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A DIFFERENT APPROACH TO THE EUROPEAN OPTION PRICING MODEL WITH NEW FRACTIONAL OPERATOR

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Abstract. In this work, we have derived an approximate solution of the fractional Black-Scholes models using an iterative method. The fractional differentiation operator used in this paper is the well-known conformable derivative. Firstly, we redefine the fractional Black-Scholes equation, conformable fractional Adomian decomposition method (CFADM) and conformable fractional modified homotopy perturbation method (CFMHPM). Then, we have solved the fractional Black-Scholes (FBS) and generalized fractional Black-Scholes (GFBS) equations by using the proposed methods, which can analytically solve the fractional partial differential equations (FPDE). In order to show the efficiencies of these methods, we have compared the numerical and exact solutions of these two option pricing problems by using in pricing the actual market data. Also, we have found out that the proposed models are very efficient and powerful techniques in finding approximate solutions of the fractional Black-Scholes models which are considered in conformable sense.

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1. Introduction

Recently, fractional differential equations have attracted much care. Some important definitions of fractional derivatives have been presented by Coimbra, Davison-Essex, Riesz, Riemann-Liouville, Hadamard, Grünwald-Letnikov, Liouville-Caputo, Caputo-Fabrizio and Atangana-Baleanu [8, 16, 39]. This is mostly due to the fact that fractional calculus has an important role in modelling and describing certain problems such as diffusion processes, chemistry, engineering, economic, material sciences and other areas of application [7, 11, 17, 18, 19, 29, 30, 41, 43]. As the characteristic feature of this study, it is important to keep in the forefront the Black-Scholes option pricing model, conformable derivative operator, ADM and HPM. Many applications of the proposed models were studied in the literature. Among them fuzzy differential equation [33], groundwater flow equation [9], poisson and biharmonic equations [10], nonlinear oscillator [32], financial equation [44], diffusion–convection–reaction equation [25], nonhomogeneous PDEs with a variable coefficient [37], with Laplace transform [34], with Sumudu transform [22, 40] and other fields [20, 28].

In addition, some authors [5, 6, 12] established concept of fractional differentiation with Mittag-Leffler kernel due to the non-locality of the dynamical system. Also, some researchers [13, 35] introduced a new fractional

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derivative called conformable derivative operator (CDO) and by using this operator, the solutions of many scientific problems have been obtained and some solution methods have been developed. Many researchers [1, 14, 15, 27] have studied on CDO in engineering, physical and applied mathematics problems. Besides in [36], Koca and Atangana defined the modified version of conformable derivative which called as Beta-derivative. Many linear and nonlinear fractional PDEs can be solved with these methods. We have solved FBSE and GFBSE with these mentioned methods and compared the numerical and approximate-analytical solutions in term of figures. When looking at the results, it is obvious that these methods are very effective and accurate in applying to the option pricing problems. Since most fractional differential equations do not have exact analytical solutions, approximation and numerical techniques are used extensively [41]. Many powerful approximate-analytical methods have been presented in finance literature, especially in modeling the option prices. For example [26, 31, 38] are relatively new approaches providing an analytical and numerical approximation to Black-Scholes option pricing equation. The financial system can be viewed as money, capital and derivative markets (options, futures, forwards, swaps, etc.). Options are widely used in global financial markets. An option is a right but not obligation. The most important benefit of the option is the ability to invest in large amounts with a very small capital.

Black and Scholes [21] in 1973, investigated in their study a model which can easily compute the prices of the options. This model also can evaluate the Greeks of the options and ratio of hedge. The Black-Scholes model that prices stock options has been applied to many different possessions and payments. This form of the pricing model is one of the most meaningful mathematical equations for a financial staple. The Black-Scholes model with respect to an option can be considered as [4, 42]:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r\left(t\right)S\frac{\partial V}{\partial S}-r\left(t\right)V=0,\quad (S,t)\in \mathrm{R}^+\times(0,T)\,, \tag{1.1}$$

where $V=V\left(S,t\right)$ shows the European option price at asset price S and time t. T represents the maturity time, $r\left(t\right)$ is the risk-free interest rate and $\sigma\left(S,t\right)$ is the volatility function of the underlying asset. In equation (1.1), we observe that

$$V\left(0,t\right)=0,\quad V\left(S,t\right)\sim S\quad \text{as}\quad S\to\infty,$$

and we can write payoff functions as:

