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Gadjieva's conjecture, K-functionals, and some applications in weighted Lebesgue spaces

Ramazan AKGÜN*
Department of Mathematics, Faculty of Science and Letters, Balıkesir University, Balıkesir, Turkey

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Abstract: We prove that Gadjieva's conjecture holds true as stated in her PhD thesis. The positive solution of this conjecture allows us to obtain improved versions of the Jackson–Stechkin type inequalities obtained in her thesis and some others. As an application, an equivalence of the modulus of smoothness with the realization functional is established. We obtain a characterization class for the modulus of smoothness.

Key words: Modulus of smoothness, Muckenhoupt weight, weighted Lebesgue spaces, characterization, K-functional

1. Introduction and results

Let $\mathcal{T}_n$ be the class of real trigonometric polynomials of degree not greater than n and γ be a weight (a.e. positive measurable function) on $T := [0, 2\pi]$. Among other weights we will consider Muckenhoupt weights. These weights have many applications in the theory of integral operators, harmonic analysis, and the theory of function spaces (see, for example, [13, 14]). We refer to the monograph of García-Cuerva and Rubio de Francia [13] for the theory of Muckenhoupt weights. A 2π-periodic weight function $\gamma : T \to (0, \infty)$ belongs to the Muckenhoupt class A_p, $p \in (1, \infty)$, if

$$\left(\frac{1}{|J|} \int_J \gamma(x)\, dx \right) \left(\frac{1}{|J|} \int_J \gamma^{-\frac{1}{p-1}}(x)\, dx \right)^{p-1} \leq C \tag{1}$$

with a finite constant C independent of J, where J is any subinterval of T and $|J|$ denotes the length of J. The least constant C satisfying (1) is called the A_p constant of γ and is denoted by $[\gamma]_{A_p}$. Let f be in the weighted Lebesgue space L^p_γ, $p \in (1, \infty)$, of measurable functions $f : T \to \mathbb{R}$ having the norm $\|f\|_{p,\gamma} := \left\{ \int_T |f(x)|^p \gamma(x)\, dx \right\}^{1/p} < \infty$ and $E_n(f)_{p,\gamma} := \inf \left\{ \|f - U\|_{p,\gamma} : U \in \mathcal{T}_n \right\}$. In 1986 in her PhD thesis [12], Gadjieva obtained, among other results, the so-called Jackson type inequality in L^p_γ, $p \in (1, \infty)$, with weights $\gamma \in A_p$:

Theorem 1 *([12, p.50, Theorem 1.4]) If $p \in (1, \infty)$, $\gamma \in A_p$, and $f \in L^p_\gamma$, then there is positive constant c*

*Correspondence: rakgun@balikesir.edu.tr
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depending only on r, p *and Muckenhoupt's* A_p *constant* $[\gamma]_{A_p}$ *of* γ *such that*

$$E_n(f)_{p,\gamma} \leq c_{r,p,[\gamma]_{A_p}} W_r\left(f, \frac{1}{n+1}\right)_{p,\gamma} \tag{2}$$

holds for $r, n \in \mathbb{N} = \{1,2,3,...\}$ *where*

$$W_r(f,\delta)_{p,\gamma} := \sup_{0 \leq h_i \leq \delta} \left\| \prod_{i=1}^{r} (I - \sigma_{h_i}) f \right\|_{p,\gamma}, \tag{3}$$

I *is the identity operator, and*

$$\sigma_h f(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt, \quad x \in T. \tag{4}$$

(2) was the first result in the literature for the Jackson type inequality for $f \in L^p_\gamma$ with $p \in (1,\infty)$ and $\gamma \in A_p$. This estimate (2) yielded several further investigations in theory. See, for example, the papers [2, 3, 5, 16,19–21, 29, 30, 33, 43]. The formulation (3) of the Butzer-Wehrens type [9, 42] modulus of smoothness $W_r(f,\cdot)_{p,\gamma}$ uses the Steklov mean (4) because the class of function L^p_γ is not necessarily translation invariant, in general, with respect to the usual shift $f(x) \to f(x+a)$ where $a \in \mathbb{R}$.

On the other hand, in the literature [11, 16, 20–28, 30, 32, 36–38, 41] there is the following type of formulation for the modulus of smoothness:

$$\Omega_r(f,\delta)_{p,\gamma} := \sup_{0 \leq h \leq \delta} \|(I - \sigma_h)^r f\|_{p,\gamma}, \quad r \in \mathbb{N}. \tag{5}$$

Note that the formulation (5) is also included in her thesis [12, p. 35]. Furthermore, the conjecture of Gadjieva is related to (5) and Peetre's K-functional, that is,

$$K_r(f,\delta,p,\gamma) := \inf_g \left\{ \|f - g\|_{p,\gamma} + \delta^r \left\|g^{(r)}\right\|_{p,\gamma} : g, g^{(r)} \in L^p_\gamma \right\} \tag{6}$$

for $r \in \mathbb{N}$, $p \in (1,\infty)$, $\gamma \in A_p$, $\delta > 0$, and $f \in L^p_\gamma$.

Conjecture 2 (Conjecture of Gadjieva) *([12, p. 35]) If* $p \in (1,\infty)$, $\gamma \in A_p$, $n \in \mathbb{N}$, *and* $f \in L^p_\gamma$, *then there is constant* $C_{[\gamma]_{A_p},r,p} > 0$ *depending only on* r, p *and* $[\gamma]_{A_p}$ *such that*

$$K_{2r}(f,\delta,p,\gamma) \leq C_{[\gamma]_{A_p},r,p} \Omega_r(f,\delta)_{p,\gamma} \tag{7}$$

holds for $r \in \mathbb{N}$.

In this work we prove that the conjecture of Gadjieva holds true as stated in [12] for functions $f \in L^p_\gamma$ with $p \in (1,\infty)$ and $\gamma \in A_p$. The main result of this paper is the following theorem consisting of an equivalence of the modulus of smoothness Ω_r and Peetre's K-functional K_{2r}, which gives a positive solution to Gadjieva's conjecture (7):

Theorem 3 *If $r \in \mathbb{N}$, $f \in L_{\gamma}^{p}$, $p \in (1,\infty)$, and $\gamma \in A_p$, then the equivalence*

$$\Omega_r(f,t)_{p,\gamma} \approx K_{2r}(f,t,p,\gamma) \tag{8}$$

holds for $t \geq 0$, where the equivalence constants depend only on r,p, and $[\gamma]_{A_p}$.

As a corollary we can obtain a Jackson–Stechkin type inequality, which improves (for $r \geq 2$) the Jackson–Stechkin type inequalities obtained in [2, 3, 16, 20, 29, 30, 43].

Theorem 4 *If $p \in (1,\infty)$, $\gamma \in A_p$, r, $n \in \mathbb{N}$, and $f \in L_{\gamma}^{p}$, then there is a positive constant depending only on r,p and $[\gamma]_{A_p}$ such that*

$$E_n(f)_{p,\gamma} \leq c_{r,p,[\gamma]_{A_p}} \Omega_r\left(f,\frac{1}{n+1}\right)_{p,\gamma}$$

holds.

We note that

$$\Omega_1(f,\cdot)_{p,\gamma} = W_1(f,\cdot)_{p,\gamma} \text{ and } \Omega_r(f,\cdot)_{p,\gamma} \leq W_r(f,\cdot)_{p,\gamma} \tag{9}$$

for $r \geq 2$. Thus, the inequality in Theorem 4 improves the inequality (2) for $r \geq 2$.

In several particular cases there were some results of the Jackson type inequality: when $\gamma \equiv 1$ and $p \in [1,\infty)$ (5) and (7) in L^p were considered in [11] and an equivalence of modulus of smoothness with Peetre's K-functional was proved. When $\gamma \equiv 1$ and $p = 2$ in L^2 Abilov and Abilova [1] obtained Theorem 4 thanks to the Parseval equality. When $r = 1$, $p \in (1,\infty)$, and $\gamma \in A_p$, Theorem 4 was investigated in some papers [2, 16, 21, 29, 43].

