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Convolution and Jackson inequalities in Musielak–Orlicz spaces

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Abstract: In the present work we prove some direct and inverse theorems for approximation by trigonometric polynomials in Musielak–Orlicz spaces. Furthermore, we get a constructive characterization of the Lipschitz classes in these spaces.

Key words: Musielak–Orlicz space, direct and inverse theorem, Lipschitz class, trigonometric approximation

1. Introduction

Musielak–Orlicz spaces are similar to Orlicz spaces but are defined by a more general function with two variables $\varphi(x,t)$. In these spaces, the norm is given by virtue of the integral

$$\int_T \varphi(x, |f(x)|)\, dx,$$

where $T := [-\pi, \pi]$. We know that in an Orlicz space, φ would be independent of $x, \varphi(|f(x)|)$. The special cases $\varphi(t) = t^p$ and $\varphi(x,t) = t^{p(x)}$ give the Lebesgue spaces L^p and the variable exponent Lebesgue spaces $L^{p(x)}$, respectively. In addition to being a natural generalization that covers results from both variable exponent and Orlicz spaces, the study of Musielak–Orlicz spaces can be motivated by applications to differential equations [13, 28], fluid dynamics [15, 23], and image processing [5, 10, 16]. Detailed information on Musielak–Orlicz spaces can be found in the book by Musielak [26].

Polynomial approximation problems in Musielak–Orlicz spaces have a long history. Orlicz spaces, which satisfy the translation invariance property, are a particular case of Musielak–Orlicz spaces. In these spaces, polynomial approximation problems were investigated by several mathematicians in [3, 11, 12, 20–25, 29, 35]. In some weighted Banach function spaces, similar problems were studied in [6, 7, 9, 17, 18, 30, 34, 36, 37]. In general, Musielak–Orlicz spaces may not attain the translation invariance property, as can be seen in the case of variable exponent Lebesgue spaces $L^{p(x)}$. Several inequalities of trigonometric polynomial approximation in $L^{p(x)}$ were obtained in [2, 4, 14, 19, 31, 33]. Note that, under the translation invariance hypothesis on Musielak–Orlicz space, Musielak obtained some trigonometric approximation inequalities in [27]. The main aim of this work is to obtain solutions to some central problems of trigonometric approximation in Musielak–Orlicz

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spaces that may not have the translation invariance property. In this work, we prove some direct and inverse theorems of approximation theory in Musielak–Orlicz spaces.

The rest of the work is organized as follows. In Section 2, we give the definition and some properties of Musielak–Orlicz spaces. In Section 3, we prove the boundedness of the Steklov operator in Musielak–Orlicz spaces and define the modulus of smoothness by means of this operator. Section 4 formulates our main results. In Section 5, we investigate the boundedness of De la Vallée Poussin and Cesaro means of the Fourier series of the functions in Musielak–Orlicz spaces. Furthermore, we prove the Bernstein inequality and the equivalence of the modulus of smoothness to the K-functional in these spaces. Section 6 contains the proofs of our main results.

We will use the following notations: $A(x) \preceq B(x) \Leftrightarrow \exists c > 0 : A(x) \leq cB(x)$ and $A(x) \approx B(x) \Leftrightarrow A(x) \preceq B(x) \ \wedge \ B(x) \preceq A(x)$.

2. Preliminaries

A function $\varphi : [0, \infty) \to [0, \infty]$ is called Φ-*function* (briefly $\varphi \in \Phi$) if φ is convex and left-continuous and

$$\varphi(0) := \lim_{t \to 0^+} \varphi(t) = 0, \ \lim_{x \to \infty} \varphi(x) = \infty.$$

A Φ-function φ is said to be an N-function if it is continuous and positive and satisfies

$$\lim_{t \to 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty.$$

Let $\Phi(T)$ be the collection of functions $\varphi : T \times [0, \infty) \to [0, \infty]$ such that:

(i) $\varphi(x, \cdot) \in \Phi$ for every $x \in T$;

(ii) $\varphi(x, u)$ is in $L^0(T)$, the set of measurable functions, for every $u \geq 0$.

A function $\varphi(\cdot, u) \in \Phi(T)$ is said to satisfy the Δ_2 condition ($\varphi \in \Delta_2$) with respect to parameter u if $\varphi(x, 2u) \leq K\varphi(x, u)$ holds for all $x \in T, \ u \geq 0$, with some constant $K \geq 2$.

Subclass $\Phi(N) \subset \Phi(T)$ consists of functions $\varphi \in \Phi(T)$ such that, for every $x \in T, \ \varphi(x, \cdot)$ is an N-function and $\varphi \in \Delta_2$.

Two functions φ and φ_1 are said to be *equivalent* (we shall write $\varphi \sim \varphi_1$) if there is $c > 0$ such that

$$\varphi_1(x, u/c) \leq \varphi(x, u) \leq \varphi_1(x, cu)$$

for all x and u.

For $\varphi \in \Phi(N)$ we set

$$\varrho_\varphi(f) := \int_T \varphi(x, |f(x)|) \, dx.$$

Musielak–Orlicz space L^φ (or generalized Orlicz space) is the class of Lebesgue measurable functions $f : T \to \mathbb{R}$ satisfying the condition

$$\lim_{\lambda \to 0} \varrho_\varphi(\lambda f) = 0.$$

The equivalent condition for $f \in L^0(T)$ to belong to L^φ is that $\varrho_\varphi(\lambda f) < \infty$ for some $\lambda > 0$. L^φ becomes a normed space with the Orlicz norm

$$\|f\|_{[\varphi]} := \sup\left\{\int\limits_T |f(x)\,g(x)|\,dx : \varrho_\psi(g) \le 1\right\}$$

and with the Luxemburg norm

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : \varrho_\varphi\left(\frac{f}{\lambda}\right) \le 1\right\},$$

where

$$\psi(t,v) := \sup_{u\ge 0}\left(uv - \varphi(t,u)\right), \quad v \ge 0, \quad t \in T$$

is the complementary function (with respect to variable v) of φ in the sense of Young. These two norms are equivalent:

$$\|f\|_\varphi \le \|f\|_{[\varphi]} \le 2\,\|f\|_\varphi .$$

Young's inequality,

$$us \le \varphi(x,u) + \psi(x,s), \tag{1}$$

holds for complementary functions $\varphi, \psi \in \Phi(N)$ where $u, s \ge 0$ and $x \in T$.