$$V_c\left(S,T\right)=\max\left(S-E,0\right)\quad \text{and}\quad V_p\left(S,T\right)=\max\left(E-S,0\right),$$

where $V_c\left(S,T\right)$ and $V_p\left(S,T\right)$ show the value of the vanilla call and put options, respectively and E is the exercise (strike) price. The closed form solution of equation (1.1) can be obtained by using the heat equation. In order to obtain the fractional Black-Scholes equation, we make the following conversions:

$$S=Ee^x,\quad t=T-\frac{2\tau}{\sigma^2},\quad V=Ev\left(x,\tau\right).$$

This yields in the equation

$$\frac{\partial^\alpha v\left(x,\tau\right)}{\partial\tau^\alpha}=\frac{\partial^2 v\left(x,\tau\right)}{\partial x^2}+\left(k-1\right)\frac{\partial v\left(x,\tau\right)}{\partial x}-kv\left(x,\tau\right),\quad \tau>0,\ x\in R,\ 0<\alpha\le 1, \tag{1.2}$$

with initial condition

$$v\left(x,0\right)=\max\left(e^x-1,0\right). \tag{1.3}$$

Equation (1.2) is called as Black-Scholes option pricing equation of fractional order (FBSE). In equation (1.2), we define $k = 2r/\sigma^2$ where k represents the balance between the interest rates and stock returns variability.

In addition to this, Cen and Le (2011) obtained the generalized fractional Black-Scholes equation [23] (GFBSE) by considering $r = 0.06$ and $\sigma = 0.4\,(2 + \sin x)$ in equation (1.2):

$$\frac{\partial^\alpha v}{\partial \tau^\alpha} + 0.08\,(2 + \sin x)^2\,\frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \quad \tau > 0,\ x \in R,\ 0 < \alpha \leq 1, \tag{1.4}$$

with the initial condition

$$v\,(x, 0) = \max\left(x - 25e^{-0.06}, 0\right). \tag{1.5}$$

The main aim of this study is to construct ADM and MHPM by using conformable derivative and is to solve equations (1.2)–(1.3) and (1.4)–(1.5) with these proposed methods.

2. Some basic motivations

In this section, we give some basic definitions of conformable fractional derivative and its properties.

Definition 2.1. Given a function $f : [0, \infty) \to$ R. Then the conformable derivative of f order $\alpha \in (0, 1]$ is defined by [35]:

$$^{con}D_*^\alpha\,(f)\,(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f\,(t)}{\varepsilon},$$

for all $t > 0$.

Definition 2.2. Let f be an n-times differentiable at t. Then the conformable derivative of f order α is defined as [1, 35]:

$$^{con}D_*^\alpha\,(f\,(t)) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}\left(t + \varepsilon t^{(\lceil \alpha \rceil - \alpha)}\right) - f^{(\lceil \alpha \rceil - 1)}\,(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (n, n+1]$. Here $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Lemma 2.3. *Let f be an n-times differentiable at t. Then*

$$^{con}D_*^\alpha\,(f\,(t)) = t^{\lceil \alpha \rceil - \alpha} f^{\lceil \alpha \rceil}\,(t)\,,$$

for all $t > 0, \alpha \in (n, n+1]$ [1, 35].

3. Adomian decomposition method defined with conformable derivative operator

Consider the following nonlinear fractional partial differential equation:

$$L_\alpha\,(v\,(x, t)) + R\,(v\,(x, t)) + N\,(v\,(x, t)) = g\,(x, t)\,, \tag{3.1}$$

where $L_\alpha = {}^{con}D_*^\alpha$ is a linear operator with conformable derivative of order $\alpha\,(n < \alpha \leq n+1)$, R is the other part of the linear operator, N is a non-linear operator and $g\,(x, t)$ is a non-homogeneous term. In equation (3.1), if we apply the linear operator to Lemma 2.3, we obtain the following equation [2]:

$$t^{\lceil \alpha \rceil - \alpha} \frac{\partial^{\lceil \alpha \rceil} v(x,t)}{\partial t^{\lceil \alpha \rceil}} + R(v(x,t)) + N(v(x,t)) = g(x,t).$$

Applying the inverse of linear operator $L_\alpha^{-1} = \int_0^t \int_0^{\gamma_1} \cdots \int_n^{\gamma_{n-1}} \frac{1}{\gamma_n^{\lceil \alpha \rceil - \alpha}} (.) \,\mathrm{d}\gamma_n \mathrm{d}\gamma_{n-1} \cdots \mathrm{d}\gamma_1$, to both sides of equation (3.1), we obtain

$$L_\alpha^{-1} L_\alpha (v(x,t)) + L_\alpha^{-1} R(v(x,t)) + L_\alpha^{-1} N(v(x,t)) = L_\alpha^{-1} g(x,t). \tag{3.2}$$

