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Some Fixed-Circle Theorems and Discontinuity at Fixed Circle

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Abstract. In this study, we give some existence and uniqueness theorems for fixed circles of self-mappings on a metric space with some illustrative examples. Recently, real-valued neural networks with discontinuous activation functions have been a great importance in practice. Hence we give some new results for discontinuity at fixed circle on a metric space.

INTRODUCTION

It is known that the fixed-point theory has been generalized by various aspects (see [1] and the references therein). Recently, a new approach so called "fixed-circle problem" has been introduced and studied on metric and S-metric spaces. For example, it was presented some existence and uniqueness theorems for fixed circles of self-mappings on metric spaces with geometric interpretation (see [2], [3] and [4]). Also there are some applications of fixed-circle problems for discontinuity at fixed point to activation functions (see [5] for more details). Therefore, it is important to study new fixed-circle theorems on various metric spaces.

On the other hand, some solutions to the open question on the existence of contractive conditions which are strong enough to generate a fixed point but which does not force the mapping to be continuous at the fixed point has been proposed and investigated (see [5], [6], [7], [8], [9] and [10]). The obtained results can be applied to neural nets and fixed-circle theory under suitable conditions (see [5], [11], [12] and [13]).

Motivated by the above studies, our aim in this paper is to obtain a new fixed-circle theorem and examine some related results. We find a new solution of the open problem related to discontinuity at fixed point on metric spaces. Also, we propose an application to activation functions used in neural networks using the obtained fixed-circle and discontinuity results.

A NEW FIXED-CIRCLE THEOREM AND SOME RELATED RESULTS

At first, we recall the following definition and the fixed-circle theorem:

Definition 1. *[2] Let (X, d) be a metric space and $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ be a circle with the center x_0 and the radius r. For a self-mapping $T : X \to X$, if $Tx = x$ for every $x \in C_{x_0,r}$ then the circle $C_{x_0,r}$ is called the fixed circle of T.*

Theorem 2. *[2] Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let us define the mapping $\varphi : X \to [0, \infty)$ as*

$$\varphi(x) = d(x, x_0),$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

1. $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$,

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2. $d(Tx, x_0) \geq r$,

for each $x \in C_{x_0,r}$, then the circle $C_{x_0,r}$ is a fixed circle of T.

Notice that the converse statement of this theorem is also true.

Now we give the following fixed-circle result for the existence of a fixed circle.

Theorem 3. *Let (X, d) be a metric space, $\mathbb{R}$ be the set of all real numbers and $C_{x_0,r}$ be any circle on X. Let us define the mapping $\varphi_r : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ as*

$$\varphi_r(u) = \begin{cases} u - r & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u = 0 \end{cases}, \tag{1}$$

for all $u \in \mathbb{R}^+ \cup \{0\}$. If there exists a self-mapping $T : X \to X$ satisfying

1. $d(Tx, x_0) = r$ *for each* $x \in C_{x_0,r}$,
2. $d(Tx, Ty) > r$ *for each* $x, y \in C_{x_0,r}$ *and* $x \neq y$,
3. $d(Tx, Ty) \leq d(x, y) - \varphi_r(d(x, Tx))$ *for each* $x, y \in C_{x_0,r}$,

then the circle $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}$ be an arbitrary point. By the condition (1), we have $Tx \in C_{x_0,r}$ for all $x \in C_{x_0,r}$. Now we prove that x is a fixed point of T. On the contrary, let us assume that $x \neq Tx$. At first, using the condition (2), we find

$$d(Tx, T^2x) > r. \tag{2}$$

Using the condition (3), we have

$$d(Tx, T^2x) \quad \leq \quad d(x, Tx) - \varphi_r(d(x, Tx)) = d(x, Tx) - d(x, Tx) + r = r. \tag{3}$$

Combining the inequalities (2) and (3), we get a contradiction. Hence it should be $Tx = x$. Consequently, the circle $C_{x_0,r}$ is a fixed circle of T. □

The elements of circles change according to the corresponding metric, the radius and the center. In this context, we give the following remarks and examples.