From Young's inequality (1) we have

$$\|f\|_{[\varphi]} \le \varrho_\varphi(f) + 1,$$

$$\|f\|_\varphi \le \varrho_\varphi(f) \text{ if } \|f\|_\varphi > 1; \text{ and } \|f\|_\varphi \ge \varrho_\varphi(f) \text{ if } \|f\|_\varphi \le 1.$$

Hölder's inequality

$$\int\limits_T |f(x)\,g(x)|\,dx \le \|f\|_\varphi \|f\|_{[\psi]} \tag{2}$$

holds for complementary functions $\varphi, \psi \in \Phi(N)$. The Jensen integral inequality can be formulated as follows. If φ is an N-function and $r(x)$ is a nonnegative measurable function, then

$$\varphi\left(\frac{1}{\int_T r(x)\,dx}\int_T f(x)\,r(x)\,dx\right) \le \frac{1}{\int_T r(x)\,dx}\int_T \varphi(f(x))\,r(x)\,dx. \tag{3}$$

Everywhere in this work we will assume that there exists a constant $A > 0$ such that for all $x, y \in T$ with $|x-y| \le 1/2$ we have

$$\frac{\varphi(x,u)}{\varphi(y,u)} \le u^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}}, \quad u \ge 1; \tag{4}$$

there exist some constants $c_1,\ c_2 > 0$ such that

$$\inf_{x\in T}\varphi(x,1) \ge c_1 \tag{5}$$

and

$$\int_T \varphi(x,1)\,dx < \infty, \qquad \psi(x,1) \le c_2 \qquad \text{a.e. on } T. \tag{6}$$

Example 1 *Let* $p : T \to [1, \infty)$ *be in* $L^0(T)$ *such that for all* $x, y \in T$ *with* $|x - y| \leq 1/2$ *we have the Dini–Lipschitz property,*

$$|p(x) - p(y)| \leq \frac{c}{\log\left(\frac{1}{|x-y|}\right)},$$

with a constant $c > 0$. *Then the following functions belong to* $\Phi(T)$ *and satisfy conditions (4), (5), and (6):*

(i) $\varphi(x, u) = u^{p(x)}$, $\sup_{x \in T} p(x) < \infty$,

(ii) $\varphi(x, u) = u^{p(x)} \log(1 + u)$,

(iii) $\varphi(x, u) = u(\log(1 + u))^{p(x)}$.

A function $\varphi \in \Phi(N)$ is in the class $\Phi(N, DL)$ if conditions (4), (5), and (6) are fulfilled.

3. Modulus of smoothness

For $f \in L^\varphi$ we define the Steklov operator A_h by

$$(A_h f)(x) := \frac{1}{h} \int_{-h/2}^{h/2} f(x - t)\, dt, \quad 0 < h < \pi, \ x \in T.$$

The characteristic function $\kappa_{[a,b]}(u)$ of a finite interval $[a, b]$ is the function on $\mathbb{R}$ defined through

$$\kappa_{[a,b]}(u) = \begin{cases} 1, & u \in [a, b], \\ 0, & u \notin [a, b]. \end{cases}$$

The operator A_h can be written as a convolution integral [9, p. 33]:

$$(A_h f)(x) = \frac{1}{2\pi} \int_T f(t) \Re_h(t - x) dt,$$

where

$$\Re_h(u) := \frac{2\pi}{h} \kappa_{\left[-\frac{h}{2}, \frac{h}{2}\right]}(u).$$

The kernel $\Re_h$ satisfies the following conditions [9, p. 33]:

$$\int_T \Re_h(u)\, du \preceq 1, \quad |\Re_h(u)| \preceq 1, \ h \leq u \leq \pi, \quad \text{and} \ \max_u |\Re_h(u)| \preceq \frac{1}{h}.$$

Lemma 2 *If* $f \in L^\varphi$ *with* $\varphi \in \Phi(N, DL)$, *then there exists a constant, independent of* n *and* f, *such that the inequality*

$$\|A_h f\|_\varphi \preceq \|f\|_\varphi$$

holds for $0 < h < \pi$.

Note that [8, p. 156, Lemma 6.1] is like Lemma 2. A necessary and sufficient condition for the translation operator in Musielak–Orlicz spaces to be continuous is well known. It was established first in [22].

Proof of Lemma 2 Let $N = \lfloor \frac{\pi}{h} \rfloor$, $x \in T$, $x_k := (kh - 1)\pi$, $U_k := [x_k, x_{k+1})$,

$$E_x := \begin{cases} T \setminus (x - \pi h, x + \pi h) & \text{, when } (x - \pi h, x + \pi h) \subset T, \\ T \setminus \{(-\pi, x + \pi h) \cup (x - \pi h + 2\pi, \pi)\} & \text{, when } x - \pi h < -\pi, \\ T \setminus \{(x - \pi h, \pi) \cup (-\pi, x + \pi h - 2\pi)\} & \text{, when } x + \pi h > \pi. \end{cases} \tag{7}$$

Then $T = \bigcup_{k=0}^{2N-1} U_k$, where the length of U_k is $l(U_k) = |x_{k+1} - x_k| = \pi/N$. Let $F(t) = f(t)/2$ and $\|F\|_\varphi \le 1$. It is necessary to show that

$$\varrho_\varphi (A_h f) = \int_T \varphi \left(x, \left| \frac{1}{\pi} \int_T F(t) \Re_h (t - x) dt \right| \right) dx \le c$$

with a constant $c > 0$ independent of f and h. From the convexity of φ we get

$$\begin{aligned} \varrho_\varphi (A_h f) &= \varrho_\varphi \left(\frac{1}{\pi} \int_T F(t) \Re_h (t - x) dt \right) \\ &\preceq \varrho_\varphi \left(\int_{x-\pi h}^{x+\pi h} F(t) \Re_h (t - x) dt \right) + \varrho_\varphi \left(\int_{E_x} F(t) \Re_h (t - x) dt \right) \\ &=: I_1 + I_2. \end{aligned}$$

When $x \in T$ and $t \in E_x$, then

$$|\Re_h (t - x)| \lesssim 1,$$

and using (2), (5), and (6) we get

$$\begin{aligned} \left| \int_{E_x} F(t) \Re_h (t - x) dt \right| &\preceq \int_T |F(t)| \, dt \\ &\preceq \|F\|_\varphi \|1\|_\psi \preceq \|1\|_\psi \preceq c + 1 \end{aligned}$$

and therefore

$$\begin{aligned} I_2 &\preceq \int_T \varphi \left(x, \left| \int_{E_x} F(t) \Re_h (t - x) dt \right| \right) dx \\ &\preceq \int_T \varphi (x, c + 1) \, dx \preceq \int_T \varphi (x, 1) \, dx \preceq 1. \end{aligned}$$

Now

$$\begin{aligned} I_1 &\preceq \int_T \varphi \left(x, \int_{x-\pi h}^{x+\pi h} |F(t)| \, |\Re_h (t - x)| \, dt \right) dx \\ &\le \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \varphi \left(x, 1 + \int_{x-\pi h}^{x+\pi h} |F(t)| \, |\Re_h (t - x)| \, dt \right) dx. \end{aligned}$$

We set

$$\varphi_k(u) := \inf\left\{\varphi(x,u) : x \in \Xi^k\right\} \leq \inf\left\{\varphi(x,u) : x \in U_k\right\}$$

for some larger set $\Xi^k \supset U_k$, which will be chosen later with the property

$$l\left(\Xi^k\right) \leq j\pi h \tag{8}$$

for some $j > 1$. On the other hand,

$$I_1 \lesssim \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \bar{A}_k(x,h)\,\varphi_k\left(1 + \int_{x-\pi h}^{x+\pi h} |F(t)|\,|\Re_h(t-x)|\,dt\right) dx,$$

where

$$\bar{A}_k(x,h) := \frac{\varphi\left(x, 1 + \int_{x-\pi h}^{x+\pi h} |F(t)|\,|\Re_h(t-x)|\,dt\right)}{\varphi_k\left(1 + \int_{x-\pi h}^{x+\pi h} |F(t)|\,|\Re_h(t-x)|\,dt\right)} := \frac{\varphi(x, \alpha(x,h))}{\varphi_k(\alpha(x,h))}.$$