The conformable ADM suggests the solution $u(x,t)$ be decomposed into the infinite series of components

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{3.3}$$

The nonlinear function in equation (3.1) is decomposed as follows:

$$N(v) = \sum_{n=0}^{\infty} A_n, (v_0, v_1, \ldots, v_n), \tag{3.4}$$

where A_n is the so-called Adomian polynomials which can be calculated for all forms of nonlinearity with respect to the algorithms developed by Adomian [3].

Substituting (3.3) and (3.4) into (3.2), we obtain

$$\sum_{n=0}^{\infty} v_n = v(x,0) + L_\alpha^{-1} g - L_\alpha^{-1} R\left(\sum_{n=0}^{\infty} v_n\right) - L_\alpha^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{3.5}$$

By using equation (3.5), the iteration terms are obtained by the following way:

$$\begin{aligned}
&v_0 = v(x,0) + L_\alpha^{-1} g, \\
&v_1 = -L_\alpha^{-1} R v_0 - L_\alpha^{-1} A_0, \\
&\vdots \\
&v_{n+1} = -L_\alpha^{-1} R v_n - L_\alpha^{-1} A_n, n \geq 0.
\end{aligned} \tag{3.6}$$

Then, the approximate-analytical solution of equation (3.1) is obtained by

$$\tilde{v}_k(x,t) = \sum_{n=0}^{k} v_n(x,t).$$

Finally, we obtain the exact solution of equation (3.1) as

$$v(x,t) = \lim_{k \to \infty} \tilde{v}_k(x,t).$$

4. Modified homotopy perturbation method defined with conformable derivative operator

In this section, some basic solution steps and properties of modified homotopy perturbation method defined with conformable derivative operator (CMHPM) are given. We introduce a solution algorithm in an effective way for the nonlinear PDEs of fractional order. Firstly, we consider the following nonlinear fractional equation:

$$ {}^{con}D_*^{\alpha} v\left(x,t\right) = L\left(v, v_x, v_{xx}\right) + N\left(v, v_x, v_{xx}\right) + g\left(x,t\right), t > 0, \tag{4.1} $$

where L is a linear operator, N is a nonlinear operator, v is a known analytical function and ${}^{con}D_*^{\alpha}, m-1 < \alpha \le m$, is the conformable time-fractional derivative of order α, subject to the initial conditions

$$ v^k\left(x,0\right) = g_k\left(x\right), \quad k = 0, 1, \ldots, m-1. $$

According to the homotopy technique, we can construct the following homotopy:

$$ \frac{\partial^m v}{\partial t^m} - L\left(v, v_x, v_{xx}\right) - g\left(x,t\right) = p\left(\frac{\partial^m v}{\partial t^m} + N\left(v, v_x, v_{xx}\right) - {}^{con}D_{*t}^{\alpha} v\right), \tag{4.2} $$

or evenly,

$$ \frac{\partial^m v}{\partial t^m} - g\left(x,t\right) = p\left(\frac{\partial^m v}{\partial t^m} + L\left(v, v_x, v_{xx}\right) + N\left(v, v_x, v_{xx}\right) - {}^{con}D_{*t}^{\alpha} v\right), \tag{4.3} $$

where $p \in [0,1]$. Considering $p = 0$, we obtain equation (4.2) as following the linearized equation

$$ \frac{\partial^m v}{\partial t^m} = L\left(v, v_x, v_{xx}\right) + g\left(x,t\right), $$

and also equation (4.3) can be formulated as follows:

$$ \frac{\partial^m v}{\partial t^m} = g\left(x,t\right). $$

If we take the homotopy parameter $p = 1$, equation (4.2) or equation (4.3) turns out to be the original differential equation of fractional order (4.1). As the basic assumption is that the solution of equation (4.3) can be written by using a power series in p :

$$ v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots . $$

At the end of the solution steps, we approximate the solution as:

$$ v\left(x,t\right) = \sum_{n=0}^{\infty} v_n\left(x,t\right). $$

5. European option pricing model with the proposed methods

In this section of the study, we implement the effectiveness and appropriateness of the CFADM and CFMHPM by applying them to FBSE and GFBSE.

