NUMERICAL INVERSE LAPLACE HOMOTOPY TECHNIQUE FOR FRACTIONAL HEAT EQUATIONS

by

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In this paper, we have aimed the numerical inverse Laplace homotopy technique for solving some interesting 1-D time-fractional heat equations. This method is based on the Laplace homotopy perturbation method, which is combined form of the Laplace transform and the homotopy perturbation method. Firstly, we have applied to the fractional 1-D PDE by using He's polynomials. Then we have used Laplace transform method and discussed how to solve these PDE by using Laplace homotopy perturbation method. We have declared that the proposed model is very efficient and powerful technique in finding approximate solutions to the fractional PDE.

Key words: *fractional heat equation, Laplace transform, Caputo derivative, homotopy perturbation method*

Introduction

Fractional differential equations have an important role in modelling and describing certain problems such as diffusion processes, chemistry, engineering, economic, material sciences, and other areas of application. Fractional calculations and approximate analytical solution methods are used extensively in the solution of real life problems, especially in mathematical and physical systems of problems. Among them, the fractional models were used to analyze the model of regularized long-wave equation arising in ion acoustic plasma waves [1], to solve KdV Burgers problem [2] to evaluate the pricing the financial instruments [3-5], to obtain the analytical solutions of non-linear coupled Boussinesq-Burger's equations arise in propagation of shallow water waves performance [6], to interpret holomorphic solutions for non-linear singular fractional differential equations [7], to calculate the numerical solutions of space-time fractional diffusion equation [8], *etc.* In addition, these fractional models were used to demonstrate the applications of approximate-analytical and numerical solution methods. Fractional techniques are also used to model the solution of optimal control problem [9], constrained optimization problem [10], portfolio optimization problem [11], diffusion-wave problem [12], a transient heat diffusion equation with relaxation term expressed through the Caputo-Fabrizio time fractional derivative [13], heat problem on radial symmetric plate [14], *etc.*

On the other hand, several researchers [2, 15-18] have applied the homotopy perturbation method (HPM), homotopy analysis method (HAM), and Laplace transform method to

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solve different kinds of fractional ODE, fractional PDE, integral equations (IE), and integro-differential equations (IDE). In addition Javidi and Ahmad [19] proposed a numerical method which is based on HPM and Laplace transform for fractional PDE. In [20], Laplace homotopy perturbation method (LHPM) which is a combination of the HPM and Laplace transform (LT) has been used for solving 1-D PDE. Talbot [21] improved direct numerical inversion of LT. The interested researchers are also referred to different applications of fractional PDE solutions [22-28].

In this paper, the method for numerical solution of fractional PDE is investigated with the HPM and Stehfest's numerical algorithm [29] in order to calculate inverse LT. When looking at the results, it is obvious that the method is very effective and accurate for solving fractional PDE.

Some preliminaries

In this section, we first give some basic definitions of fractional calculus and LT.

Definition 1. The Riemann-Liouville fractional integral and derivative of order $\alpha > 0$ for a function $f \in C_{-1}^{m}$ with $m \in N$ is defined as, respectively:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad \alpha > 0, \ t > 0 \qquad (1)$$

$$J^0 f(t) = f(t)$$

$$\mathrm{D}^{\alpha} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} J^{m-\alpha} f(t), \quad m-1 < \alpha \le m, \ m \in N \qquad (2)$$

Definition 2. The LT of $\mathrm{D}^{\alpha} f(t)$ can be defined:

$$L\left[\mathrm{D}^{\alpha} f(t)\right] = L\left[J^{m-\alpha} f^{(m)}(t)\right] = L\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,\mathrm{d}\tau\right] =$$

$$= \frac{1}{s^{m-\alpha}} L\left[f^{(m)}(t)\right] = \frac{1}{s^{m-\alpha}} \left\{ s^m L\left[f(t)\right] - s^{m-1} f(0) - s^{m-2} f'(0) - \cdots - f^{(m-1)}(0) \right\} \qquad (3)$$