Now we prove the uniform estimate $\bar{A}_k(x,h) \preceq 1$ for $x \in U_k$ where $c > 0$ is independent of x, k, and h. Indeed, since

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi_k(\varsigma_k, t)} \leq t^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}}, \quad x \in U_k, \varsigma_k \in \Xi^k$$

we get

$$\bar{A}_k(x,h) = \frac{\varphi(x, \alpha(x,h))}{\varphi_k(\alpha(x,h))} \leq \alpha(x,h)^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}}.$$

Also, $|x - \varsigma_k| \leq l\left(\Xi^k\right) \leq j\pi h$ and

$$|\alpha(x,h)| \leq \frac{1}{h}\left(\int_{x-\pi h}^{x+\pi h} |F(t)|\,dt\right) \preceq \frac{1}{h}\|F\|_{\varphi} \preceq \frac{1}{h},$$

$$\begin{aligned} \alpha(x,h)^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}} &\leq \alpha(x,h)^{\overline{\frac{A}{\log\left(\frac{h}{6j}\right)}}} \leq \left(C\frac{1}{h}\right)^{\overline{\frac{A}{\log\left(\frac{h}{6j}\right)}}} \\ &\preceq \left(\frac{1}{h}^{1/\log\left(\frac{h}{6j}\right)}\right)^{A} \preceq 1. \end{aligned}$$

Let $\mu_h = \int_{x-\pi h}^{x+\pi h} |\Re_h(t-x)|\,dt = \int_{-\pi h}^{\pi h} |\Re_h(t)|\,dt$. Then $\mu_h \preceq 1$. Without loss of generality we may assume that

$\mu_h > 0$. Using Jensen's integral inequality, we have

$$\begin{aligned} I_1 &\preceq \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \varphi_k \left(\frac{1}{\mu_h} \int_{x-\pi h}^{x+\pi h} |F(t)| \, |\Re_h(t-x)| \, dt \right) dx \\ &\preceq \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \frac{1}{\mu_h} \int_{x-\pi h}^{x+\pi h} \varphi_k \left(|F(t)| \right) |\Re_h(t-x)| \, dt dx \\ &\preceq \sum_{k=0}^{N-1} \frac{1}{\mu_h} \int_{-\pi h}^{\pi h} |\Re_h(t)| \int_{x_k}^{x_{k+1}} \varphi_k \left(|F(x+t)| \right) dx dt \\ &\preceq \frac{1}{\mu_h} \int_{-\pi h}^{\pi h} |\Re_h(t)| \sum_{k=0}^{N-1} \int_{x_k - t}^{x_{k+1}-t} \varphi_k \left(|F(x)| \right) dx dt. \end{aligned}$$

We take as Ξ^k the set (11). Clearly $\Xi^k \supset U_k$ and $l\left(\Xi^k\right) \leq 3\pi h$. Then (8) is satisfied with $j = 3$. Since each point $x \in T$ belongs simultaneously to not more than a finite number n_0 of the sets U_k, taking the maximum with respect to all the sets U_k containing x we obtain

$$\begin{aligned} I_1 &\preceq \frac{1}{\mu_h} \int_{-\pi h}^{\pi h} |\Re_h(t)| \, dt \int_{-\pi}^{\pi} \tilde{\varphi}\left(x, |F(x)|\right) dx \\ &\preceq \int_{-\pi}^{\pi} \tilde{\varphi}\left(x, |F(x)|\right) dx \end{aligned}$$

with $\tilde{\varphi}(x,u) := \max_i \varphi_i(t)$. Now using

$$\tilde{\varphi}(x,u) \leq \varphi(x,u), \quad \forall x \in T,$$

we have

$$\varrho_\varphi(A_h f) \preceq \int_{-\pi}^{\pi} \varphi\left(x, |F(x)|\right) dx \preceq \|F\|_\varphi \preceq 1.$$

This gives

$$\|A_h f\|_\varphi \preceq \|f\|_\varphi ,$$

and the result follows. □

We define the kth $(k \in \mathbb{N})$ order modulus of smoothness $\Omega_\varphi^k(\cdot, f)$ by

$$\Omega_\varphi^k(\delta, f) := \sup_{0<h_i\leq\delta} \|(I - A_{h_1}) \dots (I - A_{h_k}) f\|_\varphi , \quad \delta > 0,$$

where I is the identity operator.

4. Main results

By $E_n(f)_\varphi$ we denote the best approximation of L^φ by polynomials in $\mathcal{T}_n$, i.e.

$$E_n(f)_\varphi = \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_\varphi ,$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree $\leq n$.

Let W_{φ}^{r}, $r \in \mathbb{N}$, $\varphi \in \Phi(N, DL)$, be the class of functions $f \in L^{\varphi}$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L^{\varphi}$. W_{φ}^{r}, $\varphi \in \Phi(N, DL)$, $r \in \mathbb{N}$, becomes a Banach space with the norm $\|f\|_{W_{\varphi}^{r}} := \|f\|_{\varphi} + \left\|f^{(r)}\right\|_{\varphi}$.

Our main results are the following.

Theorem 3 *For every $f \in W_{\varphi}^{r}$, $\varphi \in \Phi(N, DL)$, $n \in \mathbb{N}$, the inequality*

$$E_n(f)_{\varphi} \preceq \frac{1}{n^r} E_n\left(f^{(r)}\right)_{\varphi}, \quad r \in \mathbb{N}$$

holds with some constant depending only on φ and r.

Theorem 4 *Let $f \in L^{\varphi}$, $\varphi \in \Phi(N, DL)$, $n \in \mathbb{N}$. Then we have the following estimate:*

$$E_n(f)_{\varphi} \preceq \Omega_{\varphi}^{r}\left(\frac{1}{n}, f\right), \quad r \in \mathbb{N}$$

with some constant depending only on φ and r.

Theorem 5 *Let $\varphi \in \Phi(N, DL)$. Then for $f \in L^{\varphi}$ and $n \in \mathbb{N}$*

$$\Omega_{\varphi}^{r}\left(\frac{1}{n}, f\right) \preceq \frac{1}{n^{2r}}\left\{E_0(f)_{\varphi} + \sum_{m=1}^{n} m^{2r-1} E_m(f)_{\varphi}\right\}, \quad r \in \mathbb{N},$$

holds with some constant depending only on φ and r.

Similar theorems were obtained in Orlicz spaces [3, 12, 18, 24, 25] and in variable exponent Lebesgue spaces [1, 4, 14, 31, 33].

From Theorems 4 and 5, we get the following Marchaud-type inequality:

Corollary 6 *Let $f \in L^{\varphi}$, $\varphi \in \Phi(N, DL)$, $n \in \mathbb{N}$. Then we have*

$$\Omega_{\varphi}^{r}(\delta, f) \preceq \delta^{2r} \int_{\delta}^{1} \frac{\Omega_{\varphi}^{r+1}(u, f)}{u^{2r}} \frac{du}{u}, \quad 0 < \delta < 1,$$

for $r \in \mathbb{N}$.