5.1. Solution of fractional Black-Scholes equation

In this subsection, we consider the linear time-fractional Black-Scholes option pricing equation (1.2) with the initial condition (1.3). Firstly, we will solve this problem by using the proposed conformable mean Adomian decomposition method. Let $L_\alpha = {}^{con}D_*^\alpha = \frac{\partial^\alpha}{\partial \tau^\alpha}$ be a linear operator, then if we apply the operator to equation (1.2) we have

$$^{con}D_*^\alpha v(x,\tau) = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \quad \tau > 0,\ x \in R,\ 0 < \alpha \le 1. \tag{5.1}$$

By using the Lemma 2.3, we can write equation (5.1) as

$$\tau^{1-\alpha}\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \quad \tau > 0,\ x \in R,\ 0 < \alpha \le 1. \tag{5.2}$$

Now, we apply the inverse of operator L_α which is $L_\alpha^{-1} = \int_0^t \frac{1}{\zeta^{1-\alpha}}(.)\, d\zeta$ to both sides of equation (5.2), we get

$$v(x,\tau) = v(x,0) + L_\alpha^{-1}\left(\frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv\right).$$

According to (3.6) and the initial condition (1.3), we can write the iterations and the decomposition series terms as by using the conformable derivative operator:

$$\begin{aligned}
v_0 &= v(x,0) = \max(e^x - 1, 0), \\
v_1 &= L_\alpha^{-1}\left(\frac{\partial^2 v_0}{\partial x^2} + (k-1)\frac{\partial v_0}{\partial x} - kv_0\right) = \frac{k\tau^\alpha}{\alpha}\left[e^x - \max(e^x-1,0)\right], \\
v_2 &= L_\alpha^{-1}\left(\frac{\partial^2 v_1}{\partial x^2} + (k-1)\frac{\partial v_1}{\partial x} - kv_1\right) = -\frac{k^2\tau^{2\alpha}}{2!\alpha^2}\left[e^x - \max(e^x-1,0)\right], \\
v_3 &= L_\alpha^{-1}\left(\frac{\partial^2 v_2}{\partial x^2} + (k-1)\frac{\partial v_2}{\partial x} - kv_2\right) = \frac{k^3\tau^{3\alpha}}{3!\alpha^3}\left[e^x - \max(e^x-1,0)\right], \\
&\vdots \\
v_n &= L_\alpha^{-1}\left(\frac{\partial^2 v_{n-1}}{\partial x^2} + (k-1)\frac{\partial v_{n-1}}{\partial x} - kv_{n-1}\right) = \left[e^x - \max(e^x-1,0)\right]\begin{cases} -\frac{k^n\tau^{n\alpha}}{n!\alpha^n}, & n = 2m \\ \frac{k^n\tau^{n\alpha}}{n!\alpha^n}, & n = 2m+1 \end{cases}.
\end{aligned} \tag{5.3}$$

So, by using the decomposition series in equation (5.3), the approximate solution of equation (1.2) obtained by Adomian decomposition method in conformable sense is

$$\tilde{v}_j(x,\tau) = \sum_{n=0}^{j} v_n(x,\tau).$$

From the last equation we obtain the approximate analytical solution of the problem as

$$v(x,\tau) = \lim_{j\to\infty}\tilde{v}_j(x,\tau) = e^{-\frac{k\tau^\alpha}{\alpha}}\max(e^x-1,0) - e^x\left(e^{-\frac{k\tau^\alpha}{\alpha}} - 1\right).$$

