On Approximation of Hexagonal Fourier Series

A. Guven

Abstract. Let the function f belong to the Hölder class $H^\alpha\left(\overline{\Omega}\right)$, $0 < \alpha \le 1$, where Ω is the spectral set of the hexagonal lattice in the Euclidean plane. Also, let $p = (p_n)$ and $q = (q_n)$ be two sequences of non-negative real numbers such that $p_n < q_n$ and $q_n \to \infty$ as $n \to \infty$. The order of approximation of f by deferred Cesàro means $D_n\left(p, q; f\right)$ of its hexagonal Fourier series is estimated in the uniform and Hölder norms.

Key Words and Phrases: deferred Cesàro mean, hexagonal Fourier series, Hölder class.

2010 Mathematics Subject Classifications: 41A25, 41A63, 42A10

---

1. Introduction

In the study of trigonometric approximation of $2\pi-$periodic functions on the real line, the partial sums and various means (Cesàro means, Abel-Poisson means, de la Vallèe-Poussin means, etc.) of Fourier series are most useful tools. Especially, there are many results on the order of approximation of continuous $2\pi-$periodic functions in the uniform norm by partial sums of trigonometric (or equivalently exponential) Fourier series and means of these sums. These results can be found in the monographs [3, 11, 13] and in the survey [7]. Also, there are several theorems about approximation of continuous $2\pi-$periodic functions by partial sums, Cesàro means and their generalizations in the Hölder norm (see, for example, [9] and [2]).

The order of approximation of functions defined on cubes of the d-dimensional Euclidean space $\mathbb{R}^d$ was studied by several authors. The common point of these studies is the assumption that the functions are $2\pi-$periodic with respect to each of their variables (see, for example [11, Sections 5.3 and 6.3] and [13, Vol. II, Chapter XVII]). But, in the case of non-tensor product domains of $\mathbb{R}^d$, other types of periodicity are needed to study approximation problems. The most

notable periodicity is the periodicity defined by lattices. The discrete Fourier analysis on lattices was developed in [8].

A lattice is the discrete subgroup $A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\}$ of the Euclidean space $\mathbb{R}^d$, where A is a non-singular $d \times d$ matrix−the generator matrix of the lattice. The lattice $A^{-tr}\mathbb{Z}^d$, where A^{-tr} is the transpose of the inverse matrix A^{-1}, is called the dual lattice of $A\mathbb{Z}^d$. A bounded open set $\Omega \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ if

$$\sum_{\alpha \in A\mathbb{Z}^d} \chi_\Omega (x + \alpha) = 1,$$

for almost all $x \in \mathbb{R}^d$. In this case the set Ω is called a spectral set for the lattice $A\mathbb{Z}^d$, written as $\Omega + A\mathbb{Z}^d = \mathbb{R}^d$. The spectral set Ω is not unique. It is specified that it contains 0 as an interior point and tiles $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ without overlapping and without gap, i.e.

$$\sum_{k \in \mathbb{Z}^d} \chi_\Omega (x + Ak) = 1,$$

for all $x \in \mathbb{R}^d$, and $\Omega + Ak$ and $\Omega + Aj$ are disjoint if $k \neq j$. For example, we can take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ for the standard lattice $\mathbb{Z}^d$ (the lattice generated by the identity matrix).

Let Ω be the spectral set of the lattice $A\mathbb{Z}^d$. $L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x) \overline{g(x)} dx,$$

where $|\Omega|$ is the $d-$dimensional Lebesgue measure of Ω. A theorem of Fuglede states that the set $\left\{e^{2\pi i \langle \alpha, x \rangle} : \alpha \in A^{-tr}\mathbb{Z}^d\right\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$, where $\langle \alpha, x \rangle$ is the usual Euclidean inner product of α and x ([4]). This theorem suggests that, by using the exponentials $e^{2\pi i \langle \alpha, x \rangle}$ $\left(\alpha \in A^{-tr}\mathbb{Z}^d\right)$ one can study Fourier series and approximation on the spectral set of the lattice $A\mathbb{Z}^d$.

A function f is said to be periodic with respect to the lattice $A\mathbb{Z}^d$ if

$$f(x + Ak) = f(x)$$

for all $k \in \mathbb{Z}^d$.

If we consider the standard lattice $\mathbb{Z}^d$ and its spectral set $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, it is clear that Fourier series with respect to this lattice coincide with usual multiple Fourier series of functions of d variables.

