holes. The corresponding regular map is an icosahedron. Also mapping onto $A_5 \cong (2, 5, 3)$, we obtain a normal subgroup $T_5(\lambda)\backslash U$ with signature $(0; \infty^{(12)}; 20)$, i.e., a sphere with 12 punctures and 20 holes. Corresponding regular map is a dodecahedron.

Furthermore, it is possible to map $H(\lambda)$ to a dihedral group $D_n \cong (2, 2, n) \cong < x, y \mid x^2 = y^2 = (xy)^n = I >$ for each $n \in \mathbb{N}$, by mapping T to x and S to y. Here we obtain a normal subgroup denoted by $W_n(\lambda)\backslash U$ with signature $(0; \infty^{(n)}; 2)$, i.e., a sphere with n punctures and two holes. Note that $D_n \cong (2, 2, n)$ contains $C_2 \cong (2, 2, 1)$.

Let us now consider the case $\lambda = 2$:

Similarly in the case $\lambda > 2$, we find the signatures of $Y_n(2)$, $S_n(2)$, $T_1(2)$, $T_2(2)$, $T_3(2)$, $T_4(2)$, $T_5(2)$, $W_n(2)$ as $(0; 2^{(n)}, \infty^{(2)})$, $(0; \infty^{(n+2)})$, $(0; \infty^{(8)})$, $(0; \infty^{(14)})$, $(0; \infty^{(14)})$, $(0; \infty^{(32)})$, $(0; \infty^{(32)})$, $(0; \infty^{(n+2)})$, respectively. Therefore, we get $S_n(2) \cong W_n(2)$, $T_2(2) \cong T_3(2)$, $T_4(2) \cong T_5(2)$. Also, $T_1(2)$, $T_2(2)$, $T_4(2)$ are $S_n(2)$ groups obtained for $n = 6$, 12, 30, respectively.

Hence we have the following result:

Theorem 4.1

i) *If* $\lambda > 2$ *then all normal subgroups of* $H(\lambda)$ *with genus* 0 *are isomorphic to one of the* $T_1(\lambda)$, $T_2(\lambda)$, $T_3(\lambda)$, $T_4(\lambda)$, $T_5(\lambda)$, $Y_n(\lambda)$, $S_n(\lambda)$, $W_n(\lambda)$, *for* $n \in \mathbb{N}$.
ii) *If* $\lambda = 2$ *then all normal subgroups of* $H(2)$ *with genus* 0 *are isomorphic to one of the* $Y_n(2)$, $S_n(2)$, *for* $n \in \mathbb{N}$.

As for every $n \in \mathbb{N}$, we can always find a normal subgroup of genus 0 we have the following result.

Corollary 4.1 *For* $\lambda \geq 2$, *all Hecke groups* $H(\lambda)$ *have infinitely many normal subgroups of genus* 0.

Remark 4.1 Note that for all $\lambda \geq 2$, $W_1(\lambda) \cong H_e(\lambda)$, $S_2(\lambda) \cong H^2(\lambda) \cong W_2(\lambda)$ and $Y_n(\lambda) \cong H^n(\lambda)$ for odd integer $n \geq 3$.

5 Free Normal Subgroups of $H(\lambda)$

As $H(\lambda)$ is the free product of a cyclic group of order 2 and a cyclic group of infinite order, it has, by the Kurosh subgroup theorem, or by considering signatures, two kinds of normal subgroups : Free ones and free products of some infinite cyclic groups with some cyclic groups of order two. Therefore the study of free normal subgroups and their group theoretical structures will be important to us. These have been done for Hecke group $H(\sqrt{5})$ by Özgür and Cangül in [14], for modular group $H(\lambda_3)$ by Newman in [11] and for Hecke groups $H(\lambda_q)$, q odd, by Cangül in [1]. Their results can be generalized to Hecke groups $H(\lambda)$, $\lambda \geq 2$.

Before giving the main theorem we need the following lemmas.