Description of the proposed method

Consider the following linear 1-D fractional PDE:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + A(x) \frac{\partial u(x,t)}{\partial x} + B(x) \frac{\partial^2 u(x,t)}{\partial x^2} + C(x) u(x,t) = v(x,t), \quad (x,t) \in [0,1] \times [0,T] \qquad (4)$$

with the initial conditions:

$$\frac{\partial^k u}{\partial t^k}(x,0) = f_k(x), \quad k = 0,1,\ldots,m-1 \qquad (5)$$

and the boundary conditions:

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad t \ge 0 \qquad (6)$$

where f_k, $k = 0,1,\ldots,m-1$, v, g_0, g_1, A, B, and C are known functions and $T > 0$ is a real number and $m-1 < \alpha \le m$. We define the method of solution for solving problem of eqs. (4)-(6).

Getting the LT of problem (4)-(6) and using eq. (3), we obtain:

$$\frac{1}{s^{m-\alpha}}\left[s^m\Psi(x,s)-s^{m-1}f_0(x)-s^{m-2}f_1(x)-\cdots-f_{m-1}(x)\right]+$$
$$+\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi(x,s)=\bar{v}(x,s) \tag{7}$$

where $\Psi(x,s)$ and $\bar{v}(x,s)$ represent the LT of $u(x,t)$ and $v(x,t)$, respectively, and we obtain the LT of boundary conditions $g_0(t)$ and $g_1(t)$:

$$\Psi(0,s)=L[g_0(t)],\quad \Psi(1,s)=L[g_1(t)]$$

Considering the eq. (7), we reach:

$$s^\alpha\Psi(x,s)=-\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi(x,s)+$$
$$+\frac{1}{s^{m-\alpha}}\left[s^{m-1}f_0(x)-s^{m-2}f_1(x)-\cdots-f_{m-1}(x)\right]+\bar{v}(x,s) \tag{8}$$

Applying the HPM [30-33] to eq. (8), we build a homotopy:

$$\Psi(x,s)=-\frac{p}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi(x,s)+$$
$$+\frac{1}{s^m}\left[s^{m-1}f_0(x)+s^{m-2}f_1(x)+\cdots+f_{m-1}(x)\right]+\frac{1}{s^\alpha}\bar{v}(x,s) \tag{9}$$

Then the solution of eq. (9) can be represented:

$$\overline{\Psi}(x,s)=\sum_{j=0}^{\infty}p^j\Psi_j(x,s) \tag{10}$$

where $\Psi_j(x,s)$, $j=0,1,2,...,$ are the unknown functions. Substituting the eq. (10) in eq. (9), we have:

$$\sum_{j=0}^{\infty}p^j\Psi_j(x,s)=-\frac{p}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\sum_{j=0}^{\infty}p^j\Psi_j(x,s)+$$
$$+\frac{1}{s^m}\left[s^{m-1}f_0(x)+s^{m-2}f_1(x)+\cdots+f_{m-1}(x)\right]+\frac{1}{s^\alpha}\bar{v}(x,s) \tag{11}$$

By comparing the coefficients of powers of p, we obtain the homotopies:

$$p^0:\Psi_0(x,s)=\frac{1}{s^m}\left[s^{m-1}f_0(x)+s^{m-2}f_1(x)+\cdots+f_{m-1}(x)\right]+\frac{1}{s^\alpha}\bar{v}(x,s)$$
$$p^1:\Psi_1(x,s)=-\frac{1}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi_0(x,s)$$
$$p^2:\Psi_2(x,s)=-\frac{1}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi_1(x,s) \tag{12}$$
$$\vdots$$
$$p^{n+1}:\Psi_{n+1}(x,s)=-\frac{1}{s^\alpha}\left[A(x)\frac{\partial}{\partial x}+B(x)\frac{\partial^2}{\partial x^2}+C(x)\right]\Psi_n(x,s)$$