Theorems 4 and 5 imply also the following estimate:

Corollary 7 *Let $f \in L^{\varphi}$, $\varphi \in \Phi(N, DL)$, and $n \in \mathbb{N}$. If*

$$E_n(f)_{L_w^{pq}} \preceq n^{-\alpha}, \quad n \in \mathbb{N}$$

for some $\alpha > 0$, then, for a given $r \in \mathbb{N}$, we have the estimations

$$\Omega_{\varphi}^{r}(\delta, f) \preceq \begin{cases} \delta^{\alpha} & , r > \alpha/2; \\ \delta^{2r} \log \frac{1}{\delta} & , r = \alpha/2; \\ \delta^{2r} & , r < \alpha/2. \end{cases}$$

Hence, if we define the Lipschitz class $Lip\,(\alpha, L^{\varphi})$ for $\alpha > 0$ and $r := \lfloor \alpha/2 \rfloor + 1,\ \lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ as

$$Lip\,(\alpha, L^{\varphi}) := \left\{ f \in L^{\varphi} : \Omega_{\varphi}^{r}\,(\delta, f) \lesssim \delta^{\alpha}, \quad \delta > 0 \right\},$$

then, from Theorem 4 and Corollary 7, we get the following constructive characterization of the class $Lip\,(\alpha, L^{\varphi})$.

Corollary 8 *Let $f \in L^{\varphi},\ \varphi \in \Phi\,(N, DL),\ n \in \mathbb{N},$ and $\alpha > 0$. The following assertions are equivalent:*

$$(i)\ \ f \in Lip\,(\alpha, L^{\varphi}), \qquad (ii)\ \ E_n(f)_{L^{\varphi}} \preceq n^{-\alpha}, \quad n \in \mathbb{N}.$$

5. Auxiliary estimates

Let

$$f\,(x) \backsim \frac{a_0\,(f)}{2} + \sum_{k=1}^{\infty} \left(a_k\,(f) \cos kx + b_k\,(f) \sin kx\right) =: \sum_{k=0}^{\infty} A_k\,(x, f) \tag{9}$$

be the Fourier series of $f \in W_{\varphi}^{1}$ and

$$S_n\,(f) := S_n\,(x, f) := \sum_{k=0}^{n} A_k\,(x, f)\,, \quad n = 0, 1, 2, \ldots.$$

be the partial sum of the Fourier series (9). In this case, for $f \in W_{\varphi}^{1}$, we have

$$f(x) = \frac{a_0\,(f)}{2} + \frac{1}{\pi} \int_T f'(t) B_r(t - x) dt,$$

where

$$B_r(u) = \sum_{k=1}^{\infty} \frac{\cos(ku + \pi/2)}{k}$$

is the Bernoulli kernel. Since $(S_n(\cdot, f))' = S_n(\cdot, f')$ we have

$$f(x) - S_n(x, f) = \frac{1}{\pi} \int_T f'(t) R_n(t - x) dt,$$

where

$$R_n(u) = \sum_{k=n+1}^{\infty} \frac{\cos(ku + \pi/2)}{k}.$$

We define the De la Vallée Poussin mean of series (9) as

$$V_m^n(f, \cdot) = \frac{1}{m+1} \sum_{i=0}^{m} S_{n+i}(\cdot, f)$$

for $n, m \in \mathbb{N} \cup \{0\}$. Then we get

$$f(x) - V_m^n(f, x) = \frac{1}{\pi} \int_T f'(t) \frac{1}{m+1} \sum_{i=0}^{m} R_{n+i}(t - x) dt.$$

Setting

$$k_{m+1}^{n}\left(u\right):=\frac{1}{m+1}\sum_{i=0}^{m}R_{n+i}(t-x),$$

we find

$$f(x)-V_{m}^{n}(f,x)=\frac{1}{\pi\left(m+1\right)}\int_{T}f^{\prime}(t)k_{m+1}^{n}(t-x)dt.$$

Let $n\in\mathbb{N}$. From [32, Lemmas 3, 4, 5] we have, for $m=n-1$ or $m=n$,

$$\int_{T}\left|k_{m+1}^{n}\left(u\right)\right|du\preceq 1,$$

$$\left|k_{m+1}^{n}\left(u\right)\right|\lesssim 1\ \text{ for }\ \left(\sqrt{n}\right)^{-1}\leq u\leq 2\pi-\left(\sqrt{n}\right)^{-1},$$

and

$$\max_{u}\left|k_{m+1}^{n}\left(u\right)\right|\lesssim n.$$

Lemma 9 *If* $f\in L^{\varphi}$ *with* $\varphi\in\Phi\left(N,DL\right)$, *then there exist some constants, independent of* n *and* f, *such that the inequalities*

$$\begin{aligned}\left\|f(\cdot)-V_{n-1}^{n}(f,\cdot)\right\|_{\varphi} &\preceq \frac{1}{n}\left\|f^{\prime}\right\|_{\varphi},\\ \left\|f(\cdot)-V_{n}^{n}(f,\cdot)\right\|_{\varphi} &\preceq \frac{1}{n}\left\|f^{\prime}\right\|_{\varphi}\end{aligned}$$

hold for any $T_{n}\in\mathcal{T}_{n}$.

Proof of Lemma 9 Let the set E_{x} be defined as in (7) with $h=1/\lfloor n^{1/2}\rfloor$. Assume that $F\left(t\right)=f^{\prime}(t)/(m+1)$ and $\left\|F\right\|_{\varphi}\leq 1$. We need to show that

$$\rho_{\varphi}\left(f-V_{m}^{n}(f,\cdot)\right)=\int\limits_{T}\varphi\left(x,\left|\frac{1}{\pi}\int_{T}F\left(t\right)k_{m+1}^{n}(t-x)dt\right|\right)dx\preceq 1$$

with $c>0$ independent of f and n. Then convexity of φ implies

$$\begin{aligned}\rho_{\varphi}\left(f-V_{m}^{n}(f,\cdot)\right) &= \rho_{\varphi}\left(\frac{1}{\pi}\int_{T}F\left(t\right)k_{m+1}^{n}(t-x)dt\right)\\ &\leq \frac{1}{\pi}\rho_{\varphi}\left(\left\{\int\limits_{x-\pi h}^{x+\pi h}+\int\limits_{E_{x}}\right\}F\left(t\right)k_{m+1}^{n}(t-x)dt\right)\\ &\preceq \rho_{\varphi}\left(\int\limits_{x-\pi h}^{x+\pi h}F\left(t\right)k_{m+1}^{n}(t-x)dt\right)+\rho_{\varphi}\left(\int\limits_{E_{x}}F\left(t\right)k_{m+1}^{n}(t-x)dt\right)\\ &=: I_{1}+I_{2}.\end{aligned}$$

If $x \in T$ and $t \in E_x$, then

$$\left|k_{m+1}^{n}(t-x)\right| \preceq 1,$$

and using Hölder's inequality (2), (5), and (6), we obtain

$$\left|\int\limits_{E_x} F(t)\, k_{m+1}^{n}(t-x)dt\right| \preceq \int\limits_{T} |F(t)|\, dt \preceq \|F\|_{\varphi} \|1\|_{\psi} \preceq \|1\|_{\psi} \preceq 1$$

and hence

$$\begin{aligned} I_2 &\preceq \rho_{\varphi}\left(\int\limits_{E_x} F(t)\, k_{m+1}^{n}(t-x)dt\right) \preceq \int\limits_{T} \varphi\left(x, \left|\int\limits_{E_x} F(t)\, k_{m+1}^{n}(t-x)dt\right|\right) dx \\ &\preceq \int\limits_{T} \varphi(x, c+1)\, dx \preceq \int\limits_{T} \varphi(x,1)\, dx \preceq 1. \end{aligned}$$