2. Hexagonal Fourier Series

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the simplest lattice is the hexagonal lattice and the simplest spectral set is the regular hexagon. Also, it is well known that the hexagonal lattice offers the densest packing of $\mathbb{R}^2$ with unit balls. Thus, the hexagonal lattice and hexagonal Fourier series have great importance in Fourier analysis.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{bmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{bmatrix}$$

and

$$\Omega_H = \left\{ (x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2, \frac{\sqrt{3}}{2} x_1 \pm \frac{1}{2} x_2 < 1 \right\}.$$

It is more convenient to use the homogeneous coordinates (t_1, t_2, t_3) that satisfy $t_1 + t_2 + t_3 = 0$. If we define

$$t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2}, \ \ t_2 := x_2, \ \ t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2}, \tag{1}$$

the hexagon Ω_H becomes

$$\Omega = \left\{ (t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \leq t_1, t_2, -t_3 < 1, \ t_1 + t_2 + t_3 = 0 \right\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}_H^3$ the plane $t_1 + t_2 + t_3 = 0$, that is

$$\mathbb{R}_H^3 = \left\{ \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0 \right\}.$$

Also we use the notation $\mathbb{Z}_H^3$ for the set of points in $\mathbb{R}_H^3$ with integer components, that is $\mathbb{Z}_H^3 = \mathbb{Z}^3 \cap \mathbb{R}_H^3$.

In the homogeneous coordinates, the inner product on $L^2(\Omega)$ becomes

$$\langle f, g \rangle_H = \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t}) \overline{g(\mathbf{t})} d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω, and the orthonormal basis of $L^2(\Omega)$ becomes

$$\left\{ \phi_{\mathbf{j}}(\mathbf{t}) = e^{\frac{2\pi i}{3} \langle \mathbf{j}, \mathbf{t} \rangle} : \mathbf{j} \in \mathbb{Z}_H^3, \ \mathbf{t} \in \mathbb{R}_H^3 \right\}.$$

Also, a function f is periodic with respect to the hexagonal lattice (or $H-$periodic) if and only if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0} \ (mod 3)$, where $\mathbf{t} \equiv \mathbf{s} \ (mod 3)$ defined as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \ (mod 3).$$

It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are $H-$periodic, and if the function f is $H-$periodic, then

$$\int_{\Omega} f(\mathbf{t} + \mathbf{s}) \, d\mathbf{t} = \int_{\Omega} f(\mathbf{t}) \, d\mathbf{t}. \ \left(\mathbf{s} \in \mathbb{R}^3_H\right)$$

For every natural number n, we define a subset of $\mathbb{Z}^3_H$ by

$$\mathbb{H}_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}^3_H : -n \le j_1, j_2, j_3 \le n \right\}.$$

Note that, $\mathbb{H}_n$ consists of all points with integer components inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathcal{H}_n := \operatorname{span} \left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n \right\}, \ (n \in \mathbb{N})$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}^3_H} \widehat{f_{\mathbf{j}}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{2}$$

where

$$\widehat{f_{\mathbf{j}}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}) \, e^{-\frac{2\pi i}{3} \langle \mathbf{j}, \mathbf{t} \rangle} \, d\mathbf{t}, \ \left(\mathbf{j} \in \mathbb{Z}^3_H\right).$$

The nth partial sum of the series (2) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f_{\mathbf{j}}} \phi_{\mathbf{j}}(\mathbf{t}). \ (n \in \mathbb{N})$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t} - \mathbf{s}) D_n(\mathbf{s}) \, d\mathbf{s}, \tag{3}$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that the Dirichlet kernel can be expressed as

$$D_n\left(\mathbf{t}\right) = \Theta_n\left(\mathbf{t}\right) - \Theta_{n-1}\left(\mathbf{t}\right), \quad \left(n \in \mathbb{N}\right), \tag{4}$$

where

$$\Theta_n\left(\mathbf{t}\right) := \frac{\sin \frac{(n+1)(t_1-t_2)\pi}{3} \sin \frac{(n+1)(t_2-t_3)\pi}{3} \sin \frac{(n+1)(t_3-t_1)\pi}{3}}{\sin \frac{(t_1-t_2)\pi}{3} \sin \frac{(t_2-t_3)\pi}{3} \sin \frac{(t_3-t_1)\pi}{3}} \tag{5}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3$ ([10, 8]).

We refer to [8] and [12] for more detailed information about Fourier analysis on lattices and hexagonal Fourier series.

We denote by $C_H\left(\overline{\Omega}\right)$ the Banach space of $H-$periodic continuous functions on $\mathbb{R}_H^3$, equipped with the uniform norm

$$\|f\|_{C_H(\overline{\Omega})} = \sup_{\mathbf{t} \in \overline{\Omega}} |f\left(\mathbf{t}\right)|,$$

and by $H^\alpha\left(\overline{\Omega}\right)$ $(0 < \alpha \le 1)$ the Hölder class, that is the class of functions $f \in C_H\left(\overline{\Omega}\right)$ for which

$$\sup_{\mathbf{t} \ne \mathbf{s}} \frac{|f\left(\mathbf{t}\right) - f\left(\mathbf{s}\right)|}{\|\mathbf{t} - \mathbf{s}\|^\alpha} < \infty$$

holds, where

$$\|\mathbf{t}\| := \max\{|t_1|, |t_2|, |t_3|\}, \quad \left(\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3\right).$$