On the other hand, a different method of trigonometric approximation in Lebesgue spaces with Muckenhoupt weights was developed by Ky ([31, 32]). He also defined a suitable weighted modulus of smoothness (see the definition of $\bar{\Omega}_r$ below). Independently of Gadjieva, Ky proved the direct and inverse theorems of trigonometric approximation in Lebesgue spaces with Muckenhoupt weights: let $x,t \in T$, $r \in \mathbb{N}$ and set

$$\Delta_t^r f(x) := \sum_{k=0}^{r} (-1)^{r+k+1} \binom{r}{k} f(x+kt), \quad f \in L^1, \tag{10}$$

where $\binom{r}{k} := \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k \geq 1$ and $\binom{r}{0} := 1$. Taking $r \in \mathbb{N}\cup\{0\}$, $p \in (1,\infty)$, $\gamma \in A_p$, $f \in L_{\gamma}^{p}$ we consider the mean $\mathcal{A}_{\delta}^{r} f(\cdot) := \frac{1}{\delta}\int_0^{\delta} |\Delta_t^r f(\cdot)|\,dt$, $\quad x \in T$. Let $r \in \mathbb{N}\cup\{0\}$, $p \in (1,\infty)$, $\gamma \in A_p$, $f \in L_{\gamma}^{p}$ and define ([32])

$$\bar{\Omega}_r(f,h)_{p,\gamma} := \sup_{|\delta|\leq h} \|\mathcal{A}_{\delta}^{r} f\|_{p,\gamma}.$$

By equivalence with the K-functional, we obtain that Ω_r and $\bar{\Omega}_{2r}$ are equivalent in the sense $\bar{\Omega}_{2r} \approx \Omega_r$ where $r \in \mathbb{N}$. Hence, Theorem 4 is equivalent to Theorem 2 of [32, (25) with $2r$].

Another part of the work concentrates on the main properties of (5). For example, we obtain that (5) has the following properties:

Theorem 5 *Let* $p \in (1, \infty)$, $\gamma \in A_p$, $f, g \in L_\gamma^p$, $\delta \geq 0$, *and* $r, k \in \mathbb{N}$. *Then*

$$\lim_{\delta \to 0^+} \Omega_r (f, \delta)_{p,\gamma} = 0, \tag{11}$$

$$\Omega_{r+k} (f, \delta)_{p,\gamma} \leq C_{r,k,p,[\gamma]_{A_p}} \Omega_k (f, \delta)_{p,\gamma}, \tag{12}$$

and for any $0 < t < 1$

$$\Omega_{r+k}(f, t)_{p,\gamma} \leq c_{r,k,p,[\gamma]_{A_p}} t^{2k} \Omega_r (f^{(2k)}, t)_{p,\gamma} \tag{13}$$

where constants are dependent only on r, k, p, *and* $[\gamma]_{A_p}$.

It is well known from Theorems 6.5 and 7.4 of [11] that

$$\Omega_r (f, t)_{p,1} \approx K_{2r} (f, t, p, 1), \quad t \geq 0 \tag{14}$$

also holds for $1 \leq p \leq \infty$, $f \in L^p$.

(8) implies the following further properties of (5).

Corollary 6 *If* $r \in \mathbb{N}$, $f \in L_\gamma^p$, $p \in (1, \infty)$, *and* $\gamma \in A_p$, *then*

$$\Omega_r (f, \lambda\delta)_{p,\gamma} \leq C (1 + \lfloor \lambda \rfloor)^{2r} \Omega_r (f, \delta)_{p,\gamma}, \quad \delta, \lambda > 0, \tag{15}$$

and

$$\Omega_r (f, \delta)_{p,\gamma} \delta^{-2r} \leq C \Omega_r (f, \delta_1)_{p,\gamma} \delta_1^{-2r}, \quad 0 < \delta_1 \leq \delta,$$

where $\lfloor z \rfloor := \max \{y \in \mathbb{Z} : y \leq z\}$.

It is well known that the basic property of moduli smoothness $\Omega_r (\cdot, \delta)_{p,\gamma}$ is the decreasing to zero of $\Omega_r (\cdot, \delta)_{p,\gamma}$ as $\delta \to 0$. Using an equivalence between $\Omega_r (\cdot, \delta)_{p,\gamma}$ and a function φ from some class Φ_a one can describe the rate (11). The class Φ_a $(a \in \mathbb{R})$ consists of functions ψ satisfying the following conditions:

(a) $\psi (t) \geq 0$ bounded on $(0, \infty)$,

(b) $\psi (t) \to 0$ as $t \to 0$,

(c) $\psi (t)$ is nondecreasing,

(d) $t^{-a} \psi (t)$ is nonincreasing.

The characterization class of (5) is given in the following theorem.

Theorem 7 *Let* $\delta \in \mathbb{R}^+$, $n, r \in \mathbb{N}$, $p \in (1, \infty)$, *and* $\gamma \in A_p$.

(a) If $f \in L_\gamma^p$ *then there exists a* $\psi \in \Phi_{2r}$ *such that*

$$\Omega_r (f, t)_{p,\gamma} \approx \psi (t) \tag{16}$$

holds for all $t \in (0, \infty)$ *with equivalence constants depending only on* r, p, *and* $[\gamma]_{A_p}$.

(b) If $\psi \in \Phi_{2r}$ *then there exists a* $f \in L_\gamma^p$ *and a positive real number* t_0 *such that*

$$\Omega_r (f, \delta)_{p,\gamma} \approx \psi (\delta) \tag{17}$$

holds for all $\delta \in (0, t_0)$ *with equivalence constants dependent only on* r, p, *and* $[\gamma]_{A_p}$.

This type of characterization theorem was proved in [40] for the spaces $L^p,\ p\in[1,\infty)$, with classical moduli of smoothness of fractional order. The class Φ_ϱ completely describes the class of all majorants for the moduli of smoothness $\omega_r(\cdot,\delta)_p$ in the space $L^p,\ p\in[1,\infty)$. For $\omega_r(\cdot,\delta)_p,\ r\in\mathbb{N}$ the characterization problem was investigated by Besov and Stechkin [7]; for $\omega_r(\cdot,\delta)_p,\ r>0$ the characterization theorem was obtained by Tikhonov [40].

Theorem 4 has a weak inverse and the following estimate is a corollary of (9) and Theorem 1.5 of [12].

Corollary 8 *If $p\in(1,\infty)$, $\gamma\in A_p$, $n\in\mathbb{N}$, and $f\in L^p_\gamma$, then there is a positive constant c depending only on r,p, and $[\gamma]_{A_p}$ such that*

$$\Omega_r\left(f,\tfrac{1}{n}\right)_{p,\gamma}\leq\tfrac{C_{r,p,[\gamma]_{A_p}}}{n^{2r}}\sum_{i=1}^{n}\frac{(i+1)^{2r}}{i+1}E_i(f)_{p,\gamma}\tag{18}$$

holds for $r\in\mathbb{N}$.

As a corollary of Theorem 4 and Corollary 8, we have the following Marchaud type inequality.