Now

$$\begin{aligned} I_1 &\preceq \int_{T} \varphi\left(x, \int_{x-\pi h}^{x+\pi h} |F(t)| \left|k_{m+1}^{n}(t-x)\right| dt\right) dx \\ &\leq \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} \varphi\left(x, 1+\int_{x-\pi h}^{x+\pi h} |F(t)| \left|k_{m+1}^{n}(t-x)\right| dt\right) dx. \end{aligned}$$

We set

$$\varphi_k(u) := \inf\left\{\varphi(x,u) : x \in \Xi^k\right\} \leq \inf\left\{\varphi(x,u) : x \in U_k\right\}$$

for some larger set $\Xi^k \supset U_k$, which will be chosen later with the property

$$l\left(\Xi^k\right) \leq j\pi/\lfloor n^{1/2} \rfloor \tag{10}$$

for some $j > 1$. On the other hand,

$$I_1 \lesssim \sum_{k=0}^{2N-1} \int_{x_k}^{x_{k+1}} A_k(x,m,n)\, \varphi_k\left(1+\int_{x-\pi h}^{x+\pi h} |F(t)| \left|k_{m+1}^{n}(t-x)\right| dt\right) dx,$$

where

$$A_k(x,m,n) := \frac{\varphi\left(x, 1+\int_{x-\pi h}^{x+\pi h} |F(t)| \left|k_{m+1}^{n}(t-x)\right| dt\right)}{\varphi_k\left(1+\int_{x-\pi h}^{x+\pi h} |F(t)| \left|k_{m+1}^{n}(t-x)\right| dt\right)} := \frac{\varphi(x, \alpha(x,m,n))}{\varphi_k(\alpha(x,m,n))}.$$

We prove the uniform estimate $A_k(x,m,n) \preceq 1$ for $x \in U_k$ where $c > 0$ is independent of x, k and m, n. Indeed, since

$$\frac{\varphi(x,t)}{\varphi_k(t)} = \frac{\varphi(x,t)}{\varphi_k(\varsigma_k, t)} \leq t^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}}, \quad x \in U_k, \varsigma_k \in \Xi^k,$$

we have

$$A_k\left(x,m,n\right)=\frac{\varphi\left(x,\alpha\left(x,m,n\right)\right)}{\varphi_k\left(\alpha\left(x,m,n\right)\right)}\leq\alpha\left(x,m,n\right)^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}}.$$

Also, $|x-\varsigma_k|\leq l\left(\Xi^k\right)\leq j\pi/\lfloor n^{1/2}\rfloor$ and

$$|\alpha\left(x,m,n\right)|\leq n\left(\int_{x-\pi h}^{x+\pi h}|F(t)|\,dt\right)\preceq n\left\|F\right\|_{\varphi}\preceq n,$$

$$\alpha\left(x,m,n\right)^{\overline{\frac{A}{\log\left(\frac{1}{|x-\varsigma_k|}\right)}}}\leq\alpha\left(x,m,n\right)^{\overline{\frac{A}{\log\left(\frac{n^{1/2}}{6j}\right)}}}\leq(Cn)^{\overline{\frac{A}{\log\left(\frac{n^{1/2}}{6j}\right)}}}\preceq\left(n^{1/\log\left(\frac{n}{6j}\right)}\right)^{A}\preceq1.$$

Let $\mu_{m,n}=\int_{x-\pi h}^{x+\pi h}\left|k_{m+1}^{n}(t-x)\right|dt=\int_{-\pi h}^{\pi h}\left|k_{m+1}^{n}(t)\right|dt$. Then $\mu_{m,n}\preceq1$. Without loss of generality we may assume that $\mu_{m,n}>0$.

By Jensen's integral inequality (3),

$$\begin{aligned}
I_1 &\lesssim \sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\varphi_k\left(C\frac{1}{\mu_{m,n}}\int_{x-\pi h}^{x+\pi h}|F(t)|\left|k_{m+1}^{n}(t-x)\right|dt\right)dx\\
&\lesssim \sum_{k=0}^{2N-1}\int_{x_k}^{x_{k+1}}\frac{1}{\mu_{m,n}}\int_{x-\pi h}^{x+\pi h}\varphi_k\left(|F(t)|\right)\left|k_{m+1}^{n}(t-x)\right|dtdx\\
&\lesssim \sum_{k=0}^{2N-1}\frac{1}{\mu_{m,n}}\int_{-\pi h}^{\pi h}\left|k_{m+1}^{n}(t)\right|\int_{x_k}^{x_{k+1}}\varphi_k\left(|F(x+t)|\right)dxdt\\
&\lesssim \frac{1}{\mu_{m,n}}\int_{-\pi h}^{\pi h}\left|k_{m+1}^{n}(t)\right|\sum_{k=0}^{2N-1}\int_{x_k-t}^{x_{k+1}-t}\varphi_k\left(|F(x)|\right)dxdt.
\end{aligned}$$

We take as Ξ^k the set

$$\bigcup_{t\in(-\pi h,\pi h)}\left\{x:x+t\in U_k\right\}. \tag{11}$$

Clearly $\Xi^k\supset U_k$ and $l\left(\Xi^k\right)\leq3\pi/\lfloor n^{1/2}\rfloor$. Then (10) is satisfied with $j=3$. Since each point $x\in T$ belongs simultaneously to not more than a finite number n_0 of the sets U_k, taking the maximum with respect to all the sets U_k containing x we obtain

$$I_1\preceq\frac{1}{\mu_{m,n}}\int_{-\pi h}^{\pi h}\left|k_{m+1}^{n}(t)\right|dt\int_{-\pi}^{\pi}\tilde{\varphi}\left(x,|F(x)|\right)dx\preceq\int_{-\pi}^{\pi}\tilde{\varphi}\left(x,|F(x)|\right)dx$$

with $\tilde{\varphi}\left(x,u\right):=\max_i\varphi_i\left(t\right)$. Now using

$$\tilde{\varphi}\left(x,u\right)\leq\varphi\left(x,u\right),\quad\forall x\in T,$$

we get

$$\rho_{\varphi}\left(f-V_m^n(f,\cdot)\right)\preceq\int_{-\pi}^{\pi}\varphi\left(x,|F(x)|\right)dx\preceq\left\|F\right\|_{\varphi}\preceq1.$$

These give the estimates

$$\left\|f(\cdot)-V_{n-1}^{n}(f,\cdot)\right\|_{\varphi} \preceq \frac{1}{n}\left\|f'\right\|_{\varphi},$$
$$\left\|f(\cdot)-V_{n}^{n}(f,\cdot)\right\|_{\varphi} \preceq \frac{1}{n}\left\|f'\right\|_{\varphi},$$

and the result follows. □

It is known that for the partial sums of the Fourier series (9) the integral representation

$$S_n(x,f)=\frac{1}{\pi}\int_T f(t)\,D_n(x-t)\,dt$$

is valid, where $D_n(t):=\frac{1}{2}+\sum_{m=1}^{n}\cos mt$ is the Dirichlet kernel.

