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SOME FIXED-POINT RESULTS ON PARAMETRIC N_b-METRIC SPACES

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Abstract. Our aim is to introduce the notion of a parametric N_b-metric and study some basic properties of parametric N_b-metric spaces. We give some fixed-point results on a complete parametric N_b-metric space. Some illustrative examples are given to show that our results are valid as the generalizations of some known fixed-point results. As an application of this new theory, we prove a fixed-circle theorem on a parametric N_b-metric space.

1. Introduction

Fixed-point theory has been studied by various methods. One of these methods is to change the contractive condition (see [2], [3], [6], [9], [10] and [15] for more details). Another method for this purpose is to generalize the metric space. For this reason, some generalized metric spaces have been introduced (see [1], [4], [5], [12], [11], [13] and [14] for more details). For example, in [1], the notion of a b-metric space was introduced as a generalization of a metric space. Also the concepts of a parametric metric space and parametric b-metric space were defined in [4] and [5], respectively. In [12], it was brought a different approach called S-metric, defined on a domain with three dimensions. The notion of an S-metric space was expanded to the notions of an S_b-metric space and a parametric S-metric space in [11] and [13], respectively. In [14], the concept of an A_b-metric space was given as a generalization of an S_b-metric space. An A_b-metric was defined on a domain with n dimensions.

In this paper, we define a new generalized metric space called a parametric N_b-metric space. In Section 2, we present the concept of a parametric N_b-metric space with some basic facts and study some relationships between the new metric space and other metric spaces. In Section 3, we extend the well known Ćirić's fixed-point result using an appropriate contractive condition defined on a complete parametric N_b-metric space. In Section 4, we give a new version

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of Kannan's fixed-point result using the notion of a parametric N_b-metric. In Section 5, we obtain a new generalization of the classical Chatterjea's fixed-point theorem. In Section 6, we prove a fixed-point theorem for a surjective self-mapping using an expansive mapping on a complete parametric N_b-metric space. In Section 7, we obtain some illustrative examples for the obtained theorems. In Section 8, we get a new approach from fixed-point theory to fixed-circle theory on a parametric N_b-metric space.

2. Parametric N_b-metric spaces

Before stating our main results we recall the definitions of an S_b-metric space and a parametric S-metric space.

Definition 2.1 ([11])**.** Let X be a nonempty set and $b \geq 1$ be a given real number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric if and only if for all $u_1, u_2, u_3, a \in X$ the following conditions are satisfied:
(S_b1) $S_b(u_1, u_2, u_3) = 0$ if and only if $u_1 = u_2 = u_3$,
(S_b2) $S_b(u_1, u_2, u_3) \leq b\,[S_b(u_1, u_1, a) + S_b(u_2, u_2, a) + S_b(u_3, u_3, a)]$.
Then the pair (X, S_b) is called an S_b-metric space.

Every S-metric is an S_b-metric with $b = 1$.

Definition 2.2 ([13])**.** Let X be a nonempty set and $P_S : X \times X \times X \times (0, \infty) \to [0, \infty)$ be a function. P_S is called a parametric S-metric on X, if
($PS1$) $P_S(u_1, u_2, u_3, t) = 0$ if and only if $u_1 = u_2 = u_3$,
($PS2$) $P_S(u_1, u_2, u_3, t) \leq P_S(u_1, u_1, a, t) + P_S(u_2, u_2, a, t) + P_S(u_3, u_3, a, t)$
for each $u_1, u_2, u_3, a \in X$ and all $t > 0$. The pair (X, P_S) is called a parametric S-metric space.

Now we give a new definition.

Definition 2.3. Let $X \neq \emptyset$, $b \geq 1$ be a given real number and $N : X^3 \times (0, \infty) \to [0, \infty)$ be a function. N is called a parametric S_b-metric on X if
(P_S^b1) $N(u_1, u_2, u_3, t) = 0$ if and only if $u_1 = u_2 = u_3$,
(P_S^b2) $N(u_1, u_2, u_3, t) \leq b\,[N(u_1, u_1, a, t) + N(u_2, u_2, a, t) + N(u_3, u_3, a, t)]$
for each $u_i, a \in X$ ($i \in \{1, 2, 3\}$) and $t > 0$. Then the pair (X, N) is called a parametric S_b-metric space.

From now on, we will denote $N(u, u, \ldots, (u)_{n-1}, v, t)$ by $N_{u,v,t}$ and define the notion of a parametric N_b-metric space as a generalization of a parametric S_b-metric space.

Definition 2.4. Let $X \neq \emptyset$, $b \geq 1$ be a given real number, $n \in \mathbb{N}$ and $N : X^n \times (0, \infty) \to [0, \infty)$ be a function. N is called a parametric N_b-metric on X if
($N1$) $N(u_1, u_2, \ldots, u_{n-1}, u_n, t) = 0$ if and only if $u_1 = u_2 = \cdots = u_{n-1} = u_n$,

$(N2)$ $N(u_1, u_2, \dots, u_{n-1}, u_n, t) \leq b[N_{u_1,a,t} + N_{u_2,a,t} + \cdots + N_{u_{n-1},a,t} + N_{u_n,a,t}]$ for each $u_i, a \in X$ $(i \in \{1, 2, \dots, n\})$ and $t > 0$. In this case, the pair (X, N) is called a parametric N_b-metric space.