$H^\alpha\left(\overline{\Omega}\right)$ $(0 < \alpha \le 1)$ becomes a Banach space with respect to the norm

$$\|f\|_{H^\alpha(\overline{\Omega})} := \|f\|_{C_H(\overline{\Omega})} + \sup_{\mathbf{t} \ne \mathbf{s}} \frac{|f\left(\mathbf{t}\right) - f\left(\mathbf{s}\right)|}{\|\mathbf{t} - \mathbf{s}\|^\alpha}.$$

Y. Xu [12] proved that the Abel-Poisson means and the sequence of Cesàro $(C, 1)$ means of the Fourier series of a function $f \in C_H\left(\overline{\Omega}\right)$ converge to this function uniformly on $\overline{\Omega}$. Later, the order of approximation by Abel-Poisson and $(C, 1)$ means of Fourier series of functions belonging to the class $H^\alpha\left(\overline{\Omega}\right)$, $(0 < \alpha \le 1)$ was investigated in uniform norm ([6]) and in the Hölder norm ([5]).

In this work, we will estimate the order of approximation by deferred Cesàro means of Fourier series of functions belonging to the Hölder class $H^\alpha\left(\overline{\Omega}\right)$ in uniform and Hölder norms, and generalize some results of [6] and [5].

3. Main results

Let $p = (p_n)$ and $q = (q_n)$ be two sequences of non-negative integers such that

$$p_n < q_n \quad \text{and} \quad \lim_{n \to \infty} q_n = \infty. \tag{6}$$

The deferred Cesàro means of the series (2) are defined by

$$D_n(p, q; f)(\mathbf{t}) := \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} S_k(f)(\mathbf{t}).$$

It is known that the $D_n(p, q)$ summability method is regular under conditions (6) and generalizes the Cesàro $(C, 1)$ method if and only if $p_n \ll q_n - p_n$ ([1]).

By considering (3) and (4) we obtain

$$
\begin{aligned}
D_n(p, q; f)(\mathbf{t}) &= \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t} - \mathbf{s}) \left(\frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} D_k(\mathbf{s}) \right) d\mathbf{s} \\
&= \frac{1}{q_n - p_n} \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t} - \mathbf{s}) \left(\sum_{k=p_n+1}^{q_n} \Theta_k(\mathbf{s}) - \Theta_{k-1}(\mathbf{s}) \right) d\mathbf{s} \\
&= \frac{1}{q_n - p_n} \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t} - \mathbf{s}) \left(\Theta_{q_n}(\mathbf{s}) - \Theta_{p_n}(\mathbf{s}) \right) d\mathbf{s}.
\end{aligned}
$$

Hence we have

$$f(\mathbf{t}) - D_n(p, q; f)(\mathbf{t}) =$$

$$= \frac{1}{(q_n - p_n)} \frac{1}{|\Omega|} \int_\Omega (f(\mathbf{t}) - f(\mathbf{t} - \mathbf{s})) \left(\Theta_{q_n}(\mathbf{s}) - \Theta_{p_n}(\mathbf{s}) \right) d\mathbf{s}, \tag{7}$$

for each $f \in L^1(\Omega)$ and $\mathbf{t} \in \mathbb{R}_H^3$.

If we take $q_n = n$ and $p_n = 0$ for $n = 1, 2, ...$, $D_n(p, q; f)$ becomes the $(C, 1)$ means $S_n^{(1)}(f)$.

Hereafter, we shall write $A \ll B$ for the quantities A and B, if there exists a constant $K > 0$ such that $A \leq KB$ holds.

The rate of approximation by $(C, 1)$ means of hexagonal Fourier series was estimated as follows:

Theorem A ([6]). *Let* $f \in H^\alpha(\overline{\Omega})$, $0 < \alpha \leq 1$. *Then*

$$\left\| f - S_n^{(1)}(f) \right\|_{C_H(\overline{\Omega})} \ll \begin{cases} \frac{1}{n^\alpha}, & \alpha < 1, \\ \frac{(\log n)^2}{n}, & \alpha = 1, \end{cases} \tag{8}$$

holds.

In this work we generalize Theorem A. Our main theorem is the following.

Theorem 1. *For each $f \in H^{\alpha}\left(\overline{\Omega}\right)$, $0 < \alpha \le 1$, the estimate*

$$\|f - D_n\left(p, q; f\right)\|_{C_H(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \begin{cases} \frac{1}{(q_n - p_n)^{\alpha}}, & \alpha < 1, \\ \frac{(\log(2(q_n - p_n)))^2}{q_n - p_n}, & \alpha = 1, \end{cases} \quad (9)$$

holds.