Remark 4. (1) *In Theorem 3, the center of the circle $C_{x_0,r}$ need not to be fixed.*

(2) *The converse statement of Theorem 3 is also true when the circle has only one element.*

Example 5. *Let $X = \mathbb{R}$ and the function $d : X \times X \to \mathbb{R}$ be defined by*

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 2\left(|x| + |y|\right) & \text{if} \quad x \neq y \end{cases},$$

for all $x, y \in \mathbb{R}$. Then d defines a metric on $\mathbb{R}$ and $(\mathbb{R}, d)$ is a metric space.

The circle $C_{1,2}$ is obtained as follows:

$$C_{1,2} = \{x \in \mathbb{R} : d(x, 1) = 2\} = \{0\}.$$

If we consider the self-mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = \begin{cases} \alpha & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x \neq 1 \end{cases},$$

for all $x \in \mathbb{R}$ and $\alpha \neq 1$, then the self-mapping T satisfies the conditions of Theorem 3 and T fixes the circle $C_{1,2}$. In other words, the self-mapping T has the unique fixed point $x = 0$. Notice that the center 1 of the circle $C_{1,2}$ is not fixed by the self-mapping T.

Remark 6. (1) *Notice that the converse statement of Theorem 3 is not true everywhen even if the condition* (1) *is satisfied by the self-mapping T (see Example 7).*

(2) *The converse statement of Theorem 2 is true while the converse statement of Theorem 3 is not true, everywhen (see Example 7).*

(3) *In Theorem 3, by the condition* (1)*, we have $T(C_{x_0,r}) \subseteq C_{x_0,r}$. In some cases, the circle need not to be fixed even if $T(C_{x_0,r}) = C_{x_0,r}$ (see Example 8).*

Now we give some illustrative examples.

Example 7. *Let $\mathbb{C}$ be the set of all complex numbers and $(\mathbb{C}, d)$ be the usual metric space. Let us consider the circle $C_{0,\frac{1}{2}}$ and define the self-mapping $T_1 : \mathbb{C} \to \mathbb{C}$ as*

$$T_1 z = \begin{cases} \frac{1}{4\overline{z}} & if \quad z \neq 0 \\ 0 & if \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$, where $\overline{z}$ denotes the complex conjugate of the complex number z. Clearly, we have $T_1(C_{0,\frac{1}{2}}) = C_{0,\frac{1}{2}}$. It can be easily checked that the self-mapping T_1 does not satisfy the condition (2) *of Theorem 3 (for example, for the points $x = \frac{\sqrt{2}-i\sqrt{2}}{4}$ and $y = \frac{1}{2}$). But, an easy computation shows that T_1 fixes the circle $C_{0,\frac{1}{2}}$. Notice that the conditions of Theorem 2 are satisfied by the self-mapping T_1.*

Example 8. *Let $(\mathbb{C}, d)$ be the usual metric space. Let us consider the circle $C_{0,\frac{1}{2}}$ and define the self-mapping $T_2 : \mathbb{C} \to \mathbb{C}$ as*

$$T_2 z = \begin{cases} \frac{1}{4z} & if \quad z \neq 0 \\ 0 & if \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$. Then we have $T_2(C_{0,\frac{1}{2}}) = C_{0,\frac{1}{2}}$. But, the self-mapping T_2 does not satisfy the conditions (2) *and* (3) *of Theorem 3 (for example, for the points $x = \frac{\sqrt{2}+i\sqrt{2}}{4}$, $y = \frac{1}{2}$ and $x = \frac{i}{2}$, $y = -\frac{i}{2}$, respectively). Clearly, the circle $C_{0,\frac{1}{2}}$ is not a fixed circle of T_2 since $T_2 \frac{i}{2} = -\frac{i}{2}$. More precisely, T_2 fixes only the points $-\frac{1}{2}$ and $\frac{1}{2}$ on the circle $C_{0,\frac{1}{2}}$.*

In the following example, we see that there exists a self-mapping having more than one fixed circle.