When the $p \to 1$, we see that the eq. (12) gives the approximate solution for the problem of eqs. (4)-(6) and the solution is given:

$$H_n(x,s) = \sum_{j=0}^{n} \Psi_j(x,s) \tag{13}$$

If we take the inverse LT of eq. (13), we obtain:

$$u(x,t) \cong u_n(x,t) = L^{-1}\left[H_n(x,s)\right] \tag{14}$$

As the last work, applying Stehfest's algorithm [29] to $H_n(x,s)$, the solution $u(x,t)$ is found:

$$u_n(x,t) = \frac{\ln(2)}{t}\sum_{j=1}^{2p} d_j H_n\left[x, j\frac{\ln(2)}{t}\right]$$

where p is a positive integer and

$$d_j = (-1)^{j+p} \sum_{i=\left[\!\left[\frac{j+1}{2}\right]\!\right]}^{\min(j,p)} \frac{i^p (2i)!}{(p-i)!(i-1)!(j-i)!(2i-j)!i!}$$

Here $[\![k]\!]$ represents the integer part of the real number k.

Numerical examples

In order to illustrate the effectiveness and appropriateness of the method proposed in this paper, several numerical examples are carried out in this section.

Example 1. Let us consider the following 1-D inhomogeneous fractional Burgers equation [34]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad t > 0, \quad x \in R, \quad 0 < \alpha \le 1 \tag{15}$$

with the initial condition:

$$u(x,0) = x^2 \tag{16}$$

We know that the exact solution of this problem for $\alpha = 1$ is given [34]:

$$u(x,t) = x^2 + t^2 \tag{17}$$

By using the proposed method and the eq. (12), we find that:

$$\begin{aligned}
&p^0 : \Psi_0(x,s) = \frac{x^2}{s} + \frac{2}{s^\alpha}\left[\frac{1}{s^{3-\alpha}} + \frac{x-1}{s}\right] \\
&p^1 : \Psi_1(x,s) = -\frac{1}{s^\alpha}\left[\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right]\Psi_0(x,s) = -\frac{1}{s^\alpha}\left[\frac{2}{s^{\alpha+1}} + \frac{2x-2}{s}\right] \\
&p^2 : \Psi_2(x,s) = -\frac{1}{s^\alpha}\left[\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\right]\Psi_1(x,s) = \frac{2}{s^{2\alpha+1}}
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&p^3:\Psi_3(x,s)=-\frac{1}{s^\alpha}\left[\frac{\partial}{\partial x}-\frac{\partial^2}{\partial x^2}\right]\Psi_2(x,s)=0\\
&\vdots\\
&p^{n+1}:\Psi_{n+1}(x,s)=-\frac{1}{s^\alpha}\left[\frac{\partial}{\partial x}-\frac{\partial^2}{\partial x^2}\right]\Psi_n(x,s)=0
\end{aligned}
\tag{18}
$$

and so on. By using the eq. (13), we see that:

$$H_n(x,s)=\frac{x^2}{s}+\frac{2}{s^3} \tag{19}$$

Taking the inverse LT of eq. (19), we observe that the approximate solution of eqs. (15) and (16) obtained by using LHPM is the same with the analytical solution in eq. (17):

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\lim_{n\to\infty}\left\{L^{-1}\left[H_n(x,s)\right]\right\}=x^2+t^2 \tag{20}$$

Example 2. We suppose the following 1-D linear inhomogeneous fractional heat equation [34]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^2 u}{\partial x^2}+u=6x^3t+\left(x^3-6x\right)t^3,\quad t>0,\quad x\in R,\quad 1<\alpha\le 2 \tag{21}$$

subject to the initial conditions:

$$u(x,0)=0,\quad \frac{\partial u}{\partial t}(x,0)=0 \tag{22}$$

By using the method presented in eq. (12), we obtain:

$$
\begin{aligned}
&p^0:\Psi_0(x,s)=\frac{6}{s^\alpha}\left[\frac{x^3}{s^2}+\frac{x^3-6x}{s^4}\right]\\
&p^1:\Psi_1(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_0(x,s)=\frac{36x-6x^3}{s^{2\alpha+2}}+\frac{72x-6x^3}{s^{2\alpha+4}}\\
&p^2:\Psi_2(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_1(x,s)=\frac{6x^3}{s^{3\alpha+2}}+\frac{6x^3-36x}{s^{3\alpha+4}}\\
&p^3:\Psi_3(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_2(x,s)=\frac{36x-6x^3}{s^{4\alpha+2}}+\frac{72x-6x^3}{s^{4\alpha+4}}\\
&\vdots\\
&p^{n+1}:\Psi_{n+1}(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_n(x,s)=\begin{cases}\dfrac{36x-6x^3}{s^{(n+1)\alpha+2}}+\dfrac{72x-6x^3}{s^{(n+1)\alpha+4}}, & n=2k+1\\[2ex] \dfrac{6x^3}{s^{(n+1)\alpha+2}}+\dfrac{6x^3-36x}{s^{(n+1)\alpha+4}}, & n=2k\end{cases}
\end{aligned}
\tag{23}
$$

As previous, using the eq. (13), we observe:

$$H_n(x,s)=6\left[\frac{x^3}{s^{\alpha+2}}+\frac{x^3-6x}{s^{\alpha+4}}+\frac{6x-x^3}{s^{2\alpha+2}}+\frac{12x-x^3}{s^{2\alpha+4}}+\frac{x^3}{s^{3\alpha+2}}+\frac{x^3-6x}{s^{3\alpha+4}}+\cdots\right] \tag{24}$$

Taking the inverse Laplace transform of eq. (24) and taking the limit $n\to\infty$, the approximate solution for problem of eqs. (21) and (22) is given:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=L^{-1}\left[\lim_{n\to\infty}H_n(x,s)\right]=$$
$$=\sum_{k=0}^{\infty}\frac{6x^3t^{(2k+1)\alpha+1}}{\Gamma\left[(2k+1)\alpha+2\right]}+\sum_{k=0}^{\infty}\frac{6\left(x^3-6x\right)t^{(2k+1)\alpha+3}}{\Gamma\left[(2k+1)\alpha+4\right]}+$$
$$+\sum_{k=1}^{\infty}\frac{6(6x-x^3)t^{(2k+1)\alpha+1}}{\Gamma\left[(2k+1)\alpha+2\right]}+\sum_{k=1}^{\infty}\frac{6(12x-x^3)t^{2k\alpha+1}}{\Gamma(2k\alpha+2)} \tag{25}$$

In eq. (25), the solution of classical Klein-Gordon equation when $\alpha=2$ is given:

$$u(x,t)=x^3t^3+\left(x^3-6x\right)\frac{6t^5}{\Gamma(6)}+36x\frac{t^5}{\Gamma(6)}-36x\frac{t^7}{\Gamma(8)}-6x^3\frac{t^5}{\Gamma(6)}-\left(x^3-6x\right)\frac{6t^7}{\Gamma(8)}+\cdots \tag{26}$$

Canceling the noise terms in eq. (26) the exact solution of eq. (21), for the special case $\alpha=2$, as follows:

$$u(x,t)=x^3t^3 \tag{27}$$

The results obtained as approximation solutions for the solution of eq. (21) with the initial conditions of eq. (22) for various values of α and t are presented in figs. 1 and 2. In figs. 1 and 2, the space variable, x is regarded as 0.5 and 1.0, respectively. According to these figures, the obtained approximation solution results coincide with the exact solution results.