Lemma 5.1 *Let N be a non-trivial normal subgroup of finite index in $H(\lambda)$. Then N is free if and only if it contains no elements of finite order.*

Proof The Kurosh subgroup theorem states that a subgroup $N \neq \{I\}$ of a free product G is itself a free product,

$$N = F * \prod_{*} G_i$$

where F is either free or $\{I\}$ and each G_i is conjugate to $\{T\}$. Thus if N contains no elements of finite order, then the free product $\prod_* G_i$ is vacuous, and so $N = F$. Since N is non-trivial, it follows that N is free. Conversely, if N is free, then by definition, it contains no elements of finite order. □

Lemma 5.2 *The only normal subgroups of finite index in $H(\lambda)$ containing elements of finite order are*

$$H(\lambda) \text{ and } Y_n(\lambda), \; n \in \mathbb{N}.$$

Proof Firstly, we consider the case $\lambda > 2$. Let N be a normal subgroup of finite index in $H(\lambda)$ containing an element of finite order. Then N contains an element of order 2. Since every element of order two in $H(\lambda)$ is conjugate to T, it follows that N contains T, since N is normal. There are then two possibilities:

(i) N contains S. Then $N = H(\lambda)$.

(ii) N contains T but not S. Thus if we map $H(\lambda)$ to $H(\lambda)/N$ then S can be mapped to a product of $n-$cycles while T goes to the identity where $n \mid \mu$ and $n \in \mathbb{N}$, μ is the index of N. Therefore TS goes to a product of $n-$cycles. By the permutation method, we obtain the signature of $N\backslash U$ as $(g; 2^{(\mu)}, \infty^{(\mu/n)}; \mu/n)$. By the Riemann-Hurwitz formula we find $g = 1 - \mu/n$. If $\mu = n$ then we have $g = 0$. In this case the factor group $H(\lambda)/N$ is isomorphic to $C_n \cong (1, n, n)$ and the signature of $N\backslash U$ is $(0; 2^{(n)}, \infty; 1)$; i.e., a sphere with one puncture, n elliptic fixed points of order 2, and one hole. Since $g \geq 0$, any other case is not possible.

Similarly, if $\lambda = 2$ then we find that the signature of $N\backslash U$ is $(0; 2^{(n)}, \infty^{(2)})$; i.e., a sphere with two punctures and n elliptic fixed points of order 2. □

Remark 5.1 Note that Lemma 5.2 implies that $H(\lambda)$ has infinitely many normal subgroups containing elements of finite order.

Now we have the following result:

Corollary 5.3 *Let N be a normal subgroup of positive genus in $H(\lambda)$. Then N is torsion-free.*

The corollary does not have a converse as there are some free normal subgroups of $H(\lambda)$ with genus 0, as we have seen in Section 4, (for example the subgroups $W_n(\lambda)$ and $S_n(\lambda)$).

Now we can characterize the freeness of a normal subgroup of $H(\lambda)$ by comparing it with the following normal subgroups :

Theorem 5.4 *Let N be a non-trivial normal subgroup of $H(\lambda)$ different from $Y_n(\lambda)$, $n \in \mathbb{N}$. Then N is free.*

Proof It can be easily seen as an immediate consequence of the lemmas. □

Theorem 5.5

i) *Let $\lambda > 2$ and let N be a free normal subgroup of $H(\lambda)$ of finite index μ. Then N has the signature*

$$\left(1 + \frac{\mu}{4} - \frac{t}{2} - \frac{u}{2}; \infty^{(t)}; u\right).$$

ii) *Let $\lambda = 2$ and let N be a free normal subgroup of $H(2)$ of finite index μ. Then N has the signature*

$$\left(1 + \frac{\mu}{4} - \frac{t}{2}; \infty^{(t)}\right).$$

Proof

i) As N is free, it has the signature $(g; \infty^{(t)}; u)$. By the Riemann-Hurwitz formula

$$2g - 2 + t + u = \mu. \left(-2 + 1 - \frac{1}{2} + 1 + 1\right)$$

and therefore

$$g = 1 + \frac{\mu}{4} - \frac{t}{2} - \frac{u}{2}.$$

ii) Similarly to the case $\lambda > 2$, we obtain the signature of N as $\left(1 + \frac{\mu}{4} - \frac{t}{2}; \infty^{(t)}\right)$. □

Notice that in Theorem 5.5 (*i*) and (*ii*), since $g \geq 0$, the possible values of u and t are $2(t + u) \leq 4 + \mu$ and $2t \leq 4 + \mu$, respectively.

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