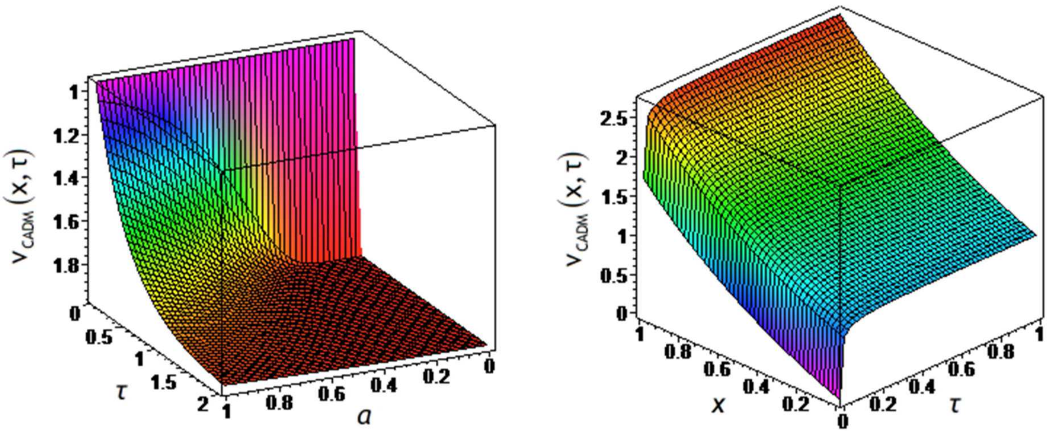

FIGURE 1. CADM solutions of fractional Black-Scholes equation.

Then the exact solution of the FBSE (1.2) obtained subject to the initial condition (1.3) for special case of $\alpha = 1$, is obtained as

$$v(x,\tau) = e^{-k\tau}\max(e^x - 1, 0) - e^x\left(e^{-k\tau} - 1\right).$$

In Figure 1 (left), CADM solutions for various values of α and τ are presented. Here we consider the vanilla call option with parameter $r = 0.04$ and $\sigma = 0.2$ [24]. Then $k = 2r/\sigma^2 = 2$. In the right figure, we take the fractional operator $\alpha = 0.35$ and $k = 2$.

Secondly, we solve the fractional Black-Scholes equation by using the modified homotopy perturbation method in conformable mean. Let us consider the initial condition (1.3) and according to the homotopy (4.2), we can construct the following linear partial differential equation system:

$$\begin{aligned}
&\frac{\partial v_0}{\partial \tau} = 0, \quad v_0(x,0) = \max(e^x - 1, 0),\\
&\frac{\partial v_1}{\partial \tau} = \frac{\partial^2 v_0}{\partial x^2} + (k-1)\frac{\partial v_0}{\partial x} - kv_0 - {}^{con}D_*^{\alpha} v_0, \quad v_1(x,0) = 0,\\
&\frac{\partial v_2}{\partial \tau} = \frac{\partial^2 v_1}{\partial x^2} + (k-1)\frac{\partial v_1}{\partial x} - kv_1 - {}^{con}D_*^{\alpha} v_1, \quad v_2(x,0) = 0,\\
&\vdots
\end{aligned} \tag{5.4}$$

By solving equation (5.4) according to v_0, v_1, v_2 and v_3, the first several components of the MHP solution for equation (1.2) are derived as follows:

$$\begin{aligned}
&v_0(x,\tau) = \max(e^x - 1, 0),\\
&v_1(x,\tau) = k\tau\left(e^x - \max(e^x - 1, 0)\right),\\
&v_2(x,\tau) = k\left(e^x - \max(e^x - 1, 0)\right)\left[\tau - \frac{k\tau^2}{2} - \frac{\tau^{2-\alpha}}{2-\alpha}\right],\\
&v_3(x,\tau) = k\left(e^x - \max(e^x - 1, 0)\right)\left[\tau - k\tau^2 - \frac{2\tau^{2-\alpha}}{2-\alpha} + \frac{k^2\tau^3}{6} + \frac{k\tau^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{k\tau^{3-\alpha}}{3-\alpha} + \frac{\tau^{3-2\alpha}}{3-2\alpha}\right],\\
&\vdots
\end{aligned}$$

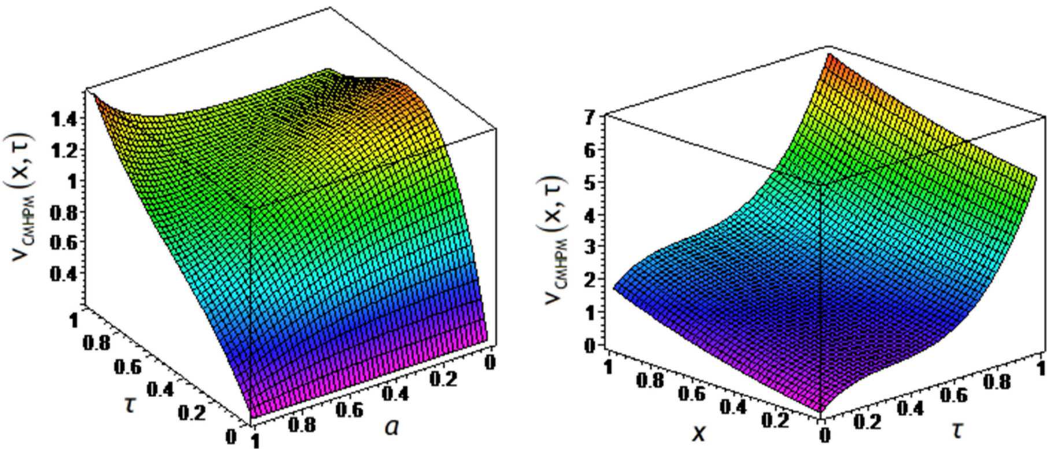

FIGURE 2. CMHPM solutions of fractional Black-Scholes equation.