Corollary 9 *If $p\in(1,\infty)$, $\gamma\in A_p$, $f\in L^p_\gamma$, r, $l\in\mathbb{R}^+$, $r<l$, and $0<t\leq 1/2$, then there exists a positive constant c depending only on r,l,p and $[\gamma]_{A_p}$ such that*

$$\Omega_r(f,t)_{p,\gamma}\leq C_{r,[\gamma]_{A_p},l,p}t^{2r}\int_t^1\tfrac{\Omega_l(f,u)_{p,\gamma}}{u^{2r}}\frac{du}{u}.$$

From Theorem 1.1 of [12, $\beta=0$] and Theorem 8 we get:

Corollary 10 *Let $p\in(1,\infty)$, $\gamma\in A_p$, $f\in L^p_\gamma$, $r\in\mathbb{N}$ and,*

$$\sum_{\nu=1}^{\infty}\frac{\nu^\alpha}{\nu}E_\nu(f)_{p,\gamma}<\infty$$

for some $\alpha>0$. In this case, for $n\in\mathbb{N}$, there exists constant $C_{\alpha,r,p,[\gamma]_{A_p}}>0$, dependent only on α, r, p, and $[\gamma]_{A_p}$, such that

$$\Omega_r\left(f^{(\alpha)},\tfrac{1}{n}\right)_{p,\gamma}\leq C_{\alpha,r,p,[\gamma]_{A_p}}\left\{\tfrac{1}{n^{2r}}\sum_{\nu=0}^{n}\frac{(\nu+1)^{\alpha+2r}}{\nu+1}E_\nu(f)_{p,\gamma}+\sum_{\nu=n+1}^{\infty}\frac{\nu^\alpha}{\nu}E_\nu(f)_{p,\gamma}\right\}$$

holds.

Realization functional $R_r(f,\delta,p,\gamma)$ is defined as

$$R_r(f,\delta,p,\gamma):=\|f-T\|_{p,\gamma}+\delta^r\left\|T^{(r)}\right\|_{p,\gamma}\tag{19}$$

where $r\in\mathbb{N}$, $T\in\mathcal{T}_n$ is a near best approximating polynomial for $f\in L^p_\gamma$, $p\in(1,\infty)$, and $\gamma\in A_p$.

Theorem 11 *If $r\in\mathbb{N}$, $f\in L_{\gamma}^{p}$, $p\in(1,\infty)$, and $\gamma\in A_{p}$, then the equivalence*

$$\Omega_{r}\left(f,1/n\right)_{p,\gamma}\approx R_{2r}\left(f,1/n,p,\gamma\right) \tag{20}$$

holds for $n\in\mathbb{N}$, where the equivalence constants depend only on r,p, and $[\gamma]_{A_{p}}$.

The rest of the work is organized as follows. In Section 2 we give some preliminary properties of weights and the modulus of smoothness (5). In Section 3 we give the proof of Gadjieva's conjecture. In Section 4 we give some properties of the modulus of smoothness (5). In Section 5 we obtain an equivalence of the modulus of smoothness (5) with Peetre's K-functional (6). In Section 6 we find a characterization class of functions for (5). Section 7 contains the proof of an equivalence of the modulus of smoothness (5) with the realization functional (19). In the final section, we consider the modulus of smoothness $\Omega_{r}\left(f,\cdot\right)_{p,\gamma}$ of fractional order $r>0$. We note that fractional smoothness is required in the literature to obtain Ul'yanov type inequalities.

Here, and in what follows, $A\lesssim B$ will mean that there exists a positive constant $C_{u,v,\ldots}$, dependent only on the parameters $u,v,\ldots$ and it can be different in different places, such that the inequality $A\leq CB$ holds. If $A\lesssim B$ and $B\lesssim A$ we will write $A\approx B$.

2. Preliminaries

We give some details for the definition of moduli of smoothness (5). If $p\in(1,\infty)$, $f\in L_{\gamma}^{p}$, and $\gamma\in A_{p}$, then the Hardy–Littlewood maximal function

$$Mf\left(x\right):=\sup_{x\in(a,b)}\frac{1}{b-a}\int_{a}^{b}\left|f\left(t\right)\right|dt$$

is bounded [34] in L_{γ}^{p}. If $p\in(1,\infty)$, $f\in L_{\gamma}^{p}$, and $\gamma\in A_{p}$, then there exists a constant $C_{p,[\gamma]_{A_{p}}}>0$, independent of h and f, such that

$$\left\Vert\sigma_{h}f\right\Vert_{p,\gamma}\leq\left\Vert Mf\right\Vert_{p,\gamma}\leq C_{p,[\gamma]_{A_{p}}}\left\Vert f\right\Vert_{p,\gamma} \tag{21}$$

and

$$\left\Vert\left(I-\sigma_{h}\right)^{r}f\right\Vert_{p,\gamma}\leq C_{p,r,[\gamma]_{A_{p}}}\left\Vert f\right\Vert_{p,\gamma}. \tag{22}$$

Now we can define the weighted modulus of smoothness as in (5): if $r\in\mathbb{N}$, $p\in(1,\infty)$, $f\in L_{\gamma}^{p}$, and $\gamma\in A_{p}$, we define

$$\Omega_{r}\left(f,\delta\right)_{p,\gamma}:=\sup_{0\leq h\leq\delta}\left\Vert\left(I-\sigma_{h}\right)^{r}f\right\Vert_{p,\gamma},\quad\Omega_{0}\left(f,\delta\right)_{p,\gamma}:=\left\Vert f\right\Vert_{p,\gamma}.$$

In this case,

$$\Omega_{r}\left(f,\delta\right)_{p,\gamma}\leq C_{p,r,[\gamma]_{A_{p}}}\left\Vert f\right\Vert_{p,\gamma} \tag{23}$$

for some constant $c>0$ dependent only on p,r and $[\gamma]_{A_{p}}$. Hence, the modulus of smoothness $\Omega_{r}\left(\cdot,\delta\right)_{p,\gamma}$ is a well-defined, nonnegative, nondecreasing function of δ on $(0,\infty)$ and satisfies the usual property $\Omega_{r}\left(f+g,\cdot\right)_{p,\gamma}\leq\Omega_{r}\left(f,\cdot\right)_{p,\gamma}+\Omega_{r}\left(g,\cdot\right)_{p,\gamma}$.

If $p \in (1,\infty)$ and $\gamma \in A_p$, then there exists (see Lemma 2 of [18]) a real number $a > 1$ such that embeddings

$$L^{\infty}, C[T] \hookrightarrow L^p_\gamma \hookrightarrow L^a, \tag{24}$$

namely

$$\|\cdot\|_1 \lesssim \|\cdot\|_a \lesssim \|\cdot\|_{p,\gamma} \lesssim \|\cdot\|_\infty, \|\cdot\|_{C[T],} \tag{25}$$

hold where $C[T]$ denotes the collection of continuous functions $f: T \to \mathbb{R}$ having the finite norm $\|f\|_{C[T]} := \max\{|f(x)| : x \in T\}$. Hence, for $p \in (1,\infty)$, $f \in L^p_\gamma$, and $\gamma \in A_p$, we have $L^p_\gamma \subset L^1$. Let

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) =: \sum_{k=0}^{\infty} A_k(x,f) \tag{26}$$

and

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k(f)\sin kx - b_k(f)\cos kx) =: \sum_{k=1}^{\infty} A_k\left(x,\tilde{f}\right) \tag{27}$$

be the *Fourier* and the *conjugate Fourier* series of $f \in L^p_\gamma$ where

$$a_k(f) = \frac{1}{\pi}\int_T f(x)\cos kx dx, \quad b_k(f) = \frac{1}{\pi}\int_T f(x)\sin kx dx \quad (k = 0,1,2,...).$$

The partial sum of Fourier series (26) of f is defined as $S_n(f) := S_n(x,f) := \sum_{k=0}^{n} A_k(x,f)$ for $n \in \mathbb{N}\cup\{0\}$. Using Fourier series (26) of $f \in L^p_\gamma$ with $p \in (1,\infty)$, $\gamma \in A_p$, and (4) we find, with $(\sin 0)/0 = 1$,

$$\sigma_h^r f(x) \backsim \sum_{k=1}^{\infty}\left(\frac{\sin kh}{kh}\right)^r A_k(x,f), \quad r \in \mathbb{N}. \tag{28}$$