Proof. Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$, by (7) we have

$$|f\left(\mathbf{t}\right) - D_n\left(p, q; f\right)\left(\mathbf{t}\right)| \ll \frac{1}{(q_n - p_n)} \frac{1}{|\Omega|} \int\limits_{\Omega} \|\mathbf{s}\|^{\alpha} |\Theta_{q_n}\left(\mathbf{s}\right) - \Theta_{p_n}\left(\mathbf{s}\right)| \, d\mathbf{s}.$$

Since the integrated function is symmetric with respect to its variables, it is sufficient to estimate the integral

$$I_n := \int\limits_{\Delta} \|\mathbf{t}\|^{\alpha} |\Theta_{q_n}\left(\mathbf{t}\right) - \Theta_{p_n}\left(\mathbf{t}\right)| \, d\mathbf{t},$$

where

$$\begin{aligned} \Delta \quad : \quad &= \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3 : 0 \le t_1, t_2, -t_3 \le 1\right\} \\ &= \left\{(t_1, t_2) : t_1 \ge 0, \ t_2 \ge 0, \ t_1 + t_2 \le 1\right\}, \end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By (5) and some simple trigonometric identities

$$\Theta_{q_n}\left(\mathbf{t}\right) - \Theta_{p_n}\left(\mathbf{t}\right) =$$

$$= 2\cos\left(\left(\frac{q_n + p_n}{2} + 1\right) \frac{(t_3 - t_1)\pi}{3}\right) \sin\left(\left(\frac{q_n - p_n}{2}\right) \frac{(t_3 - t_1)\pi}{3}\right) \times$$

$$\times \frac{\sin\left((q_n + 1) \frac{(t_1 - t_2)\pi}{3}\right) \sin\left((q_n + 1) \frac{(t_2 - t_3)\pi}{3}\right)}{\sin\frac{(t_1 - t_2)\pi}{3} \sin\frac{(t_2 - t_3)\pi}{3} \sin\frac{(t_3 - t_1)\pi}{3}} +$$

$$+ 2\cos\left(\left(\frac{q_n + p_n}{2} + 1\right) \frac{(t_2 - t_3)\pi}{3}\right) \sin\left(\left(\frac{q_n - p_n}{2}\right) \frac{(t_2 - t_3)\pi}{3}\right) \times$$

$$\times \frac{\sin\left((q_n + 1) \frac{(t_1 - t_2)\pi}{3}\right) \sin\left((p_n + 1) \frac{(t_3 - t_1)\pi}{3}\right)}{\sin\frac{(t_1 - t_2)\pi}{3} \sin\frac{(t_2 - t_3)\pi}{3} \sin\frac{(t_3 - t_1)\pi}{3}} +$$

$$+2\cos\left(\left(\frac{q_n+p_n}{2}+1\right)\frac{(t_1-t_2)\pi}{3}\right)\sin\left(\left(\frac{q_n-p_n}{2}\right)\frac{(t_1-t_2)\pi}{3}\right)\times$$

$$\times\frac{\sin\left((p_n+1)\frac{(t_2-t_3)\pi}{3}\right)\sin\left((p_n+1)\frac{(t_3-t_1)\pi}{3}\right)}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}.$$

If we use the change of variables

$$s_1:=\frac{t_1-t_3}{3},\ \ s_2:=\frac{t_2-t_3}{3}, \tag{10}$$

we obtain

$$I_n\leq 3\int_{\widetilde{\Delta}}(s_1+s_2)^\alpha\left(|L_1(s_1,s_2)|+|L_2(s_1,s_2)|+|L_3(s_1,s_2)|\right)ds_1ds_2,$$

where

$$\widetilde{\Delta}:=\{(s_1,s_2):0\leq s_1\leq 2s_2,0\leq s_2\leq 2s_1,s_1+s_2\leq 1\},$$

and

$$L_1(s_1,s_2)\ \ :\ =\frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_1\pi)\right)\sin\left((q_n+1)(s_1-s_2)\pi\right)\sin\left((q_n+1)s_2\pi\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}$$

$$L_2(s_1,s_2)\ \ :\ =\frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_2\pi)\right)\sin\left((q_n+1)(s_1-s_2)\pi\right)\sin\left((p_n+1)(s_1\pi)\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}$$

$$L_3(s_1,s_2)\ \ :\ =\frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_1-s_2)\pi\right)\sin\left((p_n+1)(s_2\pi)\right)\sin\left((p_n+1)(s_1\pi)\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, we have

$$I_n\leq 6\int_{\Delta^*}(s_1+s_2)^\alpha\left(|L_1(s_1,s_2)|+|L_2(s_1,s_2)|+|L_3(s_1,s_2)|\right)ds_1ds_2,$$

where

$$\Delta^*:=\left\{(s_1,s_2)\in\widetilde{\Delta}:s_1\leq s_2\right\},$$

i. e. the half of $\widetilde{\Delta}$. The change of variables

$$s_1:=\frac{u_1-u_2}{2},\ \ s_2:=\frac{u_1+u_2}{2} \tag{11}$$

transforms the triangle Δ^* onto triangle

$$\Gamma:=\left\{(u_1,u_2):0\leq u_2\leq\frac{u_1}{3},0\leq u_1\leq 1\right\},$$

and hence

$$I_n \leq 3 \int_\Gamma u_1^\alpha \left(|L_1^* (u_1, u_2)| + |L_2^* (u_1, u_2)| + |L_3^* (u_1, u_2)| \right) du_1 du_2,$$