We note that parametric N_b-metric spaces are a generalization of parametric S-metric spaces because every parametric S-metric is a parametric N_b-metric with $b = 1$ and $n = 3$.

Example 2.5. Let $X = \{f \mid f : (0, \infty) \to \mathbb{R} \text{ is a function}\}$, $n = 3$ and the function $N : X^3 \times (0, \infty) \to [0, \infty)$ be defined by

$$N(f, g, h, t) = \frac{1}{9} \left(|f(t) - g(t)| + |f(t) - h(t)| + |g(t) - h(t)|\right)^2$$

for each $f, g, h \in X$ and all $t > 0$. Then (X, N) is a parametric N_b-metric space with $b = 4$, but it is not a parametric S-metric space. Indeed, let us consider the following functions for each $u \in (0, \infty)$,

$$f(u) = 7,\ g(u) = 9,\ h(u) = 11 \text{ and } a(u) = 8.$$

Then the condition $(PS2)$ is not satisfied.

Lemma 2.6. *Let (X, N) be a parametric N_b-metric space. Then we have*

$$N_{u,v,t} \leq bN_{v,u,t} \text{ and } N_{v,u,t} \leq bN_{u,v,t}$$

for each $u, v \in X$ and all $t > 0$.

Proof. Using conditions $(N1)$ and $(N2)$, we get

$$N_{u,v,t} \leq b \left[\ N_{u,u,t} + N_{u,u,t} + \cdots + (N_{u,u,t})_{n-1} + N_{v,u,t}\ \right] = bN_{v,u,t}$$

and similarly

$$N_{v,u,t} \leq b \left[\ N_{v,v,t} + N_{v,v,t} + \cdots + (N_{v,v,t})_{n-1} + N_{u,v,t}\ \right] = bN_{u,v,t}$$

for each $u, v \in X$ and all $t > 0$. □

Lemma 2.7. *Let (X, N) be a parametric N_b-metric space. Then we have*

$$N_{u,v,t} \leq b\left[(n-1)N_{u,z,t} + N_{v,z,t}\right]$$

and

$$N_{u,v,t} \leq b\left[(n-1)N_{u,z,t} + bN_{z,v,t}\right]$$

for each $u, v, z \in X$ and all $t > 0$.

Proof. Using the condition $(N2)$, we obtain

$$\begin{aligned} N_{u,v,t} &\leq b \left[\ N_{u,z,t} + N_{u,z,t} + \cdots + (N_{u,z,t})_{n-1} + N_{v,z,t}\ \right] \\ &= b\left[(n-1)N_{u,z,t} + N_{v,z,t}\right] \end{aligned} \tag{2.1}$$

for each $u, v, z \in X$ and all $t > 0$. Using the inequality (2.1) and Lemma 2.6, we get

$$N_{u,v,t} \leq b\left[(n-1)N_{u,z,t} + bN_{z,v,t}\right].$$ □

Lemma 2.8. *Let (X, N) be a parametric N_b-metric space and the function $D_N : (X \times X)^n \times (0, \infty) \to [0, \infty)$ be defined by*

$$D_N((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), t) = N(u_1, u_2, \ldots, u_n, t) + N(v_1, v_2, \ldots, v_n, t)$$

for each $u_i, v_j \in X$ $(i, j \in \{1, 2, \ldots, n\})$ and all $t > 0$. Then $(X \times X, D_N)$ is a parametric N_b-metric space on $X \times X$.

Proof. Let $(u_i, v_i), (a, c) \in X \times X$. We use repeatedly condition $(N1)$. We have

$$D_N((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), t) = 0$$

if and only if

$$N(u_1, u_2, \ldots, u_n, t) + N(v_1, v_2, \ldots, v_n, t) = 0$$

if and only if

$$N(u_1, u_2, \ldots, u_n, t) = 0 \text{ and } N(v_1, v_2, \ldots, v_n, t) = 0$$

if and only if

$$u_1 = u_2 = \cdots = u_n \text{ and } v_1 = v_2 = \cdots = v_n$$

if and only if

$$(u_1, v_1) = (u_2, v_2) = \cdots = (u_n, v_n).$$

This proves $(N1)$. For condition $(N2)$

$$\begin{aligned} & D_N((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), t) \\ &= N(u_1, u_2, \ldots, u_n, t) + N(v_1, v_2, \ldots, v_n, t) \\ &\le b\,[N_{u_1,a,t} + N_{u_2,a,t} + \cdots + N_{u_n,a,t}] + b\,[N_{v_1,c,t} + N_{v_2,c,t} + \cdots + N_{v_n,c,t}] \\ &= b \begin{bmatrix} D_N((u_1, v_1), (u_1, v_1), \ldots, (a, c), t) \\ +D_N((u_2, v_2), (u_2, v_2), \ldots, (a, c), t) \\ + \cdots + D_N((u_n, v_n), (u_n, v_n), \ldots, (a, c), t) \end{bmatrix} \end{aligned}$$

and so

$$\begin{aligned} & D_N((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n), t) \\ &\le b \begin{bmatrix} D_N((u_1, v_1), (u_1, v_1), \ldots, (a, c), t) \\ +D_N((u_2, v_2), (u_2, v_2), \ldots, (a, c), t) \\ + \ldots + D_N((u_n, v_n), (u_n, v_n), \ldots, (a, c), t) \end{bmatrix}. \end{aligned}$$