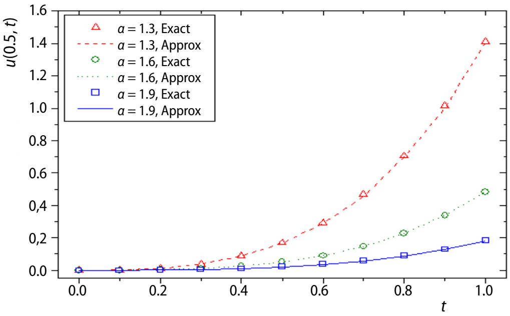

Figure 1. Numerical and exact solutions for *Example 2* at x = 0.5

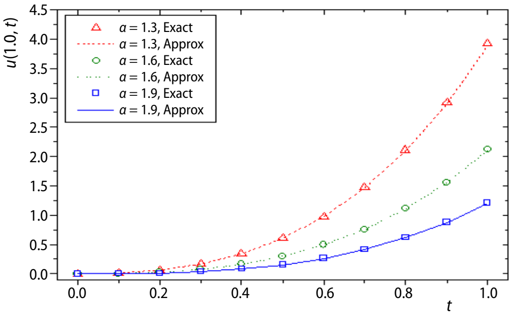

Figure 2. Numerical and exact solutions for *Example 2* at x = 1.0

Example 3. Consider the following 1-D inhomogeneous fractional heat equation [35]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+u=\frac{\partial^2 u}{\partial x^2}+\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}x(2-x)+x(2-x)t^2+2t^2,\quad t>0,\quad 0\le x\le 2,\quad 0<\alpha\le 1 \tag{28}$$

with the initial condition:

$$u(x,0)=0 \tag{29}$$

and the boundary condition:

$$u(0,t)=u(2,t)=0 \tag{30}$$

We know that the exact solution of eqs. (28)-(30) is given:

$$u(x,t)=x(2-x)t^2+\sum_{k=0}^{\infty}\frac{8t^{(2k+1)\alpha+2}}{\Gamma\left[(2k+1)\alpha+3\right]} \tag{31}$$

that equals to $u(x,t)=x(2-x)t^2$ when $\alpha=1$. Now, if we use the method in eq. (12), we have the homotopies problem of the eqs. (28)-(30):

$$\begin{aligned}
&p^0:\Psi_0(x,s)=\frac{2x(2-x)}{s^3}+\frac{2x(2-x)+4}{s^{\alpha+3}}\\
&p^1:\Psi_1(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_0(x,s)=\frac{2x^2-4x+4}{s^{\alpha+3}}+\frac{2x^2-4x}{s^{2\alpha+3}}\\
&p^2:\Psi_2(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_1(x,s)=\frac{4x-2x^2}{s^{2\alpha+3}}+\frac{4x+4-2x^2}{s^{3\alpha+3}}\\
&p^3:\Psi_3(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_2(x,s)=\frac{2x^2-4x+4}{s^{3\alpha+3}}+\frac{2x^2-4x}{s^{4\alpha+3}}\\
&\vdots\\
&p^{n+1}:\Psi_{n+1}(x,s)=-\frac{1}{s^\alpha}\left[-\frac{\partial^2}{\partial x^2}+1\right]\Psi_n(x,s)=\begin{cases}\dfrac{2x^2-4x+4}{s^{n\alpha+3}}+\dfrac{2x^2-4x}{s^{(n+1)\alpha+3}}, & n=2k+1\\[2ex] \dfrac{4x-2x^2}{s^{n\alpha+3}}+\dfrac{4-2x^2+4x}{s^{(n+1)\alpha+3}}, & n=2k\end{cases}
\end{aligned} \tag{32}$$

By means of eq. (13), we obtain:

$$H_n(x,s)=\frac{2}{s^3}\left(2x-x^2\right)+\frac{8}{s^3}\left(\frac{1}{s^\alpha}+\frac{1}{s^{3\alpha}}+\frac{1}{s^{5\alpha}}+\cdots+\frac{1}{s^{n\alpha}}\right) \tag{33}$$

Taking the inverse LT of eq. (33), the approximate solution of problem in eqs. (28)-(30) is given:

$$u_n(x,t)=L^{-1}\left[H_n(x,s)\right]=x(2-x)t^2+\sum_{k=0}^{n}\frac{8t^{(2k+1)\alpha+2}}{\Gamma\left[(2k+1)\alpha+3\right]} \tag{34}$$

that, on taking the limit $n\to\infty$, holds:

$$\begin{aligned}
u(x,t)&=\lim_{n\to\infty}u_n(x,t)=\lim_{n\to\infty}\left\{L^{-1}\left[H_n(x,s)\right]\right\}=\\
&=x(2-x)t^2+\sum_{k=0}^{\infty}\frac{8t^{(2k+1)\alpha+2}}{\Gamma\left[(2k+1)\alpha+3\right]}
\end{aligned} \tag{35}$$