where

$$L_1^* (u_1, u_2) \quad : \quad = \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) \frac{(u_1 - u_2)\pi}{2} \right) \sin \left((q_n + 1) u_2 \pi \right) \sin \left((q_n + 1) \frac{(u_1 + u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}$$

$$L_2^* (u_1, u_2) \quad : \quad = \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) \frac{(u_1 + u_2)\pi}{2} \right) \sin \left((q_n + 1) u_2 \pi \right) \sin \left((p_n + 1) \frac{(u_1 - u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}$$

$$L_3^* (u_1, u_2) \quad : \quad = \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) u_2 \pi \right) \sin \left((p_n + 1) \frac{(u_1 + u_2)\pi}{2} \right) \sin \left((p_n + 1) \frac{(u_1 - u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}.$$

If we divide the triangle Γ into three parts as

$$\Gamma_1 \quad : \quad = \left\{ (u_1, u_2) \in \Gamma : u_1 \leq \frac{1}{2 (q_n - p_n)} \right\},$$

$$\Gamma_2 \quad : \quad = \left\{ (u_1, u_2) \in \Gamma : u_1 \geq \frac{1}{2 (q_n - p_n)}, \ u_2 \leq \frac{1}{6 (q_n - p_n)} \right\},$$

$$\Gamma_3 \quad : \quad = \left\{ (u_1, u_2) \in \Gamma : u_1 \geq \frac{1}{2 (q_n - p_n)}, \ u_2 \geq \frac{1}{6 (q_n - p_n)} \right\},$$

we get

$$I_n \ll I_n^{(1)} + I_n^{(2)} + I_n^{(3)},$$

where

$$I_n^{(j)} = \int_{\Gamma_j} u_1^\alpha \left(|L_1^* (u_1, u_2)| + |L_2^* (u_1, u_2)| + |L_3^* (u_1, u_2)| \right) du_1 du_2, \quad (j = 1, 2, 3).$$

We shall need the inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \leq n, \quad (n \in \mathbb{N}), \tag{12}$$

and

$$\sin t \geq \frac{2}{\pi} t \quad \left(0 \leq t \leq \frac{\pi}{2} \right) \tag{13}$$

to estimate integrals $I_n^{(1)}, I_n^{(2)}$ and $I_n^{(3)}$.

For $(u_1, u_2) \in \Gamma$,

$$\sin\left(\frac{u_1 + u_2}{2}\pi\right) \geq \frac{\sqrt{3}}{2}\sin\frac{u_1\pi}{2},$$

and by (13) we obtain

$$\frac{1}{\sin\left(\frac{u_1+u_2}{2}\pi\right)} \leq \frac{2}{\sqrt{3}}\frac{1}{u_1} \quad (u_1 \neq 0). \tag{14}$$

By the inequality (12) we get

$$\begin{aligned}
|L_1^*(u_1, u_2)| &\ll (q_n - p_n)(q_n + 1)^2, \\
|L_2^*(u_1, u_2)| &\ll (q_n - p_n)(q_n + 1)(p_n + 1),
\end{aligned}$$

and

$$|L_3^*(u_1, u_2)| \ll (q_n - p_n)(p_n + 1)^2$$

for $(u_1, u_2) \in \Gamma_1$. Hence

$$\int_{\Gamma_1} u_1^\alpha \left|L_j^*(u_1, u_2)\right| du_1 du_2 (q_n - p_n)(q_n + 1)^2 \int_{\Gamma_1} u_1^\alpha du_1 du_2 =$$

$$= (q_n - p_n)(q_n + 1)^2 \int_0^{\frac{1}{6(q_n-p_n)}} \int_{3u_2}^{\frac{1}{2(q_n-p_n)}} u_1^\alpha du_1 du_2 \leq \frac{(q_n + 1)^2}{(q_n - p_n)^{1+\alpha}},$$

for $j = 1, 2, 3$, which implies

$$I_n^{(1)} \ll \frac{q_n^2}{(q_n - p_n)^{1+\alpha}}. \tag{15}$$

Since $u_1 - u_2 \geq \frac{2u_1}{3}$, by (13) one can easily see that

$$\frac{1}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right)} \leq \frac{3}{2u_1}, \quad (u_1, u_2) \in \Gamma_2 \cup \Gamma_3. \tag{16}$$