continuing in this way, the remaining steps of homotopy can be obtained. Then the approximate solution of equation (1.2) is given by

$$\begin{aligned} v(x,\tau) &= v_0(x,\tau) + v_1(x,\tau) + v_2(x,\tau) + v_3(x,\tau) + \cdots \\ &= \left[3\tau - \frac{3k\tau^2}{2} - \frac{3\tau^{2-\alpha}}{2-\alpha} + \frac{k^2\tau^3}{6} + \frac{k\tau^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{k\tau^{3-\alpha}}{3-\alpha} + \frac{\tau^{3-2\alpha}}{3-2\alpha}\right] \\ &\quad \times k\left(e^x - \max\left(e^x-1,0\right)\right) + \max\left(e^x-1,0\right). \end{aligned}$$

Then the exact solution of equation (1.2) with the initial condition (1.3) for special case of $\alpha = 1$, is obtained with modified homotopy perturbation method in conformable sense as

$$v(x,\tau) = e^{-k\tau}\max\left(e^x-1,0\right) - e^x\left(e^{-k\tau}-1\right).$$

Figure 2 (left) shows CMHPM solutions for various values of α and τ. Also in the figure we consider the vanilla call option with parameter $r = 0.1$ and $\sigma = 0.2$. Then $k = 2r/\sigma^2 = 5$. In the right figure, we take the fractional operator $\alpha = 0.85$ and $k = 5$.

5.2. Solution of generalized fractional Black-Scholes equation

We consider the linear time-fractional generalized Black-Scholes option pricing equation (1.4) with the initial condition (1.5) in this subsection. In order to solve this problem by using CADM, we apply the linear operator to equation (1.4), we have

$${}^{con}D_*^{\alpha} v(x,\tau) = -0.08\left(2+\sin x\right)^2 \frac{\partial^2 v}{\partial x^2} - 0.06x\frac{\partial v}{\partial x} + 0.06v, \quad \tau > 0,\ x \in R,\ 0 < \alpha \le 1,$$

or equally in view of conformable sense,

$$\tau^{1-\alpha}\frac{\partial v(x,\tau)}{\partial \tau} = -0.08\left(2+\sin x\right)^2 \frac{\partial^2 v}{\partial x^2} - 0.06x\frac{\partial v}{\partial x} + 0.06v, \quad \tau > 0,\ x \in R,\ 0 < \alpha \le 1.$$

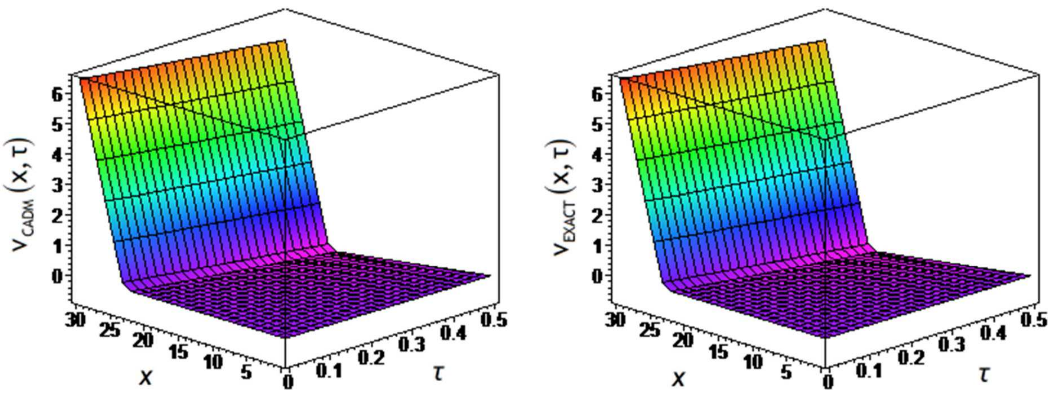

FIGURE 3. CADM and exact solutions with $\alpha = 0.95$ for GFBSE.