Consider the sequence $\{\sigma_n(\cdot,f)\}$ of the Cesaro means of the partial sums of the Fourier series (9), that is,

$$\sigma_n(x,f):=\frac{S_0(x,f)+S_1(x,f)+\cdots+S_n(x,f)}{n+1}, \quad n=\{0\}\cup\mathbb{N},$$

with $\sigma_0(x,f)=S_0(x,f):=a_0/2$. It is known that

$$\sigma_n(x,f)=\frac{1}{\pi}\int_T f(t)\,K_n(x-t)\,dt,$$

where

$$K_n(t):=\frac{1}{2}+\sum_{m=1}^{n}\left(1-\frac{m}{n+1}\right)\cos mt$$

is the Fejer kernel of order n. The Fejer kernel satisfies the following conditions [38]:

$$\int_T K_n(u)\,du \preceq 1, \quad |K_n(u)|\preceq 1,\ \frac{1}{n^{1/2}}\leq u\leq\pi \ \text{ and } \max_u |K_n(u)|\preceq n. \tag{12}$$

Taking into account these conditions (12), the following lemma is proved similarly to the previous lemma.

Lemma 10 *If $f\in L^{\varphi}$ with $\varphi\in\Phi(N,DL)$, then there exists a constant, independent of n and f, such that the inequality*

$$\|\sigma_n(x,f)\|_{\varphi}\preceq\|f\|_{\varphi}$$

holds.

Bernstein's inequality in the space L^{φ} is proved in the following lemma.

Lemma 11 *If $f\in L^{\varphi}$ with $\varphi\in\Phi(N,DL)$, then for every $T_n\in\mathcal{T}_n$ the inequality*

$$\left\|T_n^{k}\right\|_{\varphi}\preceq n^{k}\left\|T_n\right\|_{\varphi}, \qquad k\in\mathbb{N} \tag{13}$$

holds with a constant independent of n.

Proof of Lemma 11 It is sufficient to prove the lemma for $k=1$. Since

$$T_n(x) = S_n(x, T_n) = \frac{1}{\pi}\int_T T_n(u) D_n(u-x)\,du,$$

by differentiation we obtain

$$T_n'(x) = -\frac{1}{\pi}\int_T T_n(u) D_n'(u-x)\,du = \frac{1}{\pi}\int_T T_n(u+x) \sum_{m=1}^{n} m \sin mudu.$$

Taking into account

$$\int_T T_n(u+x) \sum_{m=1}^{n-1} m \sin(2n-m) udu = 0,$$

we get

$$\begin{aligned} T_n'(x) &= \frac{1}{\pi}\int_T T_n(u+x)\left[\sum_{m=1}^{n} m\sin mu + \sum_{m=1}^{n-1} m\sin(2n-m)u\right] du \\ &= \frac{1}{\pi}\int_T T_n(u+x)\, 2n\sin nu \left[\frac{1}{2} + \sum_{m=1}^{n-1} \frac{n-m}{n}\cos mu\right] du \\ &= \frac{2n}{\pi}\int_T T_n(u+x)\sin nu K_{n-1}(u)\,du. \end{aligned}$$

Since K_{n-1} is nonnegative we have

$$\begin{aligned} |T_n'(x)| &\leq \frac{2n}{\pi}\int_T |T_n(u+x)|\, K_{n-1}(u)\,du \\ &= 2n\sigma_{n-1}(x, |T_n|). \end{aligned}$$

The last inequality and the boundedness of the operator σ_n in L^φ yield the required inequality. □

Lemma 12 *If* $f \in W_\varphi^2$ *with* $\varphi \in \Phi(N, DL)$, *then*

$$\Omega_\varphi^k(\delta, f) \preceq \delta^2 \Omega_\varphi^{k-1}(\delta, f''), \qquad k \in \mathbb{N}$$

with some constant independent of δ.

Proof of Lemma 12 Setting

$$g(x) := (I - A_{h_2}) \ldots (I - A_{h_k}) f(x)$$

we get

$$(I - A_{h_1}) g(x) = (I - A_{h_1}) \ldots (I - A_{h_k}) f(x).$$

Therefore,

$$\left(I-A_{h_1}\right)...\left(I-A_{h_k}\right)f(x)=\frac{1}{2h_1}\int_{-h_1}^{h_1}\left[g(x)-g(x+t)\right]dt=-\frac{1}{8h_1}\int_0^{h_1}\int_0^{t}\int_{-u}^{u}g''(x+s)dsdudt.$$

Hence,

$$\begin{aligned}\left\|\left(I-A_{h_1}\right)...\left(I-A_{h_k}\right)f\right\|_\varphi &\preceq \frac{1}{8h_1}\sup\int_T\left|\int_0^{h_1}\int_0^{t}\int_{-u}^{u}g''(x+s)dsdudt\right|\left|v(x)\right|dx\\ &= \frac{1}{8h_1}\int_0^{h_1}\int_0^{t}2u\left\|\frac{1}{2u}\int_{-u}^{u}g''(x+s)ds\right\|_\varphi dudt\\ &\preceq \frac{1}{8h_1}\int_0^{h_1}\int_0^{t}2u\left\|g''\right\|_\varphi dudt=h_1^2\left\|g''\right\|_\varphi,\end{aligned}$$

where the supremum is taken over all $v\in L^{\psi}(T)$ with $\varrho_\psi(v)\leq 1$. Since

$$g''=\left(I-A_{h_2}\right)...\left(I-A_{h_k}\right)f'',$$

we have

$$\Omega_\varphi^k(\delta,f)\leq\sup_{0<h_i\leq\delta}ch_1^2\left\|g''\right\|_\varphi=c\delta^2\sup_{0<h_i\leq\delta}\left\|\left(I-A_{h_2}\right)...\left(I-A_{h_k}\right)f''\right\|_\varphi=c\delta^2\Omega_\varphi^{k-1}(\delta,f'').$$

□

Corollary 13 *If $f\in W_\varphi^{2k}$ with $\varphi\in\Phi(N,DL)$, then*

$$\Omega_\varphi^k(\delta,f)\preceq\delta^{2k}\left\|f^{(2k)}\right\|_\varphi,\quad k=1,2,... \tag{14}$$

with some constant independent of δ.

For an $f\in L^\varphi$ and $r\in\mathbb{N}$, Peetre's K-functional is defined as

$$K\left(f,\delta;L_\varphi,W_\varphi^r\right):=\inf_{g\in W_\varphi^r}\left\{\left\|f-g\right\|_\varphi+\delta^r\left\|g^{(r)}(x)\right\|_\varphi\right\}$$

for $\delta>0$.

Theorem 14 *If $f\in L^\varphi$ with $\varphi\in\Phi(N,DL)$, then we have*

$$\Omega_\varphi^r(\delta,f)\approx K\left(f,\delta;L_\varphi,W_\varphi^{2r}\right),\quad r\in\mathbb{N}$$

where the implied constants are independent of $\delta>0$.

Proof of Theorem 14 Let $h \in W_{\varphi}^{2r}$. From subadditivity of $\Omega_{\varphi}^{r}(\cdot, f)$ and (14) we have

$$\Omega_{\varphi}^{r}(\delta, f) \preceq \|f-h\|_{\varphi} + \delta^{2r}\left\|h^{(2r)}\right\|_{\varphi}.$$

Taking the infimum on h we get $\Omega_{\varphi}^{r}(\delta, f) \preceq K\left(f, \delta; L_{\varphi}, W_{\varphi}^{2r}\right)$.