Thus, by (12), (14) and (16) we obtain

$$\begin{aligned}
|L_1^*(u_1, u_2)| &\ll (q_n + 1)\frac{1}{u_1^2}, \\
|L_2^*(u_1, u_2)| &\ll (q_n + 1)\frac{1}{u_1^2},
\end{aligned}$$

and

$$|L_3^*(u_1, u_2)| \ll (q_n - p_n) \frac{1}{u_1^2}$$

for $(u_1, u_2) \in \Gamma_2$. For $j = 1, 2$,

$$\int_{\Gamma_2} u_1^\alpha \left| L_j^*(u_1, u_2) \right| du_1 du_2 \ll (q_n + 1) \int_0^{\frac{1}{6(q_n - p_n)}} \int_{\frac{1}{2(q_n - p_n)}}^1 u_1^{\alpha - 2} du_1 du_2$$

$$= \frac{q_n + 1}{6(q_n - p_n)} \begin{cases} \frac{1}{1-\alpha} \left(\frac{1}{(2(q_n - p_n))^{\alpha - 1}} - 1 \right), & \alpha < 1, \\ \log\left(2(q_n - p_n)\right), & \alpha = 1. \end{cases}$$

Hence

$$\int_{\Gamma_2} u_1^\alpha \left| L_j^*(u_1, u_2) \right| du_1 du_2 \ll \begin{cases} \frac{q_n}{(q_n - p_n)^\alpha}, & \alpha < 1, \\ \frac{q_n \log 2(q_n - p_n)}{q_n - p_n}, & \alpha = 1, \end{cases}$$

for $j = 1, 2$. Similarly

$$\int_{\Gamma_2} u_1^\alpha \left| L_3^*(u_1, u_2) \right| du_1 du_2 \ll (q_n - p_n) \int_0^{\frac{1}{6(q_n - p_n)}} \int_{\frac{1}{2(q_n - p_n)}}^1 u_1^{\alpha - 2} du_1 du_2$$

$$\ll \begin{cases} \frac{1}{(q_n - p_n)^{\alpha - 1}}, & \alpha < 1, \\ \log\left(2(q_n - p_n)\right), & \alpha = 1. \end{cases}$$

Thus we get

$$I_n^{(2)} \ll \begin{cases} \frac{q_n}{(q_n - p_n)^\alpha}, & \alpha < 1, \\ \frac{q_n \log(2(q_n - p_n))}{q_n - p_n}, & \alpha = 1. \end{cases} \tag{17}$$

By (13), (14) and (16) we obtain

$$\left| L_j^*(u_1, u_2) \right| \ll \frac{1}{u_1^2 u_2} \ (j = 1, 2, 3)$$

for $(u_1, u_2) \in \Gamma_3$. Thus

$$\int_{\Gamma_3} u_1^\alpha \left| L_j^*(u_1, u_2) \right| du_1 du_2 \ll \int_{\frac{1}{6(q_n - p_n)}}^{\frac{1}{3}} \int_{3u_2}^1 u_1^{\alpha - 2} \frac{1}{u_2} du_1 du_2$$

$$\ll \begin{cases} \frac{1}{(q_n-p_n)^{\alpha-1}}, & \alpha < 1, \\ \left(\log\left(2\left(q_n - p_n\right)\right)\right)^2, & \alpha = 1. \end{cases}$$

This last estimate, (15) and (17) yield

$$I_n \ll \begin{cases} \frac{q_n^2}{(q_n-p_n)^{\alpha+1}}, & \alpha < 1, \\ \frac{q_n^2}{(q_n-p_n)^2}\left(\log\left(2\left(q_n - p_n\right)\right)\right)^2, & \alpha = 1, \end{cases}$$

and the proof is completed. ◄

The following approximation theorem in Hölder norm was obtained in ([5]) :

Theorem B. *Let* $0 \le \beta < \alpha \le 1$. *Then for each* $f \in H^\alpha\left(\overline{\Omega}\right)$ *the estimate*

$$\left\| f - S_n^{(1)}\left(f\right) \right\|_{H^\beta\left(\overline{\Omega}\right)} \ll \begin{cases} \frac{1}{n^{\alpha-\beta}}, & \alpha < 1, \\ \frac{(\log n)^2}{n^{1-\beta}}, & \alpha = 1. \end{cases} \tag{18}$$

holds.

We generalize Theorem B as follows:

Theorem 2. *Let* $0 \le \beta < \alpha \le 1$ *and* $f \in H^\alpha\left(\overline{\Omega}\right)$. *Then*

$$\left\| f - D_n\left(p, q; f\right) \right\|_{H^\beta\left(\overline{\Omega}\right)} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \begin{cases} \frac{1}{(q_n-p_n)^{\alpha-\beta}}, & \alpha < 1, \\ \frac{(\log(2(q_n-p_n)))^2}{(q_n-p_n)^{1-\beta}}, & \alpha = 1. \end{cases} \tag{19}$$

Proof. Set $e_n\left(\mathbf{t}\right) := f\left(\mathbf{t}\right) - D_n\left(p, q; f\right)\left(\mathbf{t}\right)$. Hence

$$\left\| f - D_n\left(p, q; f\right) \right\|_{H^\beta\left(\overline{\Omega}\right)} = \left\| e_n \right\|_{C_H\left(\overline{\Omega}\right)} + \sup_{\mathbf{t} \ne \mathbf{s}} \frac{\left| e_n\left(\mathbf{t}\right) - e_n\left(\mathbf{s}\right) \right|}{\left\| \mathbf{t} - \mathbf{s} \right\|^\beta}.$$