Consequently, $(X \times X, D_N)$ is a parametric N_b-metric space on $X \times X$. □

Remark 2.9. 1) If we take $n = 3$ in Lemma 2.8, then we have

$$D_N((u_1, v_1), (u_2, v_2), (u_3, v_3), t) = N(u_1, u_2, u_3, t) + N(v_1, v_2, v_3, t)$$

for each $u_i, v_j \in X$ $(i, j \in \{1, 2, 3\})$ and all $t > 0$, and $(X \times X, D_N)$ is a parametric S_b-metric space.

2) If we take $n = 3$ and $b = 1$ in Lemma 2.8, then we have

$$D_N((u_1, v_1), (u_2, v_2), (u_3, v_3), t) = P_S(u_1, u_2, u_3, t) + P_S(v_1, v_2, v_3, t)$$

for each $u_i, v_j \in X$ $(i, j \in \{1, 2, 3\})$ and all $t > 0$, and $(X \times X, D_N)$ is a parametric S-metric space.

Definition 2.10. Let (X, N) be a parametric N_b-metric space and $\{u_k\}$ be a sequence in X. Then

(1) $\{u_k\}$ converges to u in X if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have $N_{u_k,u,t} \leq \varepsilon$, that is, $\lim_{k\to\infty} N_{u_k,u,t} = 0$. We will write $\lim_{k\to\infty} u_k = u$.

(2) $\{u_k\}$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, l \geq n_0$, we have $N_{u_k,u_l,t} \leq \varepsilon$, that is, $\lim_{k,l\to\infty} N_{u_k,u_l,t} = 0$.

(3) (X, N) is called complete if every Cauchy sequence is a convergent sequence.

Lemma 2.11. *Let (X, N) be a parametric N_b-metric space. If the sequence $\{u_k\}$ in X converges to u, then u is unique.*

Proof. Let $\{u_k\}$ converges to u and v with $u \neq v$. Then for each $\varepsilon > 0$, there exist $k_1, k_2 \in \mathbb{N}$ such that for all $k_1, k_2 \geq n_0$,

$$N_{u_k,u,t} < \frac{\varepsilon}{2b^2(n-1)} \text{ and } N_{u_k,v,t} < \frac{\varepsilon}{2b^2}$$

for all $t > 0$ and $b \geq 1$. If we put $n_0 = \max\{k_1, k_2\}$, then using the conditions $(N1)$, $(N2)$ and Lemma 2.7, for every $k \geq n_0$ we obtain

$$\begin{aligned} N_{u,v,t} &\leq b(n-1)N_{u,u_k,t} + bN_{v,u_k,t} \leq b^2(n-1)N_{u_k,u,t} + b^2 N_{u_k,v,t} \\ &< b^2(n-1)\frac{\varepsilon}{2b^2(n-1)} + b^2\frac{\varepsilon}{2b^2} = \varepsilon \end{aligned}$$

and we get $N_{u,v,t} = 0$, that is $u = v$. □

Lemma 2.12. *Let (X, N) be a parametric N_b-metric space. If the sequence $\{u_k\}$ in X converges to u, then $\{u_k\}$ is a Cauchy sequence.*

Proof. Since the sequence $\{u_k\}$ in X converges to u then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that for all $k \geq n_1$, $l \geq n_2$,

$$N_{u_k,u,t} < \frac{\varepsilon}{2b(n-1)} \text{ and } N_{u_l,u,t} < \frac{\varepsilon}{2b}$$

for all $t > 0$ and $b \geq 1$. If we put $n_0 = \max\{n_1, n_2\}$, then for every $k, l \geq n_0$ we get

$$N_{u_k,u_l,t} \leq b(n-1)N_{u_k,u,t} + bN_{u_l,u,t} < \varepsilon.$$

Therefore $\{u_k\}$ is Cauchy. □

Lemma 2.13. *Let (X, N) be a parametric N_b-metric space and $\{u_k\}$, $\{v_k\}$ be two convergent sequences to u and v, respectively. Then we have*

$$\frac{1}{b^2}N_{u,v,t} \leq \liminf_{k\to\infty} N_{u_k,v_k,t} \leq \limsup_{k\to\infty} N_{u_k,v_k,t} \leq b^2 N_{u,v,t}$$

for all $t > 0$. In particular, if $\{v_k\}$ is a constant sequence such that $v_k = v$, then we get

$$\frac{1}{b^2} N_{u,v,t} \leq \liminf_{k\to\infty} N_{u_k,v,t} \leq \limsup_{k\to\infty} N_{u_k,v,t} \leq b^2 N_{u,v,t}$$

for all $t > 0$. Also if $u = v$, then we have

$$\lim_{k\to\infty} N_{u_k,v,t} = 0$$

for all $t > 0$.