From the relations (4) and (28) we obtain

$$(I-\sigma_h)^r f(x) \backsim \sum_{k=0}^{\infty}\left(1 - \frac{\sin kh}{kh}\right)^r A_k(x,f), \quad r \in \mathbb{N}.$$

3. Properties of the modulus of smoothness $\Omega_r(f,\cdot)_{p,\gamma}$

The following weighted Marcinkiewicz multiplier theorem was proved in [6, Theorem 4.4]:

Lemma 12 *Let a sequence* $\{\lambda_\mu\}$ *of real numbers satisfy*

$$|\lambda_\mu| \le A, \qquad \sum_{\mu=2^{m-1}}^{2^m-1} |\lambda_\mu - \lambda_{\mu+1}| \le A \tag{29}$$

for all $m \in \mathbb{N}$, $\mu \in \mathbb{N}\cup\{0\}$. *If* $p \in (1,\infty)$, $\gamma \in A_p$, *and* $f \in L^p_\gamma$ *with the Fourier series (26), then there is a function* $G \in L^p_\gamma$ *such that the series* $\sum_{k=0}^{\infty} \lambda_k A_k(x,f)$ *is Fourier series for* F *and*

$$\|G\|_{p,\gamma} \lesssim A\|f\|_{p,\gamma} \tag{30}$$

where the constant does not depend on f.

For the proof of Theorem 5, we need the following lemma.

Lemma 13 *Let* $p \in (1, \infty)$, $\gamma \in A_p$, $n \in \mathbb{N}$, $U_n \in \mathcal{T}_n$, *and* $r, k \in \mathbb{N} \cup \{0\}$. *Then for any* $0 < t < 1/n$ *there exists a constant* $C_{p,r,k,[\gamma]_{A_p}} > 0$ *depending only on* p, r, k, *and* $[\gamma]_{A_p}$ *such that*

$$\Omega_{r+k}(U_n, t)_{p,\gamma} \leq C_{p,r,k,[\gamma]_{A_p}} t^{2k} \Omega_r(U_n^{(2k)}, t)_{p,\gamma}$$

holds.

Proof of Lemma 13 (i) For $k = 0$, Lemma 13 is obvious. (ii) For $r = 0$ and $k \in \mathbb{N}$ we set $U_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx) = \sum_{j=0}^{n} A_j(x, U_n)$ with $a_0, a_j, b_j \in \mathbb{R}$, $j \in \mathbb{N}$. Then

$$\begin{aligned} U_n^{(2r)}(x) &= \sum_{j=1}^{n} A_j\left(x, U_n^{(2r)}\right) = \sum_{j=1}^{n} j^{2r} A_j\left(x + \frac{r\pi}{j}, U_n\right) \\ &= \sum_{j=1}^{n} j^{2r}\left(\cos r\pi A_j(x, U_n) - \sin r\pi A_j\left(x, \widetilde{U_n}\right)\right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} A_j(x, U_n) &= a_j \cos j\left(x + \frac{r\pi}{j} - \frac{r\pi}{j}\right) + b_j \sin j\left(x + \frac{r\pi}{j} - \frac{r\pi}{j}\right) \\ &= A_j\left(x + \frac{r\pi}{j}, U_n\right)\cos r\pi + A_j\left(x + \frac{r\pi}{j}, \widetilde{U_n}\right)\sin r\pi. \end{aligned}$$

Setting

$$\text{sinc}t := \begin{cases} \frac{\sin t}{t} & , t > 0 \\ 1 & , t = 0 \end{cases}$$

we have the obvious inequality

$$1 - \text{sinc}t \leq t^2 \text{ for } t \geq 0.$$

We get for $0 < \delta \leq t$ that

$$\begin{aligned} \left\|(I - \sigma_\delta)^r U_n\right\|_{p,\gamma} &= \left\|\sum_{j=0}^{n} (1 - \text{sinc}j\delta)^r A_j(x, U_n)\right\|_{p,\gamma} \\ &= \left\|\sum_{j=1}^{n} \left(\frac{1 - \text{sinc}j\delta}{(j\delta)^2}\right)^r (j\delta)^{2r} A_j(x, U_n)\right\|_{p,\gamma} \\ &\leq t^{2r}\left\|\sum_{j=1}^{n} \left(\frac{1 - \text{sinc}j\delta}{(j\delta)^2}\right)^r j^{2r} A_j(x, U_n)\right\|_{p,\gamma}. \end{aligned}$$

We define

$$h_j := \begin{cases} \frac{\left(1 - \text{sinc}\frac{j}{n}\right)^r}{\left(\frac{j}{n}\right)^{2r}} & , j = 1, 2, \cdots, n, \\ 0 & , j > n. \end{cases}$$

For $j = 1, 2, 3, \ldots,$ $\{h_j\}$ satisfies (29) with $A = (0, 17)^r$. Now using Lemma 12 we obtain

$$\begin{aligned}\|(I-\sigma_\delta)^r U_n\|_{p,\gamma} &\leq ct^{2r}\left\|\sum_{j=1}^{n} j^{2r} A_j(x, U_n)\right\|_{p,\gamma} \\ &= ct^{2r}\left\|\sum_{j=1}^{n} j^{2r}\left[A_j\left(x+\frac{r\pi}{j}, U_n\right)\cos r\pi + A_j\left(x+\frac{r\pi}{j}, \widetilde{U_n}\right)\sin r\pi\right]\right\|_{p,\gamma} \\ &\leq ct^{2r}\left(\left\|\sum_{j=1}^{n} j^{2r} A_j\left(x+\frac{r\pi}{j}, U_n\right)\right\|_{p,\gamma} + \left\|\sum_{j=1}^{n} j^{2r} A_j\left(x+\frac{r\pi}{j}, \widetilde{U_n}\right)\right\|_{p,\gamma}\right).\end{aligned}$$

Note that [20, p.161]

$$A_j\left(x, U_n^{(2r)}\right) = j^{2r} A_j\left(x+\frac{r\pi}{j}, U_n\right), \quad j \in \mathbb{N}.$$

Using [17, Theorem 1] we find

$$\left\|\widetilde{U_n^{(2r)}}\right\|_{p,\gamma} \leq C_{p,r,[\gamma]_{A_p}}\left\|U_n^{(2r)}\right\|_{p,\gamma}.$$

Also, (27) and (31) imply that

$$\widetilde{U_n}^{(2r)} = \widetilde{U_n^{(2r)}}.$$

Summing up, we find

$$\begin{aligned}\Omega_r(U_n, t)_{p,\gamma} &= \sup_{0\leq\delta\leq t}\|(I-\sigma_\delta)^r U_n\|_{p,\gamma} \\ &\leq ct^{2r}\left(\left\|U_n^{(2r)}\right\|_{p,\gamma} + \left\|\widetilde{U_n}^{(2r)}\right\|_{p,\gamma}\right) \\ &= ct^{2r}\left(\left\|U_n^{(2r)}\right\|_{p,\gamma} + \left\|\widetilde{U_n^{(2r)}}\right\|_{p,\gamma}\right) \leq ct^{2r}\left\|U_n^{(2r)}\right\|_{p,\gamma}.\end{aligned}$$

(iii) Let both r and k not be equal to zero. Using Lemma 12 we have for $0<h\leq t$

$$\begin{aligned}
\left\|(I-\sigma_h)^{r+k}U_n\right\|_{p,\gamma} &= \left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r+k}A_j(x,U_n)\right\|_{p,\gamma} \\
&\leq ch^{2k}\left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r}j^{2k}A_j(x,U_n)\right\|_{p,\gamma} \\
&\leq ct^{2k}\left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r}j^{2k}A_j\left(x+\frac{k\pi}{j},U_n\right)\cos\beta\pi\right\|_{p,\gamma} \\
&\quad +ct^{2k}\left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r}j^{2k}A_j\left(x+\frac{k\pi}{j},\widetilde{U_n}\right)\sin\beta\pi\right\|_{p,\gamma}.
\end{aligned}$$