Example 9. *Let $(\mathbb{R}, d)$ be the usual metric space. Let us consider the circles $C_{0,2}$ and $C_{3,1}$ and the self-mapping $T_3 : \mathbb{R} \to \mathbb{R}$ as*

$$T_3 x = \begin{cases} \frac{x}{x+3} & if \quad x \in (-\infty, 2) \\ \frac{57x-36}{3x+36} & if \quad x \in (2, \infty) \\ 2 & if \quad x = 2 \end{cases},$$

for all $x \in \mathbb{R}$. It can be easily checked that the self-mapping T_3 satisfies the conditions of Theorem 3 and that the circles $C_{0,2}$ and $C_{3,1}$ are the fixed circles of T_3.

Remark 10. (1) *In Example 9, we see that the self-mapping T has two fixed circles, that is, the fixed circle does not have to be unique.*

(2) *If we consider the circles $C_{0,2}$ and $C_{3,1}$ given in Example 9, then the fixed circles do not have to be disjoint.*

Now we give a uniqueness condition of a fixed circle.

Theorem 11. *Let (X, d) be a metric space and $C_{x_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping which fixes the circle $C_{x_0,r}$ and*

$$N^*(x, y) = \max \{d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), \alpha d(Ty, x)\}, \tag{4}$$

where $\alpha \in [0, 1)$. If the condition

$$d(Tx, Ty) < N^*(x, y), \tag{5}$$

is satisfied by T for all $x \in C_{x_0,r}$ and $y \in X \setminus C_{x_0,r}$, then $C_{x_0,r}$ is the unique fixed circle of T.

Proof. Assume that there exists another fixed circle $C_{x_1,\rho}$ of the self-mapping T. Let $x \in C_{x_0,r}$, $y \in C_{x_1,\rho}$ and $x \neq y$ be arbitrary points. We show that $d(x,y) = 0$ and so $x = y$. Using the inequality (5), we get

$$\begin{aligned} d(x,y) &= d(Tx,Ty) < N^*(x,y) \\ &= \max\{d(x,y), d(Tx,x), d(Ty,y), d(Tx,y), \alpha d(Ty,x)\} \\ &= \max\{d(x,y), d(x,x), d(y,y), d(x,y), \alpha d(y,x)\} \\ &= d(x,y), \end{aligned}$$

which is a contradiction. Consequently, it should be $x = y$ for all $x \in C_{x_0,r}$, $y \in C_{x_1,\rho}$ and so $C_{x_0,r}$ is the unique fixed circle of T. □

Now we consider the identity map $I_X : X \to X$ defined as $I_X(x) = x$ for all $x \in X$. We note that the identity map satisfies the conditions of Theorem 2 but does not satisfy the conditions of Theorem 3 in general. Therefore we investigate a condition which excludes the identity map in Theorem 2 (resp. Theorem 3). For this purpose, using the definition of the functions φ and φ_r, we obtain the following theorems.

Theorem 12. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping having a fixed circle $C_{x_0,r}$ and the mapping φ be defined as in Theorem 2. The self-mapping T satisfies the condition*

$$d(x,Tx) \leq h\left[\varphi(x) - \varphi(Tx)\right], \tag{6}$$

for all $x \in X$ and some $h \in (0,1)$ if and only if $T = I_X$.

Proof. Let $x \in X$ be any point and assume that $Tx \neq x$. Then using the inequality (6) and the triangle inequality we obtain

$$\begin{aligned} d(x,Tx) &\leq h\left[\varphi(x) - \varphi(Tx)\right] = h\left[d(x,x_0) - d(Tx,x_0)\right] \\ &\leq h\left[d(x,Tx) + d(Tx,x_0) - d(Tx,x_0)\right] \\ &= hd(x,Tx), \end{aligned}$$

which is a contradiction since $h \in (0,1)$. Hence we get $Tx = x$ and $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (6). □

Corollary 13. *If a self-mapping $T : X \to X$ satisfies the conditions of Theorem 2 (resp. Theorem 3) and does not satisfy the condition (6) then $T \neq I_X$.*

Theorem 14. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping having a fixed circle $C_{x_0,r}$ and the mapping φ_r be defined as in Theorem 3. The self-mapping $T : X \to X$ satisfies the condition*

$$d(x,Tx) < \varphi_r(d(x,Tx)) + r, \tag{7}$$

for all $x \in X$ if and only if $T = I_X$.