Proof. Using the condition $(N2)$, Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned} N_{u,v,t} &\leq b(n-1)N_{u,u_k,t} + bN_{v,u_k,t} \\ &\leq b(n-1)N_{u,u_k,t} + b^2(n-1)N_{v,v_k,t} + b^2 N_{u_k,v_k,t} \\ &\leq b^2(n-1)N_{u_k,u,t} + b^3(n-1)N_{v_k,v,t} + b^2 N_{u_k,v_k,t} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} N_{u_k,v_k,t} &\leq b(n-1)N_{u_k,u,t} + bN_{v_k,u,t} \\ &\leq b(n-1)N_{u_k,u,t} + b^2(n-1)N_{v_k,v,t} + b^2 N_{u,v,t} \end{aligned} \tag{2.3}$$

for all $t > 0$. Taking lower limit for $k \to \infty$ in the inequality (2.2) and upper limit for $k \to \infty$ in the inequality (2.3), we get

$$\frac{1}{b^2} N_{u,v,t} \leq \liminf_{k\to\infty} N_{u_k,v_k,t} \leq \limsup_{k\to\infty} N_{u_k,v_k,t} \leq b^2 N_{u,v,t}$$

for all $t > 0$. If $v_k = v$, then we find

$$N_{u,v,t} \leq b(n-1)N_{u,u_k,t} + bN_{v,u_k,t} \leq b^2(n-1)N_{u_k,u,t} + b^2 N_{u_k,v,t} \tag{2.4}$$

and

$$N_{u_k,v,t} \leq b(n-1)N_{u_k,u,t} + bN_{v,u,t} \leq b(n-1)N_{u_k,u,t} + bN_{u,v,t} \tag{2.5}$$

for all $t > 0$. Taking lower limit for $k \to \infty$ in the inequality (2.4) and upper limit for $k \to \infty$ in the inequality (2.5), we get the desired result. It can be easily seen that $u = v$ then we have

$$\lim_{k\to\infty} N_{u_k,v,t} = 0.$$

□

Lemma 2.14. *Let (X, N) be a parametric N_b-metric space. If there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that*

$$\lim_{k\to\infty} N_{u_k,v_k,t} = 0,$$

whenever $\{u_k\}$ is a convergent sequence in X such that $\lim_{k\to\infty} u_k = u_0$ for some $u_0 \in X$, then we have $\lim_{k\to\infty} v_k = u_0$.

Proof. Using the condition $(N2)$, Lemmas 2.6 and 2.7, we have

$$N_{v_k,u_0,t} \le b(n-1)N_{v_k,u_k,t} + bN_{u_0,u_k,t} \le b^2(n-1)N_{u_k,v_k,t} + b^2 N_{u_k,u_0,t}$$

and so taking upper limit for $k \to \infty$ we get

$$\limsup_{k\to\infty} N_{v_k,u_0,t} \le b^2(n-1)\limsup_{k\to\infty} N_{u_k,v_k,t} + b^2 \limsup_{k\to\infty} N_{u_k,u_0,t}$$

and so we obtain $\lim_{k\to\infty} v_k = u_0$. □

3. A new generalization of Ćirić's fixed-point result

In this section we extend the known Ćirić's fixed-point result [3] using an appropriate contractive condition defined on a complete parametric N_b-metric space. We prove the following theorem.

Theorem 3.1. *Let (X, N) be a complete parametric N_b-metric space and T be a self-mapping of X satisfying*

$$N_{Tu,Tv,t} \le h \max\left\{ N_{u,v,t}, N_{Tu,u,t}, N_{Tv,v,t}, N_{Tv,u,t}, N_{Tu,v,t} \right\} \tag{3.1}$$

for each $u, v \in X$, all $t > 0$ and some $0 \le h < \frac{1}{b+b^2(n-1)}$. Then T has a unique fixed point in X.

Proof. Let $u_0 \in X$ and the sequence $\{u_k\}$ be defined as

$$Tu_0 = u_1,\ Tu_1 = u_2, \ldots,\ Tu_k = u_{k+1}, \ldots.$$

Assume that $u_k \neq u_{k+1}$ for all k. Using the condition (3.1), we get

$$\begin{aligned} N_{u_k,u_{k+1},t} &= N_{Tu_{k-1},Tu_k,t} \\ &\le h \max\left\{N_{u_{k-1},u_k,t}, N_{u_k,u_{k-1},t}, N_{u_{k+1},u_k,t}, N_{u_{k+1},u_{k-1},t}, N_{u_k,u_k,t}\right\} \\ &= h \max\left\{N_{u_{k-1},u_k,t}, N_{u_k,u_{k-1},t}, N_{u_{k+1},u_k,t}, N_{u_{k+1},u_{k-1},t}\right\}. \end{aligned} \tag{3.2}$$

By Lemma 2.7, we obtain

$$N_{u_{k+1},u_{k-1},t} \le b(n-1)N_{u_{k+1},u_k,t} + bN_{u_{k-1},u_k,t}. \tag{3.3}$$