Since the conjugate operator is linear and bounded [17] in L_γ^p for $p\in(1,\infty)$ and $\gamma\in A_p$, we have

$$\begin{aligned}
\Omega_{r+k}(U_n,t)_{p,\gamma} &= \sup_{0\leq h\leq t}\left\|(I-\sigma_h)^{r+k}U_n\right\|_{p,\gamma} \\
&\leq ct^{2k}\sup_{0\leq h\leq t}\left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r}j^{2k}A_j\left(x+\frac{k\pi}{j},U_n\right)\right\|_{p,\gamma} \\
&\quad +ct^{2k}\sup_{0\leq h\leq t}\left\|\sum_{j=0}^{n}(1-\mathrm{sinc}jh)^{r}j^{2k}A_j\left(x+\frac{k\pi}{j},\widetilde{U_n}\right)\right\|_{p,\gamma} \\
&= ct^{2k}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma}+Ct^{2k}\sup_{0\leq h\leq t}\left\|\left[(I-\sigma_h)^{r}U_n^{(2k)}\right]^{\sim}\right\|_{p,\gamma} \\
&\leq ct^{2k}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma}+Ct^{2k}\sup_{0\leq h\leq t}\left\|(I-\sigma_h)^{r}U_n^{(2k)}\right\|_{p,\gamma} \\
&\leq ct^{2k}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma}.
\end{aligned}$$

□

Proof of Theorem 5 The proof of (11) follows from (23). The proof of (12) is a consequence of (22) and the property

$$(I-\sigma_h)^{\alpha+\beta}f=(I-\sigma_h)^{\alpha}(I-\sigma_h)^{\beta}f,$$

which can be proved easily. Now we prove (13). Since $0<t<1$ there exists some $n\in\mathbb{N}$ so that $(1/n)<t\leq(2/n)$ holds. Then we have

$$\begin{aligned}
\Omega_{r+k}(f,t)_{p,\gamma} &\leq \Omega_{r+k}(U_n,t)_{p,\gamma}+\Omega_{r+k}(f-U_n,t)_{p,\gamma} \\
&\leq C_{r,k,p,[\gamma]_{A_p}}t^{2k}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma}+C_{r,k,p,[\gamma]_{A_p}}E_n(f)_{p,\gamma}.
\end{aligned}$$

On the other hand, using Theorem 1 of [4] and Theorem 4 we get

$$E_n(f)_{p,\gamma} \le \frac{C_{k,p,[\gamma]_{A_p}}}{n^{2k}} E_n\left(f^{(2k)}\right)_{p,\gamma} \le \frac{C_{r,k,p,[\gamma]_{A_p}}}{n^{2k}} \Omega_r\left(f^{(2k)},1/n\right)_{p,\gamma}$$

and

$$\begin{aligned}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma} &\le \Omega_r\left(U_n^{(2k)}-f^{(2k)},t\right)_{p,\gamma}+\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}\\ &\le C_{r,k,p,[\gamma]_{A_p}} E_n\left(f^{(2k)}\right)_{p,\gamma}+\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}\\ &\le C_{r,k,p,[\gamma]_{A_p}} \Omega_r\left(f^{(2k)},1/n\right)_{p,\gamma}+\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\Omega_{r+k}(f,t)_{p,\gamma} &\le C_{r,k,p,[\gamma]_{A_p}} t^{2k}\Omega_r\left(U_n^{(2k)},t\right)_{p,\gamma}+\frac{C_{r,k,p,[\gamma]_{A_p}}}{n^{2k}}\Omega_r\left(f^{(2k)},1/n\right)_{p,\gamma}\\ &\le C_{r,k,p,[\gamma]_{A_p}}\left[t^{2k}\Omega_r\left(f^{(2k)},\tfrac{1}{n}\right)_{p,\gamma}+t^{2k}\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}+\tfrac{1}{n^{2k}}\Omega_r\left(f^{(2k)},\tfrac{1}{n}\right)\right]_{p,\gamma}\\ &\le C_{r,k,p,[\gamma]_{A_p}}\left[t^{2k}\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}+t^{2k}\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}+t^{2k}\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}\right]\\ &= C_{r,k,p,[\gamma]_{A_p}} t^{2k}\Omega_r\left(f^{(2k)},t\right)_{p,\gamma}.\end{aligned}$$

□

4. Proof of the conjecture of Gadjieva

(1.20) of [12, p. 37] and (9) give the following:

Lemma 14 *Let $p\in(1,\infty)$, $\gamma\in A_p$, $f\in L_\gamma^p$, and $r\in\mathbb{N}$. Then for any $0<t<1$, the following inequality holds:*

$$\Omega_r(f,t)_{p,\gamma}\le C_{r,p,[\gamma]_{A_p}} t^{2r}\|f^{(2r)}\|_{p,\gamma},$$

with some constant depending only on r,p and $[\gamma]_{A_p}$.

We can start with the following Bernstein–Nikolski inequality.

Lemma 15 *Let $r,n\in\mathbb{N}$, $p\in(1,\infty)$, $\gamma\in A_p$, and $U_n\in\mathcal{T}_n$. Then*

$$h^{2r}\left\|U_n^{(2r)}\right\|_{p,\gamma}\lesssim\left\|(I-\sigma_h)^r U_n\right\|_{p,\gamma}$$

holds for any $h\in(0,\pi/n]$ with some constant depending only on r,p and $[\gamma]_{A_p}$.

Proof of Lemma 15 Let $U_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ with $a_0, a_k, b_k \in \mathbb{R},\ k \in \mathbb{N},\ h \in (0, \pi/n]$. Then

$$\begin{aligned}
h^{2r} \left\| U_n^{(2r)} \right\|_{p,\gamma} &= h^{2r} \left\| \sum_{k=1}^{n} k^{2r} A_k \left(x + \frac{r\pi}{k}, U_n \right) \right\|_{p,\gamma} \\
&= h^{2r} \left\| \sum_{k=1}^{n} k^{2r} \left(\cos r\pi A_k (x, U_n) - \sin r\pi A_k \left(x, \widetilde{U_n} \right) \right) \right\|_{p,\gamma} \\
&\leq h^{2r} \left\| \sum_{k=1}^{n} k^{2r} \cos r\pi A_k (x, U_n) \right\|_{p,\gamma} \\
&\quad + h^{2r} \left\| \sum_{k=1}^{n} k^{2r} \sin r\pi A_k \left(x, \widetilde{U_n} \right) \right\|_{p,\gamma} \\
&= \left\| \sum_{k=1}^{n} \cos r\pi \left(\frac{(kh)^2}{(1 - \text{sinc}kh)} \right)^r (1 - \text{sinc}kh)^r A_k (x, U_n) \right\|_{p,\gamma} \\
&\quad + \left\| \sum_{k=1}^{n} \sin r\pi \left(\frac{(kh)^2}{(1 - \text{sinc}kh)} \right)^r (1 - \text{sinc}kh)^r A_k \left(x, \widetilde{U_n} \right) \right\|_{p,\gamma}.
\end{aligned}$$

We will use Lemma A once more. Let

$$\lambda_j := \begin{cases} \dfrac{\left(\frac{j}{n}\right)^{2r}}{\left(1 - \frac{\sin\frac{j}{n}}{\frac{j}{n}}\right)^r} & , \text{for } 1 \leq j \leq n, \\ 0 & , \text{for } j > n. \end{cases}$$