Proof. Let $x \in X$ be any point and assume that $Tx \neq x$. Then using the inequality (7), we get

$$d(x,Tx) < \varphi_r(d(x,Tx)) + r = d(x,Tx) - r + r,$$

which is a contradiction. Hence we have $Tx = x$ and $T = I_X$.

Conversely, it is clear that the identity map I_X satisfies the condition (7). □

Corollary 15. *If a self-mapping $T : X \to X$ satisfies the conditions of Theorem 2 (resp. Theorem 3) and does not satisfy the condition (7) then $T \neq I_X$.*

Theorem 16. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping having a fixed circle $C_{x_0,r}$, the mapping φ be defined as in Theorem 2 and the mapping φ_r be defined as in Theorem 3. T satisfies the condition (6) if and only if T satisfies the condition (7).*

Proof. If T satisfies the condition (6) then by Theorem 12 we have $T = I_X$ and so by the converse statement of Theorem 14, T satisfies the condition (7). Using similar arguments, the converse statement follows easily. □

A DISCONTINUITY RESULT AT FIXED POINT AND AN APPLICATION

In this section, we obtain a new discontinuity result at fixed point and give an application of the obtained result to activation function. At first we recall the following definition given in [5]:

$$N(x,y) = \max\{d(x,y), d(Tx,x), d(Ty,y), d(Tx,y), d(Ty,x)\}. \tag{8}$$

Now we give a new discontinuity result using the uniqueness condition (4) and the above condition (8).

Theorem 17. *Let (X,d) be a complete metric space, $N^*(x,y)$ be defined as in (4), $N(x,y)$ be defined as in (8) and T be a self-mapping on X satisfying the following conditions:*

1. *There exists a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(t) < t$ for each $t > 0$ and $d(Tx,Ty) \leq \frac{1}{2}\psi(N^*(x,y))$.*
2. *There exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < N(x,y) < \varepsilon + \delta$ implies $d(Tx,Ty) \leq \varepsilon$ for a given $\varepsilon > 0$.*

Then T has a unique fixed point $u \in X$ and $T^n x \to u$ for each $x \in X$. Also, T is discontinuous at y_0 if and only if $\lim\limits_{x\to u} N^*(x,u) \neq 0$.

Proof. Let $x_0 \in X$, $x \neq Tx$ and the sequence $\{x_n\}$ be defined as $T^n x_0 = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Using the condition (1), we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \frac{1}{2}\psi(N^*(x_{n-1}, x_n)) < \frac{1}{2}N^*(x_{n-1}, x_n) \\ &< \frac{1}{2}[d(x_{n-1}, x_n) + d(x_{n+1}, x_n)] \end{aligned}$$

and so

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

If we put $d(x_n, x_{n+1}) = s_n$ then in view of the above inequality we have

$$s_n < s_{n-1}, \tag{9}$$

that is, s_n is a strictly decreasing sequence of positive real numbers and so the sequence s_n tends to a limit $s \geq 0$. Assume that $s > 0$. There exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$s < s_n < s + \delta(s). \tag{10}$$

Using the condition (2) and the inequality (9), we obtain

$$d(x_n, x_{n+1}) = s_n < s, \tag{11}$$

for $n \geq k$. The inequality (11) contradicts to the inequality (10). Hence it should be $s = 0$.