By (7) we have

$$e_n\left(\mathbf{t}\right) - e_n\left(\mathbf{s}\right) = \frac{1}{\left(q_n - p_n\right)} \frac{1}{|\Omega|} \int\limits_{\Omega} \varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right) \left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right) d\mathbf{u},$$

where

$$\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right) := f\left(\mathbf{t}\right) - f\left(\mathbf{t} - \mathbf{u}\right) - f\left(\mathbf{s}\right) + f\left(\mathbf{s} - \mathbf{u}\right).$$

Thus

$$\left| e_n\left(\mathbf{t}\right) - e_n\left(\mathbf{s}\right) \right| \ll \frac{1}{q_n - p_n} J_n,$$

where

$$J_n := \int\limits_{\Omega} \left| \varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right) \right| \left| \left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right) \right| d\mathbf{u}.$$

Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$, we have

$$\left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \ll \left\|\mathbf{u}\right\|^{\alpha},$$

and by Theorem 1,

$$
\begin{aligned}
J_n^{1-\frac{\beta}{\alpha}} &= \left(\int_{\Omega} \left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \left|\left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\
&\ll \left(\int_{\Omega} \left\|\mathbf{u}\right\|^{\alpha} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\
&\ll \begin{cases} \left(\frac{q_n^2}{(q_n-p_n)^{\alpha+1}}\right)^{1-\frac{\beta}{\alpha}}, & \alpha < 1, \\[2mm] \left[\frac{q_n^2}{(q_n-p_n)^2}\left(\log\left(2\left(q_n-p_n\right)\right)\right)^2\right]^{1-\beta}, & \alpha = 1. \end{cases}
\end{aligned}
$$

We also have

$$\left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \ll \left\|\mathbf{t}-\mathbf{s}\right\|^{\alpha},$$

and hence

$$
\begin{aligned}
J_n^{\frac{\beta}{\alpha}} &\ll \left(\int_{\Omega} \left\|\mathbf{t}-\mathbf{s}\right\|^{\alpha} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\
&= \left\|\mathbf{t}-\mathbf{s}\right\|^{\beta} \left(\int_{\Omega} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right) - \Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}}.
\end{aligned}
$$

As in proof of Theorem 1, it is sufficient to estimate the integral

$$\int_{\Delta} \left|\left(\Theta_{q_n}\left(\mathbf{t}\right) - \Theta_{p_n}\left(\mathbf{t}\right)\right)\right| d\mathbf{t}.$$

By the transforms (10) and (11),

$$\int_{\Delta} \left|\left(\Theta_{q_n}\left(\mathbf{t}\right) - \Theta_{p_n}\left(\mathbf{t}\right)\right)\right| d\mathbf{t} \ll$$

$$\ll \int_{\Gamma} \left(\left|L_1^*\left(u_1, u_2\right)\right| + \left|L_2^*\left(u_1, u_2\right)\right| + \left|L_3^*\left(u_1, u_2\right)\right|\right) du_1 du_2.$$

For $j = 1, 2, 3,$

$$\int\limits_{\Gamma_1} \left|L_j^*(u_1, u_2)\right| du_1 du_2 \ll (q_n - p_n)(q_n + 1)^2 \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{3u_2}^{\frac{1}{2(q_n - p_n)}} du_1 du_2$$

$$\ll \frac{q_n^2}{q_n - p_n}.$$

For $j = 1, 2,$

$$\int\limits_{\Gamma_2} \left|L_j^*(u_1, u_2)\right| du_1 du_2 \ll (q_n + 1) \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{\frac{1}{2(q_n - p_n)}}^1 \frac{1}{u_1^2} du_1 du_2 \ll q_n,$$

and

$$\int\limits_{\Gamma_2} \left|L_3^*(u_1, u_2)\right| du_1 du_2 \ll (q_n - p_n) \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{\frac{1}{2(q_n - p_n)}}^1 \frac{1}{u_1^2} du_1 du_2 \ll q_n - p_n.$$

Also,

$$\int\limits_{\Gamma_3} \left|L_j^*(u_1, u_2)\right| du_1 du_2 \ll \int\limits_{\frac{1}{6(q_n - p_n)}}^{\frac{1}{3}} \int\limits_{3u_2}^1 \frac{1}{u_1^2 u_2} du_1 du_2 \ll q_n - p_n$$

for $j = 1, 2, 3$. By combining these inequalities we obtain

$$\int\limits_{\Delta} \left|(\Theta_{q_n}(\mathbf{t}) - \Theta_{p_n}(\mathbf{t}))\right| d\mathbf{t} \ll \frac{q_n^2}{q_n - p_n}.$$