For $j = 1, 2, 3, \ldots,$ $\{\lambda_j\}$ satisfies (29) with $A = (1 - \sin 1)^{-r}$. Using the Marcinkiewicz multiplier theorem [6] for Lebesgue spaces with Muckenhoupt weight, we have

$$\begin{aligned}
h^{2r} \left\| U_n^{(2r)} \right\|_{p,\gamma} &\lesssim \left\| \sum_{k=0}^{n} (1 - \text{sinc}kh)^r A_k (x, U_n) \right\|_{p,\gamma} + \left\| \sum_{k=0}^{n} (1 - \text{sinc}kh)^r A_k \left(x, \widetilde{U_n} \right) \right\|_{p,\gamma} \\
&= \left\| \sum_{k=0}^{n} (1 - \text{sinc}kh)^r A_k (x, U_n) \right\|_{p,\gamma} + \left\| \left(\sum_{k=0}^{n} (1 - \text{sinc}kh)^r A_k (x, U_n) \right)^{\sim} \right\|_{p,\gamma}.
\end{aligned}$$

In the last step we used the linear property of the conjugate operator. Thus, from the boundedness of the conjugate (see, e.g., [17]) operator, we get

$$h^{2r} \left\| U_n^{(2r)} \right\|_{p,\gamma} \lesssim \left\| \sum_{k=0}^{n} (1 - \text{sinc}kh)^r A_k (x, U_n) \right\|_{p,\gamma} = \left\| (I - \sigma_h)^r U_n \right\|_{p,\gamma}.$$

□

Proof of Theorem 3 From (9) and the right-hand side of inequality (1.27) in [12, p. 46] we get $\Omega_r (f, \delta)_{p,\gamma} \leq C_{r,p,[\gamma]_{A_p}} K_{2r} (\delta, f, p, \gamma)$. If $\delta > 0$ there exists $n \in N$ such that $\frac{n}{\pi} \leq 1/\delta < 2\frac{n}{\pi}$. Let U_n be the near best approximating trigonometric polynomial to f. From Theorem 4,

$$\|f-U_n\|_{p,\gamma} \lesssim E_n(f)_{p,\gamma} \lesssim \Omega_r\left(f,\frac{\pi}{n}\right)_{p,\gamma}.$$

Thus, using Lemma 15,

$$\begin{aligned}\delta^{2r}\left\|U_n^{(2r)}\right\|_{p,\gamma} &\lesssim \|(I-\sigma_\delta)^r U_n\|_{p,\gamma} \lesssim \Omega_r(U_n,\pi/n)_{p,\gamma} \\ &\lesssim \Omega_r(U_n-f,\pi/n)_{p,\gamma}+\Omega_r(f,\pi/n)_{p,\gamma} \\ &\lesssim \|f-U_n\|_{p,\gamma}+\Omega_r(f,\pi/n)_{p,\gamma} \lesssim \Omega_r(f,\pi/n)_{p,\gamma}\end{aligned}$$

and

$$\|f-U_n\|_{p,\gamma}+\delta^{2r}\left\|U_n^{(2r)}\right\|_{p,\gamma} \lesssim \Omega_r(f,\pi/n)_{p,\gamma}. \tag{32}$$

Now

$$K_{2r}(\delta,f,p,\gamma) \le \|f-U_n\|_{p,\gamma}+\delta^{2r}\left\|U_n^{(2r)}\right\|_{p,\gamma} \lesssim \Omega_r(f,\delta)_{p,\gamma}.$$

Thus, (8) is proved. □

5. Proof of the Jackson type inequality

Below we give a lemma required for the proof of Theorem 4.

Lemma 16 *Let $p\in(1,\infty)$, $\gamma\in A_p$, $F\in L_\gamma^p$, and $r\in\mathbb{N}$. Then there exists a number $\delta\in(0,1)$, depending only on p and $[\gamma]_{A_p}$ such that*

$$\|(I-\sigma_h)^r F\|_{p,\gamma} \lesssim C\delta^{mr}\|F\|_{p,\gamma}+C(m)\,C_{r,p,[\gamma]_{A_p}}\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}$$

holds for any $h\in(0,1)$ and $m\in\mathbb{N}$ where the constants $C>0$, $C_{r,p,[\gamma]_{A_p}}$ depending only on r,p and $[\gamma]_{A_p}$ and the constant $C(m)$ satisfy $C(m)=\sum_{i=0}^{m-1}(\delta^r)^i$.

Proof For any $h>0$ there exists (see, e.g., (21)) a constant $\mathfrak{C}>1$ such that

$$\|\sigma_h F\|_{p,\gamma} \le \mathfrak{C}\|F\|_{p,\gamma}.$$

We set $\delta:=\mathfrak{C}/(1+\mathfrak{C})$. Now, for any $h\in(0,1)$, we prove

$$\|(I-\sigma_h)^r F\|_{p,\gamma} \le \delta^r\|\left(I-\sigma_h^2\right)^r F\|_{p,\gamma}+c\Omega_{r+1}(F,h)_{p,\gamma}. \tag{33}$$

To prove (33) we observe

$$I-\sigma_h = 2^{-1}(I-\sigma_h)(I+\sigma_h)+2^{-1}(I-\sigma_h)^2$$

and

$$\sigma_h(I-\sigma_h) = 2^{-1}(I-\sigma_h)(I+\sigma_h)-2^{-1}(I-\sigma_h)^2.$$

Hence, for $g \in L^p_\gamma$

$$\|(I-\sigma_h)g\|_{p,\gamma}+\|\sigma_h(I-\sigma_h)g\|_{p,\gamma} \leq \|(I-\sigma_h)(I+\sigma_h)g\|_{p,\gamma}+\|(I-\sigma_h)^2 g\|_{p,\gamma}. \tag{34}$$

On the other hand,

$$\|(I-\sigma_h)^r F\|_{p,\gamma} = \delta\left((1/\mathfrak{C})\|(I-\sigma_h)^r F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right)$$

$$\begin{aligned}
&\leq \delta\left(\|(I-\sigma_h)^r F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right)\\
&= \delta\left(\|(I-\sigma_h)(I-\sigma_h)^{r-1}F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right)\\
&= \delta\left(\|\left(\sigma_h(I-\sigma_h)+(I-\sigma_h)^2\right)(I-\sigma_h)^{r-1}F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right)\\
&\leq \delta\left(\|\sigma_h(I-\sigma_h)(I-\sigma_h)^{r-1}F\|_{p,\gamma}+\|(I-\sigma_h)^2(I-\sigma_h)^{r-1}F\|_{p,\gamma}\right)\\
&\quad +\delta\|(I-\sigma_h)^r F\|_{p,\gamma}\\
&\leq \delta\left(\|\sigma_h(I-\sigma_h)^r F\|_{p,\gamma}+\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right).
\end{aligned} \tag{35}$$

Taking $g := (I-\sigma_h)^{r-1}F$ in (34) we have

$$\|\sigma_h(I-\sigma_h)^r F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma} \leq \|(I-\sigma_h)^r(\sigma_h+I)F\|_{p,\gamma}+\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}$$

and, using this in (35),

$$\begin{aligned}
\|(I-\sigma_h)^r F\|_{p,\gamma} &\leq \delta\left(\|\sigma_h(I-\sigma_h)^r F\|_{p,\gamma}+\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}+\|(I-\sigma_h)^r F\|_{p,\gamma}\right)\\
&\leq \delta\left(\|(I-\sigma_h)^r(\sigma_h+I)F\|_{p,\gamma}+\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}\right)\\
&\quad +\delta\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}\\
&\leq \delta\|(I-\sigma_h)^r(\sigma_h+I)F\|_{p,\gamma}+2\delta\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}.
\end{aligned} \tag{36}$$