We define an operator L_{δ} on L^{φ} as

$$(L_{\delta} f)(x) := 3\delta^{-3} \int_{0}^{\delta} \int_{0}^{u} \int_{-t}^{t} f(x+s)\, ds dt du, \quad x \in T.$$

From [1, p. 15],

$$\frac{d^{2r}}{dx^{2r}}\left(L_{\delta}^{r} f\right) = \frac{c}{\delta^{2r}}\left(I-A_{\delta}\right)^{r}, \quad r \in \mathbb{N}.$$

Because of estimates

$$\|L_{\delta} f\|_{\varphi} \preceq 3\delta^{-3} \int_{0}^{\delta} \int_{0}^{u} 2t\, \|A_{t} f\|_{\varphi}\, dt du \preceq \|f\|_{\varphi},$$

the operator L_{δ} is bounded in L^{φ}.

Defining another operator $\mathcal{L}_{\delta}^{r}$ as

$$\mathcal{L}_{\delta}^{r} := I - \left(I - L_{\delta}^{r}\right)^{r},$$

we obtain

$$\left\|\frac{d^{2r}}{dx^{2r}} \mathcal{L}_{\delta}^{r} f\right\|_{\varphi} \preceq \left\|\frac{d^{2r}}{dx^{2r}} L_{\delta}^{r} f\right\|_{\varphi} = \frac{1}{\delta^{2r}}\left\|\left(I-A_{\delta}\right)^{r} f\right\|_{\varphi} \preceq \frac{1}{\delta^{2r}} \Omega_{\varphi}^{r}(\delta, f).$$

Since L_{δ} is bounded in L^{φ} and $I - L_{\delta}^{r} = \left(I - L_{\delta}\right) \sum_{j=0}^{r-1} L_{\delta}^{j}$ we have

$$\left\|\left(I-L_{\delta}^{r}\right) g\right\|_{\varphi} \preceq \left\|\left(I-L_{\delta}\right) g\right\|_{\varphi} \preceq \delta^{-3} \int_{0}^{\delta} \int_{0}^{u} 2t\, \left\|\left(I-A_{t}\right) g\right\|_{\varphi} dt du \preceq \sup_{0<t \leq \delta}\left\|\left(I-A_{t}\right) g\right\|_{\varphi}$$

for any $g \in L^{\varphi}$.

Applying this inequality r times in $\|f - \mathcal{L}_{\delta}^{r} f\|_{\varphi} = \left\|\left(I - L_{\delta}^{r}\right)^{r} f\right\|_{\varphi}$ we obtain

$$\begin{aligned}
\|f-\mathcal{L}_{\delta}^{r} f\|_{\varphi} & \preceq \sup_{0<t_{1} \leq \delta}\left\|\left(I-A_{t_{1}}\right)\left(I-L_{\delta}^{r}\right)^{r-1} f\right\|_{\varphi} \\
& \preceq \sup_{0<t_{1}, t_{2} \leq \delta}\left\|\left(I-A_{t_{1}}\right)\left(I-A_{t_{2}}\right)\left(I-L_{\delta}^{r}\right)^{r-2} f\right\|_{\varphi} \\
& \preceq \ldots \preceq \sup_{0<t_{i} \leq \delta}\left\|\left(I-A_{t_{1}}\right) \ldots\left(I-A_{t_{r}}\right) f\right\|_{\varphi} = \Omega_{\varphi}^{r}(t, f).
\end{aligned}$$

This gives the reverse estimate and completes the proof. □

6. Proofs of main results

Proof of Theorem 3 It is enough to prove $E_n(f)_\varphi \preceq \frac{1}{n} E_n(f')_\varphi$. For this we need

$$E_j(f)_\varphi \preceq \frac{1}{j}\|f'\|_\varphi \tag{15}$$

with $j \in \mathbb{N}$. If $j = 2n$, then

$$E_j(f)_\varphi = E_{2n}(f)_\varphi \le \|f(\cdot) - V_n^n(f,\cdot)\|_\varphi \preceq \frac{1}{n}\|f'\|_\varphi \preceq \frac{1}{j}\|f'\|_\varphi .$$

If $j = 2n-1$, then

$$E_j(f)_\varphi = E_{2n-1}(f)_\varphi \le \left\|f(\cdot) - V_{n-1}^n(f,\cdot)\right\|_\varphi \preceq \frac{1}{n}\|f'\|_\varphi \preceq \frac{1}{j}\|f'\|_\varphi .$$

We obtained (15). Now suppose that $E_n(f')_\varphi = \|f' - \Theta_n(f')\|_\varphi$ and

$$\mathcal{F}(x) := \int_0^x \Theta_n(f')(t)\,dt.$$

Then $\mathcal{F} \in \mathcal{T}_n$ and $\mathcal{F}'(x) = \Theta_n(f')(x)$. Thus,

$$\begin{aligned} E_n(f)_\varphi &= E_n(f-\mathcal{F})_\varphi \preceq \frac{1}{n}\left\|(f-\mathcal{F})'\right\|_\varphi = \frac{1}{n}\|f' - \mathcal{F}'\|_\varphi \\ &= \frac{1}{n}\|f' - \Theta_n(f')\|_\varphi \preceq \frac{1}{n}E_n(f')_\varphi . \end{aligned}$$

□

Corollary 15 *For every $f \in W_\varphi^r$, $\varphi \in \Phi(N, DL)$, $n \in \mathbb{N}$, the inequality*

$$E_n(f)_\varphi \preceq \frac{1}{n^r}\left\|f^{(r)}\right\|_\varphi, \quad r \in \mathbb{N}, \tag{16}$$

holds with some constant depending only on φ and r.

Proof of Theorem 4 Let $h \in W_\varphi^{2r}$. From (16) and Theorem 14

$$\begin{aligned} E_n(f)_\varphi &= E_n(f-h+h)_\varphi \le E_n(f-h)_\varphi + E_n(h)_\varphi \\ &\lesssim \|f-h\|_\varphi + n^{-2r}\left\|h^{(2r)}\right\|_\varphi \lesssim \Omega_\varphi^r\left(\frac{1}{n}, f\right). \end{aligned}$$

□

Proof of Theorem 5 Let $f \in L^\varphi$, $\delta := 1/n$ and let $T_n \in \mathcal{T}_n$ be the best approximating polynomial to f. We have

$$\Omega_\varphi^k(\delta, f) \le \Omega_\varphi^k(\delta, f - T_{2^{j+1}}) + \Omega_\varphi^k(\delta, T_{2^{j+1}}), \qquad j \in \mathbb{N}$$

and

$$\Omega_{\varphi}^{k}\left(\delta, f-T_{2^{j+1}}\right) \preceq \left\|f-T_{2^{j+1}}\right\|_{\varphi}=E_{2^{j+1}}(f)_{\varphi}.$$