Now we prove that $\{x_n\}$ is a Cauchy sequence. Let us fix an $\varepsilon > 0$. Without loss of generality, we can assume that $\delta(\varepsilon) < \varepsilon$. There exists $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) = s_n < \frac{\delta}{2},$$

for $n \geq k$ since $s_n \to 0$. Following Jachymski's method (see [14] and [15] for more details), using the mathematical induction, we get

$$d(x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2}. \tag{12}$$

Using the inequality (12), definition of $N(x,y)$ and following similar arguments used in the proof of Theorem 2.7 given in [5] we have

$$N(x_k, x_{k+n}) < \varepsilon + \delta.$$

From the condition (2), we find

$$d(Tx_k, Tx_{k+n}) \leq \varepsilon.$$

Therefore the inequality (12) implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Also we get $Tx_n \to u$. Now we show that $Tu = u$. On the contrary, assume that u is not a fixed point of T, that is, $Tu \neq u$. Then using the condition (1) we get

$$\begin{aligned} d(Tu, Tx_n) &\leq \frac{1}{2}\psi\left(N^*(u, x_n)\right) < \frac{1}{2}N^*(u, x_n) \\ &= \frac{1}{2}\max\left\{ \begin{array}{c} d(u, x_n), d(Tu, u), d(Tx_n, x_n), \\ d(Tu, x_n), \alpha d(Tx_n, u) \end{array} \right\} \end{aligned}$$

and so taking limit for $n \to \infty$ we find

$$d(Tu, u) < \frac{1}{2}d(Tu, u),$$

which is a contradiction. Thus, u is a fixed point of T. By the condition (1), it can be easily seen that u is unique. Following the similar arguments used in the proof of Theorem 2.7 given in [5], it can be easily checked that T is discontinuous at u if and only if $\lim\limits_{x \to u} N^*(x, u) \neq 0$. □

Remark 18. *The case* $\alpha = 1$ *was given in Theorem 2.7 (for more details see page 7 in [5]).*

If we consider Theorem 3 together with $N^*(x, y)$, then we get the following proposition:

Proposition 19. *Let* (X, d) *be a metric space,* T *be a self-mapping on* X *and* $C_{x_0,r}$ *be a fixed circle of* T*. Then* T *is discontinuous at any* $u \in C_{x_0,r}$ *if and only if* $\lim\limits_{x \to u} N^*(x, u) \neq 0$*.*

In [16], it was introduced a general class of discontinuous activation functions for the problem of multistability of competitive neural networks. Now using this general class we give an application of Proposition 19 to discontinuous activation functions. For this purpose, we construct the following discontinuous function:

$$T_4 x = \left\{ \begin{array}{ccc} 5 & \text{if} & -\infty < x < -3 \\ x + 8 & \text{if} & -3 \leq x \leq 1 \\ -x + 10 & \text{if} & 1 < x \leq 5 \\ 12 & \text{if} & 5 < x < +\infty \end{array} \right. .$$

The function T_4 satisfies the conditions of Theorem 3 for the circle $C_{\frac{17}{2},\frac{7}{2}} = \{5, 12\}$. Hence T_4 fixes the circle $C_{\frac{17}{2},\frac{7}{2}}$. We obtain that the function T_4 is discontinuous at any $u \in C_{\frac{17}{2},\frac{7}{2}}$ if and only if $\lim\limits_{x \to u} N^*(x, u) \neq 0$ by Proposition 19. Using this, it can be easily checked that T_4 is continuous at the point $u_1 = 12$ but it is discontinuous at $u_2 = 5$.

CONCLUSION AND FUTURE WORKS

In this paper, we obtain a new fixed-circle theorem and some related results. We note that the converse statement of Theorem 3 is not true everywhen. Especially, in Theorem 3, the self-mapping T maps the circle $C_{x_0,r}$ into (or onto) itself by the condition (1).

Theorem 3 gives no results about the continuity of the self-mapping T on a fixed circle. In Example 7, we have seen that the self-mapping T_1 is continuous at the whole fixed circle. The self-mapping T_4 defined in the previous section is continuous at one of the points of the fixed circle while is discontinuous at the other point. At this context we propose the following open problem:

Open Problem C: What are the conditions which make a self-mapping T is continuous on a fixed circle?

Finally, the obtained results can be lead to new applications for neural nets and fixed-circle theory under suitable conditions.

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