Repeating r times the last inequality we have

$$\begin{aligned}
\|(I-\sigma_h)^r F\|_{p,\gamma} &\leq \delta\|(I-\sigma_h)^r(\sigma_h+I)F\|_{p,\gamma}+2\delta\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}\\
&\leq \delta^2\|(I-\sigma_h)^r(\sigma_h+I)^2F\|_{p,\gamma}+2\delta^2\|(I-\sigma_h)^{r+1}(\sigma_h+I)F\|_{p,\gamma}\\
&\quad +2\delta\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}\\
&\leq \ldots \leq \delta^r\|(I-\sigma_h)^r(\sigma_h+I)^r F\|_{p,\gamma}\\
&\quad +2\sum\nolimits_{k=1}^{r}\delta^k\|(I-\sigma_h)^{r+1}(\sigma_h+I)^{k-1}F\|_{p,\gamma}\\
&= \delta^r\|\left(I-\sigma_h^2\right)^r F\|_{p,\gamma}+2\sum\nolimits_{k=1}^{r}\delta^k\|(I-\sigma_h)^{r+1}(\sigma_h+I)^{k-1}F\|_{p,\gamma}.
\end{aligned}$$

Hence,

$$\|(I-\sigma_h)^r F\|_{p,\gamma} \leq \delta^r\|\left(I-\sigma_h^2\right)^r F\|_{p,\gamma}+C\left(r,p,[\gamma]_{A_p}\right)\|(I-\sigma_h)^{r+1}F\|_{p,\gamma}$$

and the proof of (33) is finished. Using the last inequality recursively we obtain

$$\| (I-\sigma_h)^r F\|_{p,\gamma} \leq \delta^r \| \left(I-\sigma_h^2\right)^r F\|_{p,\gamma} + C\left(r,p,[\gamma]_{A_p}\right) \| (I-\sigma_h)^{r+1} F\|_{p,\gamma}$$

$$\leq \delta^{2r} \| \left(I-\sigma_h^4\right)^r F\|_{p,\gamma} + (\delta^r+1)\, C\left(r,p,[\gamma]_{A_p}\right) \| (I-\sigma_h)^{r+1} F\|_{p,\gamma} \leq$$

$$\leq \delta^{4r} \| \left(I-\sigma_h^8\right)^r F\|_{p,\gamma} + \left(\delta^{2r}+\delta^r+1\right) C\left(r,p,[\gamma]_{A_p}\right) \| (I-\sigma_h)^{r+1} F\|_{p,\gamma} \leq \cdots$$

$$\leq \cdots \leq \delta^{mr} \| \left(I-\sigma_h^{2^m}\right)^r F\|_{p,\gamma} + C\left(r,p,[\gamma]_{A_p}\right) \left(\sum_{j=0}^{m-1} \delta^{rj}\right) \| (I-\sigma_h)^{r+1} F\|_{p,\gamma}. \tag{37}$$

Using

$$\|MF\|_{C[T]} \leq \|F\|_{C[T]}$$

([15, p. 78]) we have

$$\| \left(I-\sigma_h^{2^m}\right)^r F\|_{C[T]} = \left\| \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left(\sigma_h^{2^m}\right)^k (F) \right\|_{C[T]}$$

$$\leq \left\| \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left(M^{2^m}\right)^k (F) \right\|_{C[T]} \leq \sum_{k=0}^{r} \left| \binom{r}{k} \right| \left\| \left(M^{2^m}\right)^k (F) \right\|_{C[T]}$$

$$\leq \sum_{k=0}^{r} \left| \binom{r}{k} \right| \|F\|_{C[T]} \leq 2^r \|F\|_{C[T]}.$$

From this and a transference result we get that

$$\| \left(I-\sigma_h^{2^m}\right)^r F\|_{p,\gamma} \leq C_{p,r,[\gamma]_{A_p}} \|F\|_{p,\gamma}.$$

The last inequality and (37) gives

$$\| (I-\sigma_h)^r F\|_{p,\gamma} \lesssim C_{r,p,[\gamma]_{A_p}} \delta^{mr} \|F\|_{p,\gamma} + C(m)\, C_{r,p,[\gamma]_{A_p}} \| (I-\sigma_h)^{r+1} F\|_{p,\gamma}.$$

□

Proof of Theorem 4 First we prove inequality for $r = 1, 2, 3, 4, \ldots$. Following the idea of [10], for this purpose we will use induction on r. We know from Theorems 1 and 4 that

$$E_n(f)_{p,\gamma} \leq C_{p,[\gamma]_{A_p}} \Omega_1 \left(f, \frac{1}{n}\right)_{p,\gamma}.$$

We suppose that the inequality

$$E_n(f)_{p,\gamma} \leq \mathcal{C} \Omega_r \left(f, \frac{1}{n}\right)_{p,\gamma}, \quad r \in \mathbb{N} \tag{38}$$

holds for any $f \in L^p_\gamma$ with some constant $\mathcal{C} > 0$. We set $u(\cdot) := f(\cdot) - S_n f(\cdot)$. First we will show that

$$\|f - S_n f\|_{p,\gamma} \le C_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(f, \frac{1}{n}\right)_{p,\gamma}. \tag{39}$$

Then (39) will give (38). We have

$$\begin{aligned} S_n(u)(\cdot) &= S_n(f - S_n f)(\cdot) = \left(S_n(f) - S_n(S_n f)\right)(\cdot) \\ &= \left(S_n(f) - S_n(f)\right)(\cdot) = 0. \end{aligned}$$

Since $S_n f$ is the near best approximant for f, using induction hypothesis (38),

$$\|u\|_{p,\gamma} = \|u - S_n(u)\|_{p,\gamma} \le C_{p,[\gamma]_{A_p}} E_n(u)_{p,\gamma} \le \mathcal{C} C_{p,r,[\gamma]_{A_p}} \Omega_r\left(u, \frac{1}{n}\right)_{p,\gamma}.$$

We know from Lemma 16 that for $m \in \mathbb{N}$

$$\|\left(I - \sigma_h\right)^r u\|_{p,\gamma} \le C'_{p,r,[\gamma]_{A_p}} \delta^{mr} \|u\|_{p,\gamma} + C\left(m\right) C''_{p,r,[\gamma]_{A_p}} \|\left(I - \sigma_h\right)^{r+1} u\|_{p,\gamma}$$

and thus

$$\|u\|_{p,\gamma} \le \mathcal{C} C_{p,r,[\gamma]_{A_p}} C'_{p,r,[\gamma]_{A_p}} \delta^{mr} \|u\|_{p,\gamma} + \mathcal{C} C\left(m\right) C_{p,r,[\gamma]_{A_p}} C''_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(u, \frac{1}{n}\right)_{p,\gamma}.$$

Choosing m so big that $\mathcal{C} C_{p,[\gamma]_{A_p}} C'_{p,r,[\gamma]_{A_p}} \delta^{mr} \le 1/2$, from the last inequality we obtain

$$\|u\|_{p,\gamma} \le C_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(u, \frac{1}{n}\right)_{p,\gamma}.$$

From boundedness [17] of operator $f \longmapsto S_n f$ in L^p_γ for $p \in (1, \infty)$ and $\gamma \in A_p$ we have

$$\Omega_{r+1}\left(u, \frac{1}{n}\right)_{p,\gamma} \le C_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(f, \frac{1}{n}\right)_{p,\gamma}$$

and the result

$$\begin{aligned} E_n\left(f\right)_{p,\gamma} &\le \|f - S_n f\|_{p,\gamma} = \|u\|_{p,\gamma} \le C_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(u, \frac{1}{n}\right)_{p,\gamma} \\ &\le C_{p,r,[\gamma]_{A_p}} \Omega_{r+1}\left(f, \frac{1}{n}\right)_{p,\gamma} \end{aligned}$$

holds. Then (38) holds for any $r \in \mathbb{N}$. □

6. Characterization class of $\Omega_r(f, \cdot)_{p,\gamma}$

Let $\omega_r\left(\cdot, \delta\right)_p$, $(1 \le p \le \infty)$, be the usual nonweighted modulus of smoothness:

$$\omega_r\left(g, \delta\right)_p := \sup_{0 \le h \le \delta} \|\left(I - T_h\right)^r g\|_p, \quad g \in L^p,$$

where $T_h g(\cdot) := g(\cdot + h)$. By (1.31) of [12, p. 50], (8), and (14) there exist positive constants depending only on r, p such that

$$\omega_r(g,\delta)_p \approx \Omega_r(g,\delta)_{p,1} \tag{40}$$

holds for $1 \leq p \leq \infty$ and $g \in L^p$.