The results obtained for the solution of eq. (28) for various values of α are presented in tab. 1 and fig. 3. According to tab. 1, it can be say that the numerical results found are very close to the exact solution results. Also it is obvious that fig. 3 says that the numerical results are very similar to the exact solution one.

Table 1. Absolute errors $|u(x,t) - u_n(x,t)|$ by LPHM for *Example 3*

x	α	t				
		0.1	0.3	0.5	0.7	0.9
0.1	α = 0.25	8.20E–10	1.70E–10	1.90E–09	4.00E–10	1.70E–09
	α = 0.40	2.35E–09	5.74E–09	2.15E–09	1.11E–09	2.07E–08
	α = 0.75	2.12E–07	1.12E–08	2.93E–09	2.15E–08	2.78E–08
0.3	α = 0.25	5.43E–08	6.59E–09	2.32E–09	3.15E–09	2.45E–07
	α = 0.40	3.24E–09	1.52E–08	2.56E–09	5.74E–07	5.69E–09
	α = 0.75	2.62E–08	9.24E–09	3.14E–08	7.15E–09	4.47E–09
0.5	α = 0.25	1.87E–09	1.87E–09	2.09E–09	6.85E–09	2.00E–09
	α = 0.40	4.54E–07	7.16E–09	2.89E–07	4.23E–07	1.96E–08
	α = 0.75	9.22E–09	8.12E–08	1.00E–10	3.45E–09	4.78E–08
0.7	α = 0.25	1.30E–09	1.00E–10	2.47E–09	1.70E–07	2.00E–09
	α = 0.40	2.87E–09	3.75E–09	2.76E–07	1.87E–08	6.45E–09
	α = 0.75	3.91E–09	7.77E–08	3.15E–09	1.69E–09	5.65E–07
0.9	α = 0.25	8.29E–10	1.32E–08	2.40E–07	1.70E–08	2.00E–09
	α = 0.40	2.19E–09	1.45E–09	9.12E–08	2.41E–09	2.19E–08
	α = 0.75	1.13E–08	5.41E–09	2.00E–10	2.75E–09	7.00E–09

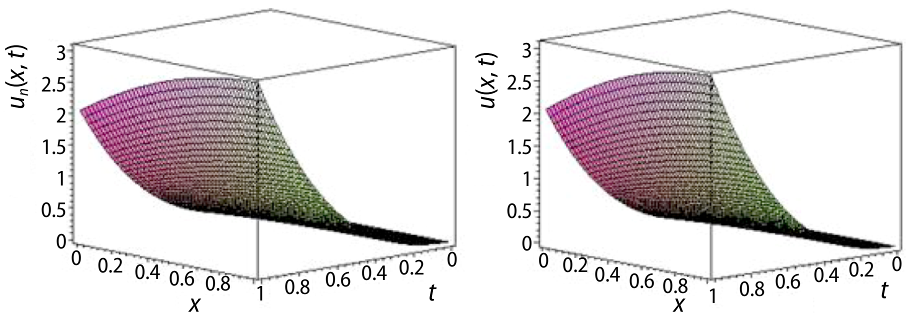

Figure 3. Solution functions of $u_n(x,t)$ and $u(x,t)$ for *Example 3*

Concluding remarks

The fundamental aim of this paper is to construct an approximation solution of linear PDE of fractional order. The goal has been achieved by applying the HPM with LT to several fractional PDE. The present work has verified the validity and effectiveness of the LHPM using the three different examples. At the first example, LHPM has a very important result because it has given the exact solution of the problem. At the second and third examples, the proposed method has given very close results to the exact solutions.

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