Using the inequalities (3.2), (3.3) and Lemma 2.6, we have

$$\begin{aligned} N_{u_k,u_{k+1},t} &\le h \max\left\{ \begin{array}{c} N_{u_{k-1},u_k,t}, bN_{u_{k-1},u_k,t}, bN_{u_k,u_{k+1},t}, \\ b^2(n-1)N_{u_k,u_{k+1},t} + bN_{u_{k-1},u_k,t} \end{array} \right\} \\ &= hb^2(n-1)N_{u_k,u_{k+1},t} + hbN_{u_{k-1},u_k,t} \end{aligned}$$

and so

$$(1 - hb^2(n-1))N_{u_k,u_{k+1},t} \le hbN_{u_{k-1},u_k,t},$$

which implies

$$N_{u_k,u_{k+1},t} \le \frac{hb}{1-hb^2(n-1)} N_{u_{k-1},u_k,t}. \tag{3.4}$$

Let $a = \frac{hb}{1-hb^2(n-1)}$. Then $a < 1$ since $hb + hb^2(n-1) < 1$. Notice that $1 - hb^2(n-1) \neq 0$ since $0 \leq h < \frac{1}{b+b^2(n-1)}$. For $k \in \{1, 2, \ldots\}$, using the inequality (3.4) and mathematical induction, we find

$$N_{u_k,u_{k+1},t} \leq a^k N_{u_0,u_1,t}. \tag{3.5}$$

Now we show that the sequence $\{u_k\}$ is a Cauchy sequence. Then for all $k, l \in \mathbb{N}$ with $l > k$, using the inequality (3.5), the condition $(N2)$, Lemmas 2.6 and 2.7, we get

$$\begin{aligned}
N_{u_k,u_l,t} &\leq b(n-1)N_{u_k,u_{k+1},t} + bN_{u_l,u_{k+1},t} \leq b(n-1)N_{u_k,u_{k+1},t} + b^2 N_{u_{k+1},u_l,t} \\
&\leq b(n-1)N_{u_k,u_{k+1},t} + b^3(n-1)N_{u_{k+1},u_{k+2},t} + b^3 N_{u_l,u_{k+2},t} \\
&\leq b(n-1)N_{u_k,u_{k+1},t} + b^3(n-1)N_{u_{k+1},u_{k+2},t} + b^4 N_{u_{k+2},u_l,t} \\
&\leq b(n-1)N_{u_k,u_{k+1},t} + b^3(n-1)N_{u_{k+1},u_{k+2},t} \\
&\quad + b^5(n-1)N_{u_{k+2},u_{k+3},t} + b^5 N_{u_l,u_{k+3},t} \\
&\leq b(n-1)N_{u_k,u_{k+1},t} + b^3(n-1)N_{u_{k+1},u_{k+2},t} \\
&\quad + b^5(n-1)N_{u_{k+2},u_{k+3},t} + b^7(n-1)N_{u_{k+3},u_{k+4},t} \\
&\quad + \cdots \\
&\quad + b^{2l-2k-3}(n-1)N_{u_{l-2},u_{l-1},t} + b^{2l-2k-2} N_{u_{l-1},u_l,t} \\
&\leq (n-1)\left[ba^k + b^3 a^{k+1} + b^5 a^{k+2} + \cdots + b^{2l-2k-3} a^{l-2}\right] \\
&\quad \times N_{u_0,u_1,t} + b^{2l-2k-2} a^{l-1} N_{u_0,u_1,t} \\
&= (n-1)ba^k \left[1 + b^2 a + b^4 a^2 + \cdots + b^{2l-2k-4} a^{l-k-2}\right] \\
&\quad \times N_{u_0,u_1,t} + ba^k b^{2l-2k-3} a^{l-k-1} N_{u_0,u_1,t} \\
&\leq (n-1)ba^k \left[1 + b^2 a + b^4 a^2 + \cdots\right] N_{u_0,u_1,t} \\
&\leq (n-1)\frac{ba^k}{1-b^2 a} N_{u_0,u_1,t}. \qquad (3.6)
\end{aligned}$$

By the inequality (3.6), we have

$$\lim_{k,l\to\infty} N_{u_k,u_l,t} = 0$$

and so $\{u_k\}$ is a Cauchy sequence. From the completeness hypothesis, there exists $u \in X$ such that $\lim_{k\to\infty} u_k = u$. Now we prove that u is a fixed point of T. Suppose that u is not a fixed point of T, that is, $Tu \neq u$. Using the condition (3.1), we get

$$\begin{aligned}
N_{u_k,Tu,t} &= N_{Tu_{k-1},Tu,t} \\
&\leq h \max\left\{N_{u_{k-1},u,t}, N_{u_k,u_{k-1},t}, N_{Tu,u,t}, N_{Tu,u_{k-1},t}, N_{u_k,u,t}\right\}
\end{aligned}$$

and so taking limit for $k \to \infty$, using Lemma 2.6 and the condition $(N1)$, we have

$$\begin{aligned} N_{u,Tu,t} &\leq h \max\{N_{u,u,t}, N_{u,u,t}, N_{Tu,u,t}, N_{Tu,u,t}, N_{u,u,t}\} \\ &= hN_{Tu,u,t} \leq hbN_{u,Tu,t}, \end{aligned}$$

which implies $N_{u,Tu,t} = 0$ and $Tu = u$ since $0 \leq h < \frac{1}{b+b^2(n-1)}$.