Using (13) and (14) and considering that the sequence of the best approximations is decreasing, we obtain

$$\begin{aligned}
\Omega_{\varphi}^{k}\left(\delta, T_{2^{j+1}}\right) & \preceq \delta^{2k}\left\|T_{2^{j+1}}^{(2k)}\right\|_{\varphi} \\
& \preceq \delta^{2k}\left\{\left\|T_{1}^{(2k)}-T_{0}^{(2k)}\right\|_{\varphi}+\sum_{i=0}^{j}\left\|T_{2^{i+1}}^{(2k)}-T_{2^{i}}^{(2k)}\right\|_{\varphi}\right\} \\
& \preceq \delta^{2k}\left\{\left\|T_{1}-T_{0}\right\|_{\varphi}+\sum_{i=0}^{j} 2^{2(i+1)2k}\left\|T_{2^{i+1}}-T_{2^{i}}\right\|_{\varphi}\right\} \\
& \preceq \delta^{2k}\left\{E_{0}(f)_{\varphi}+2^{2k} E_{1}(f)_{\varphi}+\sum_{i=1}^{j} 2^{2(i+1)2k} E_{2^{i}}(f)_{\varphi}\right\} \\
& \preceq \delta^{2k}\left\{E_{0}(f)_{\varphi}+\sum_{m=1}^{2^{j}} m^{2k-1} E_{m}(f)_{\varphi}\right\}.
\end{aligned}$$

Selecting j such that $2^{j} \leq n \leq 2^{j+1}$ we have

$$E_{2^{j+1}}(f)_{\varphi} \leq \frac{2^{4k}}{n^{2k}} \sum_{m=2^{j-1}+1}^{2^{j}} m^{2k-1} E_{m}(f)_{\varphi}.$$

□

References

[1] Akgün R. Trigonometric approximation of functions in generalized Lebesgue spaces with variable exponent. Ukr Math J 2011; 63: 1-26.

[2] Akgün R. Polynomial approximation of functions in weighted Lebesgue and Smirnov spaces with nonstandard growth. Georgian Math J 2011; 18: 203-235.

[3] Akgün R. Inequalities for one sided approximation in Orlicz spaces. Hacet J Math Stat 2011; 40: 231-240.

[4] Akgün R, Kokilashvili V. Refined estimates of trigonometric approximation for functions with generalized derivatives in weighted variable exponent Lebesgue spaces. Georgian Math J 2012; 19: 611-626.

[5] Alaouia MK, Nabilab T, Altanjia M. On some new non-linear diffusionmodels for the image filtering. Appl Anal 2014; 93: 269-280.

[6] Akgün R, Yıldırır YE. Jackson-Stechkin type inequality in weighted Lorentz spaces. Math Inequal Appl 2015; 18: 1283-1293.

[7] Benkirane A, Douieb J, Ould Mohamedhen Val M. An approximation theorem in Musielak-Orlicz-Sobolev spaces. Comment Math 2011; 51: 109-120.

[8] Bennet C, Sharpley R. Interpolation of Operators. Boston, MA, USA: Academic Press, 1968.

[9] Butzer PL, Nessel RJ. Fourier Analysis and Approximation, Volume 1. Basel, Switzerland: Birkhauser Verlag, 1971.

[10] Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image restoration. SIAM J Appl Math 2006; 66: 1383-1406.

[11] Cohen E. On the degree of approximation of a function by the partial sums of its Fourier series. T Am Math Soc 1978; 235: 35-74.

[12] Garidi W. On approximation by polynomials in Orlicz spaces. Approx Theor Appl 1991; 7: 97-110.

[13] Giannetti F, Passarelli di Napoli A. Regularity results for a new class of functionals with nonstandard growth conditions. J Differ Equations 2013; 254: 1280-1305.

[14] Guven A, Israfilov DM, Trigonometric approximation in generalized Lebesgue spaces $L^{p(x)}$. J Math Inequal 2010; 4: 285-299.

[15] Gwiazda AS. Nonlinear parabolic problems in Musielak–Orlicz spaces. Nonlinear Anal 2014; 98: 48-65.

[16] Harjulehto P, Hästö P, Latvala V, Toivanen O. Critical variable exponent functionals in image restoration. Appl Math Lett 2013; 26: 56-60.

[17] Hudzik H. On generalized Orlicz-Sobolev space. Funct Approximateo Comment Math 1976; 4: 37-51.

[18] Israfilov DM, Guven A. Approximation by trigonometric polynomials in weighted Orlicz spaces. Stud Math 2006; 174: 147-168.

[19] Israfilov DM, Kokilashvili V, Samko S. Approximation in weighted Lebesgue and Smirnov spaces with variable exponents. Proc A Razmadze Math Inst 2007; 143: 25-35.

[20] Jafarov SZ. Approximation by polynomials and rational functions in Orlicz spaces. J Comput Anal Appl 2011; 5: 953 962.

[21] Jafarov SZ, Mamedkhanov JI. On approximation by trigonometric polynomials in Orlicz spaces. Georgian Math J 2012; 4: 687 695.

[22] Kaminska A. On the translation in the Orlicz-Musielak sequence spaces. Comment Math Prace Mat 1984; 24: 77-84.

[23] Kaminska AW. Existence result for the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces. Nonlinearity 2014; 27: 685-716.

[24] Kokilashvili VM. An inverse theorem of the constructive theory of functions in Orlicz spaces. Soob Akad Nauk Gruzin SSR 1965; 37: 263-270 (in Russian).

[25] Kokilashvili VM. On approximations of periodic functions in Orlicz spaces. Bull Acad Polon Sci Sér Sci Math Astronom Phys 1966; 14: 77-80.

[26] Musielak J. Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics. Berlin, Germany: Springer, 1983.

[27] Musielak J. Approximation in modular function spaces. Dedicated to Roman Taberski on the occasion of his 70th birthday. Funct Approximateo Comment Mat 1997; 25: 45-57.

[28] Ok J. Gradient estimates for elliptic equations with $L^{p(\cdot)}LogL$ growth. Calc Var Partial Dif 2016; 55: 1-30.

[29] Ramazanov AR. On approximation by polynomials and rational functions in Orlicz spaces. Anal Math 1984; 10: 117-132.

[30] Samko S. Denseness of $C_0^\infty(R^N)$ in the generalized Sobolev spaces $W^{M,P(X)}(R^N)$. In: Gilbert RP, editor. Direct and Inverse Problems of Mathematical Physics. Dordrecht, the Netherlands: Springer, 2000, pp. 333-342.

[31] Sharapudinov II. Some problems in approximation theory in the spaces $L^{p(x)}(E)$. Anal Math 2007; 33: 135-153 (in Russian).

[32] Sharapudinov II. Approximation of smooth functions in $L_{2\pi}^{p(x)}$ by Vallee–Poussin means. Izv Saratov Univ Mat Mekh Inform 2013; 13: 45-49.

[33] Sharapudinov II. Approximation of functions in $L_{2\pi}^{p(x)}$ by trigonometric polynomials. Izv Ross Akad Nauk Ser Mat 2013; 77: 197-224 (in Russian).

[34] Taberski R. Differences, moduli and derivatives of fractional orders. Comment Math Prace Mat 1977; 19: 389-400.

[35] Tsyganok II. A generalization of a theorem of Jackson. Mat Sb 1966; 71: 257-260 (in Russian).

[36] Yildirir YE. Approximation of periodic functions in weighted Orlicz spaces. Glas Mat 2012; 47: 401-413.

[37] Yildirir YE, Israfilov DM. The properties of convolution type transforms in weighted Orlicz spaces. Glas Mat 2010; 45: 461-474.

[38] Zygmund A. Trigonometric Series. Cambridge, UK: Cambridge University Press, 1968.