Applying the inverse of operator L_α to both sides of the last equation, we obtain

$$v(x,\tau) = v(x,0) - L_\alpha^{-1}\left(0.08\,(2+\sin x)^2\,\frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v\right).$$

Considering equation (3.6) and the initial condition (1.5), we can obtain the iterations by using the conformable derivative operator and the decomposition series terms as:

$$\begin{aligned}
v_0 &= v(x,0) = \max\left(x - 25e^{-0.06}, 0\right),\\
v_1 &= -L_\alpha^{-1}\left(0.08\,(2+\sin x)^2\,\frac{\partial^2 v_0}{\partial x^2} + 0.06x\frac{\partial v_0}{\partial x} - 0.06v_0\right) = -\frac{0.06\tau^\alpha}{\alpha}\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right],\\
v_2 &= -L_\alpha^{-1}\left(0.08\,(2+\sin x)^2\,\frac{\partial^2 v_1}{\partial x^2} + 0.06x\frac{\partial v_1}{\partial x} - 0.06v_1\right) = -\frac{(0.06)^2\,\tau^{2\alpha}}{2!\alpha^2}\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right],\\
v_3 &= -L_\alpha^{-1}\left(0.08\,(2+\sin x)^2\,\frac{\partial^2 v_2}{\partial x^2} + 0.06x\frac{\partial v_2}{\partial x} - 0.06v_2\right) = -\frac{(0.06)^3\,\tau^{3\alpha}}{3!\alpha^3}\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right],\\
&\vdots\\
v_n &= -L_\alpha^{-1}\left(0.08\,(2+\sin x)^2\,\frac{\partial^2 v_{n-1}}{\partial x^2} + 0.06x\frac{\partial v_{n-1}}{\partial x} - 0.06v_{n-1}\right) = -\frac{(0.06)^n\,\tau^{n\alpha}}{n!\alpha^n}\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right].
\end{aligned} \tag{5.5}$$

Then, by using (5.5) the approximate solution of GFBSE obtained as

$$\tilde{v}_j(x,\tau) = \sum_{n=0}^{j} v_n(x,\tau) = \left[\max\left(x - 25e^{-0.06}, 0\right) - x\right]\sum_{n=0}^{j}\frac{(0.06)^n\,\tau^{n\alpha}}{n!\alpha^n}.$$

From the last equation, we have the approximate analytical solution as:

$$v(x,\tau) = \lim_{j\to\infty}\tilde{v}_j(x,\tau) = \max\left(x - 25e^{-0.06}, 0\right) + \left(1 - e^{\frac{0.06\tau^\alpha}{\alpha}}\right)\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right].$$

The exact solution of equation (1.4) with the initial condition (1.5) for special case of $\alpha = 1$, is obtained as

$$v(x,\tau) = \max\left(x - 25e^{-0.06}, 0\right) e^{0.06\tau} + x\left(1 - e^{0.06\tau}\right).$$

Figure 3 (left) presents CADM and exact solutions for various values of x and τ. Besides we regard as $\alpha = 0.95$. Considering the outcomes of Figure 3 we can say that the numerical results obtained with ADM in conformable mean are very close to the exact solution values.

Now, let us consider the solution of the GFBSE with CMHPM. For the solution, firstly we use the homotopy in order to obtain the following set of linear partial differential equations:

$$\begin{aligned}
&\frac{\partial v_0}{\partial \tau} = 0, \quad v_0(x,0) = \max\left(x - 25e^{-0.06}, 0\right), \\
&\frac{\partial v_1}{\partial \tau} = -0.08\left(2 + \sin x\right)^2 \frac{\partial^2 v_2}{\partial x^2} - 0.06x\frac{\partial v_2}{\partial x} + 0.06v_2 - {}^{con}D_*^{\alpha} v(x,\tau), v_1(x,0) = 0, \\
&\frac{\partial v_2}{\partial \tau} = -0.08\left(2 + \sin x\right)^2 \frac{\partial^2 v_1}{\partial x^2} - 0.06x\frac{\partial v_1}{\partial x} + 0.06v_1 - {}^{con}D_*^{\alpha} v(x,\tau), v_2(x,0) = 0, \\
&\vdots
\end{aligned} \tag{5.6}$$