Proof of Theorem 7 (i) Note that if $F \in C[T]$ then

$$\|(I-\sigma_t)^r F\|_{p,\gamma} \leq C_{p,[\gamma]_{A_p}} \|(I-\sigma_t)^r F\|_{C(T)}. \tag{41}$$

Using Theorem 2.5 (A) of [40], (40), (14), (8), and (41) there exists $\psi \in \Phi_{2r}$ such that

$$\Omega_r(F,\delta)_{p,\gamma} \leq C_{p,[\gamma]_{A_p}} \Omega_r(F,\delta)_{\infty,1} \leq C_{p,[\gamma]_{A_p}} \omega_{2r}(F,\delta)_\infty \leq C_{r,p,[\gamma]_{A_p}} \psi(\delta).$$

If $p \in (1,\infty)$, $\gamma \in A_p$, $f \in L^p_\gamma$, by Lemma 4 of [20, $M(x) = x^p$], for any $\varepsilon > 0$ there exists a $F \in C[T]$ such that $\|f - F\|_{p,\gamma} < \varepsilon$. Thus,

$$\begin{aligned} \Omega_r(f,\delta)_{p,\gamma} &\leq \Omega_r(f-F,\delta)_{p,\gamma} + \Omega_r(F,\delta)_{p,\gamma} \\ &\leq C_{r,p,[\gamma]_{A_p}} \|f-F\|_{p,\gamma} + C_{r,p,[\gamma]_{A_p}} \psi(\delta). \end{aligned}$$

Letting $\varepsilon \to 0^+$ we get

$$\Omega_r(f,\delta)_{p,\gamma} \leq C_{r,p,[\gamma]_{A_p}} \psi(\delta).$$

On the other hand, from (8) and Theorem 2.5 (A) of [40],

$$\psi(\delta) \leq C_{r,p,[\gamma]_{A_p}} \omega_{2r}(f,\delta)_1 \leq C_{r,p,[\gamma]_{A_p}} \Omega_r(f,\delta)_{p,\gamma}$$

and equivalence (16) is established.

(ii) For the equivalence (17) let $\psi \in \Phi_{2r}$. By Theorem 2.5 (B) and Remark 2.7 (1) of [40] there exist $f \in L^\infty$ and a positive real number t_0 such that

$$\omega_{2r}(f,\delta)_p \approx \psi(\delta), \quad p = 1, \infty$$

holds for all $\delta \in (0, t_0)$ with equivalence constants depending only on r. Then by (8), (40), and (24) we get

$$\begin{aligned} \psi(\delta) &\leq C_r \omega_{2r}(f,\delta)_1 \leq C_r \Omega_r(f,\delta)_{1,1} \leq C_{r,p,[\gamma]_{A_p}} \Omega_r(f,\delta)_{p,\gamma} \\ &\leq C_{r,p,[\gamma]_{A_p}} \Omega_r(f,\delta)_{\infty,1} \leq C_{r,p,[\gamma]_{A_p}} \omega_{2r}(f,\delta)_\infty \leq C_{r,p,[\gamma]_{A_p}} \psi(\delta) \end{aligned}$$

for all $\delta \in (0, t_0)$. □

7. Realization functional

Proof of Theorem 11 Let U_n be the near best approximating trigonometric polynomial to f. By (32) and (15)

$$\|f - U_n\|_{p,\gamma} + \tfrac{1}{n^{2r}} \left\|U_n^{(2r)}\right\|_{p,\gamma} \lesssim \Omega_r(f,\pi/n)_{p,\gamma} \lesssim \Omega_r(f,1/n)_{p,\gamma}$$

and hence $R_{2r}(f,1/n,p,\gamma) \lesssim \Omega_r(f,1/n)_{p,\gamma}$. For the reverse inequality we use (23) and Lemma 15 (with $h = 1/n$):

$$\begin{aligned}\Omega_r(f,1/n)_{p,\gamma} &\leq \Omega_r(f-U_n,1/n)_{p,\gamma} + \Omega_r(U_n,1/n)_{p,\gamma} \\ &\lesssim \|f-U_n\|_{p,\gamma} + \tfrac{1}{n^{2r}}\left\|U_n^{(2r)}\right\|_{p,\gamma} = R_{2r}(f,1/n,p,\gamma).\end{aligned}$$

□

8. Fractional order modulus of smoothness

Fractional order modulus of smoothness is not a new concept. Classical nonweighted fractional smoothness $\omega_r(f,\cdot)_p$, $r>0$, was defined by Butzer et al. [8] and Taberski [39]. See also [35]. Here we consider fractional smoothness $\Omega_r(\cdot,\delta)_{p,\gamma}$, $r>0$, suitable for some weighted spaces. Letting $x\in T$, $r,t>0$, $N\in\mathbb{N}$, $p\in(1,\infty)$, $\gamma\in A_p$, and $f\in L_\gamma^p$, we define the quantity

$$\begin{aligned}\Xi_t^r f(x) \quad &: \quad = (I-\sigma_t)^r f(x) = \sum_{k=0}^{\infty}\binom{r}{k}(-1)^k \sigma_t^k f(x) \qquad (42)\\ &= \lim_{N\to\infty}\sum_{k=0}^{N}\binom{r}{k}(-1)^k (\sigma_t f)^k(x)\end{aligned}$$

where $\binom{r}{k} := \frac{r(r-1)\ldots(r-k+1)}{k!}$ for $k>1$ and $\binom{r}{1} := r$ and $\binom{r}{0} := 1$ are Binom coefficients. Note that when $r\in\mathbb{N}$ (42) turns into (5).

If $F\in C[T]$ then we know that $\|\sigma_t F\|_{C[T]} \leq \|F\|_{C[T]}$ and $\|\Xi_t^r F\|_{C[T]} \leq 2^r\|F\|_{C[T]}$. From the last inequality and a transference result we can obtain that there exists a constant C independent of t such that

$$\|\Xi_t^r f\|_{p,\gamma} \leq C_{p,[\gamma]_{A_p},r}\|f\|_{p,\gamma} \qquad (43)$$

holds for $r>0$ with $p\in(1,\infty)$, $\gamma\in A_p$, and $f\in L_\gamma^p$.

Now we can define the weighted fractional modulus of smoothness: if $r\in\mathbb{R}^+$, $p\in(1,\infty)$, $f\in L_\gamma^p$, and $\gamma\in A_p$ we define

$$\Omega_r(f,\delta)_{p,\gamma} := \sup_{0\leq t\leq\delta}\|\Xi_t^r f\|_{p,\gamma}, \quad \Omega_0(f,\delta)_{p,\gamma} := \|f\|_{p,\gamma}.$$

In this case,

$$\Omega_r(f,\delta)_{p,\gamma} \leq C_{p,r,[\gamma]_{A_p}}\|f\|_{p,\gamma} \qquad (44)$$

for some constant $c>0$ dependent only on p,r and $[\gamma]_{A_p}$. Hence, the modulus of smoothness $\Omega_r(\cdot,\delta)_{p,\gamma}$ is a well-defined, nonnegative, nondecreasing function of δ on $(0,\infty)$ and satisfies the usual property $\Omega_r(f+g,\cdot)_{p,\gamma} \leq \Omega_r(f,\cdot)_{p,\gamma} + \Omega_r(g,\cdot)_{p,\gamma}$.

Remark 17 *(44) implies that all the results given in the introduction above also hold for replacement of $r\in\mathbb{N}$ by $r\in\mathbb{R}^+$. Indeed, (i) for Theorem 4 see Proposition 1 of [2]. For other theorems see the results given in [3].*

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