Finally we show that the fixed point u is unique. On the contrary, let u and v be two fixed points of T, that is, $Tu = u$ and $Tv = v$. Using the conditions (3.1), $(N1)$ and Lemma 2.6, we obtain

$$\begin{aligned} N_{u,v,t} &= N_{Tu,Tv,t} \\ &\leq h \max\{N_{u,v,t}, N_{u,u,t}, N_{v,v,t}, N_{v,u,t}, N_{u,v,t}\} \\ &\leq h \max\{N_{u,v,t}, bN_{u,v,t}\} = hbN_{u,v,t}, \end{aligned}$$

which implies $N_{u,v,t} = 0$, that is, $u = v$. Consequently, T has a unique fixed point in X. □

Remark 3.2. If we take $n = 3$, $b = 1$ and set the function $N_b : X \times X \times X \to [0, \infty)$ in Theorem 3.1, then we get Corollary 2.21 given in [10] on page 123 on a complete S-metric space. Since S-metric spaces are generalizations of metric spaces, Theorem 3.1 is another generalization of the known Ćirić's fixed-point result.

4. A new generalization of Kannan's fixed point result

In this section we introduce a new generalized version of Kannan's fixed-point result [6] using a parametric N_b-metric.

Theorem 4.1. *Let (X, N) be a complete parametric N_b-metric space and T be a self-mapping of X satisfying*

$$N_{Tu,Tv,t} \leq h\,[N_{u,Tu,t} + N_{v,Tv,t}] \tag{4.1}$$

for each $u, v \in X$, all $t > 0$ and some $0 \leq h < \frac{1}{2}$. Then T has a unique fixed point in X.

Proof. Let $u_0 \in X$ and the sequence $\{u_k\}$ be defined as

$$Tu_0 = u_1,\ Tu_1 = u_2, \ldots, Tu_k = u_{k+1}, \ldots.$$

Assume that $u_k \neq u_{k+1}$ for all k. Using the condition (4.1), we get

$$N_{u_k,u_{k+1},t} = N_{Tu_{k-1},Tu_k,t} \leq h\left[N_{u_{k-1},u_k,t} + N_{u_k,u_{k+1},t}\right]$$

and so

$$(1-h)N_{u_k,u_{k+1},t} \leq hN_{u_{k-1},u_k,t},$$

which implies

$$N_{u_k,u_{k+1},t} \leq \frac{h}{1-h} N_{u_{k-1},u_k,t}. \tag{4.2}$$

Let $a = \frac{h}{1-h}$. Then $a < 1$ since $2h < 1$. Notice that $1 - h \neq 0$ since $0 \leq h < \frac{1}{2}$. For $k \in \{1, 2, \ldots\}$, using the inequality (4.2) and mathematical induction, we find

$$N_{u_k,u_{k+1},t} \leq a^k N_{u_0,u_1,t}.$$

Using similar arguments as in the proof of Theorem 3.1, we can easily see that the sequence $\{u_k\}$ is a Cauchy sequence. From the completeness hypothesis, there exists $u \in X$ such that $\lim_{k\to\infty} u_k = u$. Now we prove that u is a fixed point of T. Suppose that u is not a fixed point of T, that is, $Tu \neq u$. Using the condition (4.1), we get

$$N_{u_k,Tu,t} = N_{Tu_{k-1},Tu,t} \leq h\left[N_{u_{k-1},u_k,t} + N_{u,Tu,t}\right]$$

and so taking limit for $k \to \infty$, using the condition $(N1)$, we have

$$N_{u,Tu,t} \leq hN_{u,Tu,t},$$

which implies $N_{u,Tu,t} = 0$ and $Tu = u$ since $h \in \left[0, \frac{1}{2}\right)$.

Finally, we show that the fixed point u is unique. On the contrary, let u and v be two fixed points of T, that is, $Tu = u$ and $Tv = v$. Using the conditions (4.1) and $(N1)$, we obtain

$$N_{u,v,t} = N_{Tu,Tv,t} \leq h\left[N_{u,u,t} + N_{v,v,t}\right] = 0,$$

which implies $u = v$. Consequently, T has a unique fixed point in X. □

Remark 4.2. If we take $n = 3$, $b = 1$ and set the function $N_b : X \times X \times X \to [0, \infty)$ in Theorem 4.1, then we get Corollary 2.8 given in [10] on page 118 on a complete S-metric space. Hence Theorem 4.1 is another generalization of the known Kannan's fixed-point result.

5. A new generalization of Chatterjea's fixed-point result

In this section we give a generalization of the classical Chatterjea's fixed-point theorem [2].

Theorem 5.1. *Let (X, N) be a complete parametric N_b-metric space and T be a self-mapping of X satisfying*

$$N_{Tu,Tv,t} \leq h\left[N_{u,Tv,t} + N_{v,Tu,t}\right] \tag{5.1}$$

for each $u, v \in X$, all $t > 0$ and some $0 \leq h < \frac{1}{(n-1)b+b^2}$. Then T has a unique fixed point in X.