Hence

$$J_n^{\frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^\beta \left(\frac{q_n^2}{q_n - p_n}\right)^{\frac{\beta}{\alpha}}.$$

Let $\alpha < 1$.

$$J_n = J_n^{\frac{\beta}{\alpha}} J_n^{1 - \frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^\beta \left(\frac{q_n^2}{q_n - p_n}\right)^{\frac{\beta}{\alpha}} \left(\frac{q_n^2}{(q_n - p_n)^{\alpha + 1}}\right)^{1 - \frac{\beta}{\alpha}}$$

$$= \|\mathbf{t} - \mathbf{s}\|^\beta \frac{q_n^2}{(q_n - p_n)^{\alpha - \beta + 1}}.$$

This implies

$$|e_n(\mathbf{t}) - e_n(\mathbf{s})| \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha - \beta}} \|\mathbf{t} - \mathbf{s}\|^\beta,$$

and hence

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\beta} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha - \beta}}$$

for every $\mathbf{t}, \mathbf{s} \in \mathbb{R}_H^3$ with $\mathbf{t} \neq \mathbf{s}$. This and Theorem 1 give

$$\|f - D_n(p, q; f)\|_{H^\beta(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha - \beta}}.$$

Now let $\alpha = 1$. In this case,

$$
\begin{aligned}
J_n &= J_n^{\frac{\beta}{\alpha}} J_n^{1 - \frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^\beta \left(\frac{q_n^2}{q_n - p_n}\right)^\beta \left[\frac{q_n^2}{(q_n - p_n)^2} \left(\log\left(2\left(q_n - p_n\right)\right)\right)^2\right]^{1 - \beta} \\
&= \|\mathbf{t} - \mathbf{s}\|^\beta \frac{q_n^2}{(q_n - p_n)^{2 - \beta}} \left(\log\left(2\left(q_n - p_n\right)\right)\right)^{2(1 - \beta)},
\end{aligned}
$$

which implies

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\beta} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{\left(\log\left(2\left(q_n - p_n\right)\right)\right)^{2(1 - \beta)}}{(q_n - p_n)^{1 - \beta}}, \quad \mathbf{t} \neq \mathbf{s}.$$

By this inequality and by Theorem 1 we obtain

$$\|f - D_n(p, q; f)\|_{H^\beta(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{\left(\log\left(2\left(q_n - p_n\right)\right)\right)^{2(1 - \beta)}}{(q_n - p_n)^{1 - \beta}},$$

which finishes the proof. ◄

Remark. If we take $p_n = n - 1$ and $q_n = n + k - 1$, where $k \in \mathbb{N}$ satisfies $n \ll k$, then the summability method $D_n(p, q)$ generalizes the $(C, 1)$ method and $D_n(p, q; f)$ becomes

$$D_n(p, q; f) = S_{n,k}^{(1)}(f) := \left(1 + \frac{n}{k}\right) S_{n+k-1}^{(1)}(f) - \frac{n}{k} S_{n-1}^{(1)}(f),$$

which is called the delayed arithmetic mean ([13, Vol. I, p.80]). These means give the same approximation order as Cesàro $(C, 1)$ means.

Acknowledgment

References

[1] R.P. Agnew, *On deferred Cesàro means*, Ann. of Math., **33**, 1932, 413-241.

[2] J. Bustamante, M.A. Jimenez, *Trends in Hölder approximation*, In: Approximation, optimization and mathematical economics, M. Lassonde (ed), Springer, 2001.

[3] R.A. DeVore, G.G. Lorentz, *Constructive approximation*, Springer-Verlag, 1993.

[4] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Functional Analysis, **16**, 1974, 101-121.

[5] A. Guven, *Approximation by means of hexagonal Fourier series in Hölder norms*, J. Classical Anal., **1**, 2012, 43-52.

[6] A. Guven, *Approximation by $(C,1)$ and Abel-Poisson means of Fourier series on hexagonal domains*, Math. Inequal. Appl., **16**, 2013, 175-191.

[7] A.S.B. Holland, *A survey of degree of approximation of continuous functions*, SIAM Rev., **23**, 1981, 344-379.

[8] H. Li, J. Sun and Y. Xu, *Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle*, SIAM J. Numer. Anal., **46**, 2008, 1653-1681.

[9] S. Prössdorf, *Zur konvergenz der Fourierreihen hölderstetiger funktionen*, Math. Nachr., **69**, 1975, 7-14.

[10] J. Sun, *Multivariate Fourier series over a class of non tensor-product partition domains*, J. Comput. Math., **21**, 2003, 53-62.

[11] A.F. Timan, *Theory of approximation of functions of a real variable*, Pergamon Press, 1963.

[12] Y. Xu, *Fourier series and approximation on hexagonal and triangular domains*, Constr. Approx., **31**, 2010, 115-138.

[13] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, 1959.

Ali Guven
Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, 10145,
Balikesir, Turkey
E-mail: guvennali@gmail.com