If we solve equation (5.6) according to v_0, v_1, v_2 and v_3, the first few components of the modified homotopy perturbation solution for equation (1.4) are obtained as follows:

$$\begin{aligned}
v_0(x,\tau) &= \max\left(x - 25e^{-0.06}, 0\right), \\
v_1(x,\tau) &= -0.06\tau\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right], \\
v_2(x,\tau) &= -0.06\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right]\left[\frac{0.06\tau^2}{2} - \frac{\tau^{2-\alpha}}{2-\alpha}\right], \\
v_3(x,\tau) &= -0.06\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right]\left[\frac{(0.06)^2\tau^3}{6} - \frac{0.06\tau^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{0.06\tau^{3-\alpha}}{3-\alpha} + \frac{\tau^{3-2\alpha}}{3-2\alpha}\right], \\
&\vdots
\end{aligned}$$

continuing in this way, the remaining steps of homotopy can be calculated. Therefore, the approximate solution of equation (1.4) is given by

$$\begin{aligned}
v(x,t) &= v_0(x,\tau) + v_1(x,\tau) + v_2(x,\tau) + v_3(x,\tau) + \cdots \\
&= \max\left(x - 25e^{-0.06}, 0\right) - 0.06\left[x - \max\left(x - 25e^{-0.06}, 0\right)\right] \\
&\quad \times \left[\tau + \frac{0.06\tau^2}{2} - \frac{\tau^{2-\alpha}}{2-\alpha} + \frac{(0.06)^2\tau^3}{6} - \frac{0.06\tau^{3-\alpha}}{(2-\alpha)(3-\alpha)} + \frac{0.06\tau^{3-\alpha}}{3-\alpha} + \frac{\tau^{3-2\alpha}}{3-2\alpha} + \cdots\right],
\end{aligned}$$

Then, for the special value of $\alpha = 1$, the exact solution of equation (1.4) subject to the initial condition (1.5) is obtained with CMHPM as

$$v(x,\tau) = \max\left(x - 25e^{-0.06}, 0\right) e^{0.06\tau} + x\left(1 - e^{0.06\tau}\right),$$

which is the same solution with obtained CADM one.

In Figure 4, the graphs of solution functions of GFBSE with respect to the CMHPM and the exact solution for $\alpha = 0.50$ are shown. According to Figure 4 we can say that the proposed model solves the GFBSE accurately.

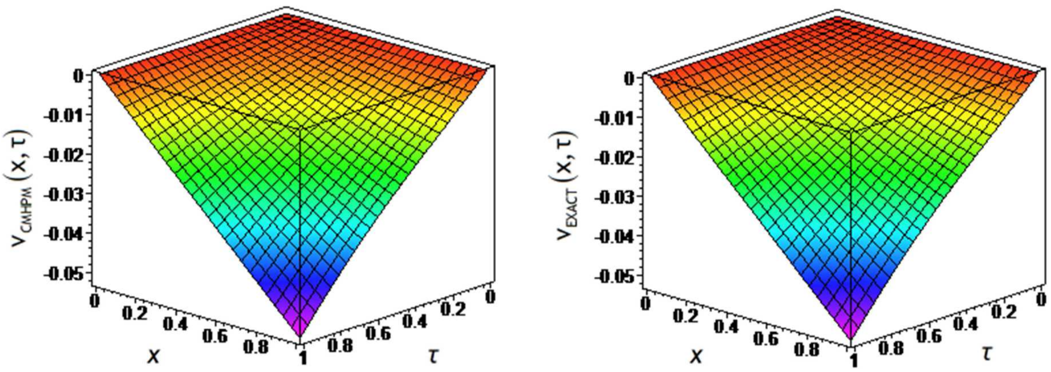

FIGURE 4. CMHPM and exact solutions with $\alpha = 0.50$ for GFBSE.

6. Concluding remarks

In the present paper, approximate-analytical solutions with two numerical methods for linear partial differential equations of time-fractional order have been employed. These methods are based on conformable derivative (CD) which is extremely popular in the last years. We have demonstrated the efficiencies and accuracies of the proposed methods by applying them to the fractional Black-Scholes option pricing models with the initial conditions satisfied by the standard European vanilla call option. The fractional model mentioned in this study can model the price of different financial derivatives like swaps, warrant, etc. The successful applications of the proposed models prove that these models are in complete agreement with the corresponding exact solutions. According to the solution graphs, we can obtain how the option is priced for fractional cases of European call option pricing models. Besides, in view of their usability, our methods are applicable to many initial-boundary value problems and linear-nonlinear partial differential equations of fractional order.

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