Proof. Let $u_0 \in X$ and the sequence $\{u_k\}$ be defined as

$$Tu_0 = u_1,\ Tu_1 = u_2, \ldots, Tu_k = u_{k+1}, \ldots.$$

Assume that $u_k \neq u_{k+1}$ for all k. Using the conditions (5.1), $(N2)$ and Lemma 2.6, we get

$$\begin{aligned} N_{u_k,u_{k+1},t} &= N_{Tu_{k-1},Tu_k,t} \leq h\left[N_{u_{k-1},u_{k+1},t} + N_{u_k,u_k,t}\right] \\ &= hN_{u_{k-1},u_{k+1},t} \leq (n-1)hbN_{u_{k-1},u_k,t} + hbN_{u_{k+1},u_k,t} \end{aligned}$$

$$\leq (n-1)hbN_{u_{k-1},u_k,t} + hb^2 N_{u_k,u_{k+1},t},$$

which implies

$$N_{u_k,u_{k+1},t} \leq \frac{(n-1)hb}{1-hb^2} N_{u_{k-1},u_k,t}. \tag{5.2}$$

Let $a = \frac{(n-1)hb}{1-hb^2}$. Then $a < 1$ since $h((n-1)b+b^2) < 1$. Notice that $1-hb^2 \neq 0$ since $0 \leq h < \frac{1}{(n-1)b+b^2}$. For $k \in \{1,2,\ldots\}$, using the inequality (5.2) and mathematical induction, we find

$$N_{u_k,u_{k+1},t} \leq a^k N_{u_0,u_1,t}.$$

Using similar arguments as in the proof of Theorem 3.1, we can easily see that the sequence $\{u_k\}$ is a Cauchy sequence. From the completeness hypothesis, there exists $u \in X$ such that $\lim\limits_{k\to\infty} u_k = u$. Now we prove that u is a fixed point of T. Suppose that u is not a fixed point of T, that is, $Tu \neq u$. Using the condition (5.1), we get

$$N_{u_k,Tu,t} = N_{Tu_{k-1},Tu,t} \leq h\left[N_{u_{k-1},Tu,t} + N_{u,u_k,t}\right]$$

and so taking limit for $k \to \infty$, using the condition $(N1)$, we have

$$N_{u,Tu,t} \leq hN_{u,Tu,t},$$

which implies $N_{u,Tu,t} = 0$ and $Tu = u$ since $h \in \left[0, \frac{1}{(n-1)b+b^2}\right)$.

Finally, we show that the fixed point u is unique. On the contrary, let u and v be two fixed points of T, that is, $Tu = u$ and $Tv = v$. Using the conditions (5.1), $(N1)$ and Lemma 2.6, we get

$$N_{u,v,t} = N_{Tu,Tv,t} \leq h\left[N_{u,v,t} + N_{v,u,t}\right] \leq h(1+b)N_{u,v,t},$$

which implies $u = v$ since $h(1+b) < 1$. Consequently, T has a unique fixed point in X. □

Remark 5.2. If we take $n = 3$, $b = 1$ and set the function $N_b : X \times X \times X \to [0,\infty)$ in Theorem 5.1, then we get Corollary 2.15 given in [10] on page 121 on a complete S-metric space. Therefore Theorem 5.1 is a new generalization of the known Chatterjea's fixed-point result.

6. A new fixed-point theorem for an expansive mapping

In this section we prove a fixed-point theorem for a surjective self-mapping using an expansive mapping on a complete parametric N_b-metric space.

Theorem 6.1. *Let (X,N) be a complete parametric N_b-metric space and T be a surjective self-mapping of X satisfying the following condition*:

There exist real numbers $h_i (i = 1,2,3)$ satisfying $h_1 > b^2$ and $h_2, h_3 \geq 0$ such that

$$N_{Tu,Tv,t} \geq h_1 N_{u,v,t} + h_2 N_{Tu,u,t} + h_3 N_{Tv,v,t} \tag{6.1}$$

for each $u, v \in X$ and all $t > 0$.

Then T has a unique fixed point in X.

Proof. Using the condition (6.1), if we take $Tu = Tv$, then we get

$$0 = N_{Tu,Tu,t} = N_{Tu,Tv,t} \geq h_1 N_{u,v,t} + h_2 N_{Tu,u,t} + h_3 N_{Tv,v,t}$$

for all $t > 0$ and so we have $N_{u,v,t} = 0$, that is, $u = v$ since $h_1 > b^2$. Hence T is an injective self-mapping of X.

Let F be the inverse mapping of T and $u_0 \in X$. Let us define the sequence $\{u_k\}$ as

$$Fu_k = u_{k+1}.$$

Assume that $u_k \neq u_{k+1}$ for all k. Using the condition (6.1), we obtain

$$\begin{aligned} N_{u_{k-1},u_k,t} &= N_{TT^{-1}u_{k-1},TT^{-1}u_k,t} \\ &\geq h_1 N_{T^{-1}u_{k-1},T^{-1}u_k,t} + h_2 N_{TT^{-1}u_{k-1},T^{-1}u_{k-1},t} + h_3 N_{TT^{-1}u_k,T^{-1}u_k,t} \\ &= h_1 N_{Fu_{k-1},Fu_k,t} + h_2 N_{u_{k-1},Fu_{k-1},t} + h_3 N_{u_k,Fu_k,t} \\ &= h_1 N_{u_k,u_{k+1},t} + h_2 N_{u_{k-1},u_k,t} + h_3 N_{u_k,u_{k+1},t} \\ &= (h_1 + h_3) N_{u_k,u_{k+1},t} + h_2 N_{u_{k-1},u_k,t}, \end{aligned}$$

which implies

$$N_{u_k,u_{k+1},t} \leq \frac{1 - h_2}{h_1 + h_3} N_{u_{k-1},u_k,t}, \tag{6.2}$$

since $h_1 + h_3 \neq 0$. If we put $a = \frac{1-h_2}{h_1+h_3}$, then we have $a < \frac{1}{b^2}$ since $h_1 + h_2 + h_3 > b^2$. Using the inequality (6.2), we get

$$N_{u_k,u_{k+1},t} \leq a^k N_{u_0,u_1,t} \tag{6.3}$$

for all $t > 0$.

Now we show that the sequence $\{u_k\}$ is a Cauchy sequence. For all $k, l \in \mathbb{N}$ with $l > k$, using the inequality (6.3), the condition $(N2)$ and Lemma 2.6, we find

$$N_{u_k,u_l,t} \leq \frac{(n-1)ba^k}{1 - b^2 a} N_{u_0,u_1,t}. \tag{6.4}$$

If we take limit for $k, l \to \infty$, we obtain

$$\lim_{k,l\to\infty} N_{u_k,u_l,t} = 0.$$

Hence $\{u_k\}$ is Cauchy. Using the completeness hypothesis, there exists $u \in X$ such that

$$\lim_{k\to\infty} u_k = u.$$

From the surjectivity hypothesis, there exists a point $x \in X$ such that $Tx = u$. By the condition (6.1), we get

$$N_{u_k,u,t} = N_{Tu_{k-1},Tx,t} \geq h_1 N_{u_{k-1},x,t} + h_2 N_{u_k,u_{k-1},t} + h_3 N_{u,x,t}. \tag{6.5}$$

If we take limit for $k \to \infty$ in the inequality (6.5), we have

$$0 = N_{u,u,t} \geq (h_1 + h_3) N_{u,x,t},$$

which implies $u = x$, that is, $Tu = u$. Now we show that the fixed point u is unique. On the contrary, let v be another fixed point of T such that $u \neq v$. Using the conditions (6.1) and $(N1)$, we find

$$N_{u,v,t} = N_{Tu,Tv,t} \geq h_1 N_{u,v,t} + h_2 N_{u,u,t} + h_3 N_{v,v,t} = h_1 N_{u,v,t},$$

which implies $u = v$ since $h_1 > 1$. Consequently, T has a unique fixed point in X. □

If we take $h_1 = h$ and $h_2 = h_3 = 0$ in Theorem 6.1, then we get the following corollary.

Corollary 6.2. *Let (X, N) be a complete parametric N_b-metric space and T be a surjective self-mapping of X. If there exists a real number $h > b^2$ such that*

$$N_{Tu,Tv,t} \geq h N_{u,v,t}$$

for each $u, v \in X$ and all $t > 0$. Then T has a unique fixed point in X.

Remark 6.3. 1) If we take $n = 3$, $b = 1$ and set the function $N_b : X \times X \times X \times (0, \infty) \to [0, \infty)$ in Theorem 6.1, then we get Theorem 21 given in [13] on page 4 on a complete parametric S-metric space.

2) If we take $n = 3$, $b = 1$ and set the function $N_b : X \times X \times X \times (0, \infty) \to [0, \infty)$ in Corollary 6.2, then we get Corollary 25 given in [13] on page 5 on a complete parametric S-metric space.

7. Some illustrative examples

In this section we give some illustrative examples of the obtained theorems. Now we give an example of Theorem 3.1 and Theorem 4.1.

Example 7.1. Let $X = \mathbb{R}^+ \cup \{0\}$ and the function $N : X^4 \times (0, \infty) \to [0, \infty)$ be defined by

$$N(u_1, u_2, u_3, u_4, t) = \begin{cases} 0 & ; \quad \text{if } u_1 = u_2 = u_3 = u_4 \\ n(t) \max\{u_1, u_2, u_3, u_4\} & ; \quad \text{otherwise} \end{cases}$$

for each $u_1, u_2, u_3, u_4 \in X$ and $t > 0$, where $n : (0, \infty) \to (0, \infty)$ is a continuous function. Then (X, N) is a complete parametric N_b-metric space with $b = 2$. Let us define the self-mapping $T : X \to X$ as

$$Tu = \begin{cases} \frac{u^2}{16} & ; \quad u \in [0, a) \\ \frac{u}{15} & ; \quad u \in [a, \infty) \end{cases}$$

for all $u \in X$ with $\frac{1}{4} < a < 1$. Then T satisfies the inequality (3.1) with $h = \frac{1}{15}$. Also T satisfies the inequality (4.1) with $h = \frac{1}{2}$. Therefore T has a unique fixed point $u = 0$ in X.

In the following example we show a self-mapping satisfying the conditions of Theorem 5.1.

Example 7.2. Let $X = \mathbb{R}$ and the function $N : X^3 \times (0,\infty) \to [0,\infty)$ be defined by

$$N(u_1,u_2,u_3,t) = t^3 \left(|u_1 - u_2| + |u_1 - u_3| + |u_2 - u_3|\right)^2$$

for each $u_1, u_2, u_3 \in X$ and $t > 0$. Then (X, N) is a complete parametric N_b-metric space with $b = 4$. Let us define the self-mapping $T : X \to X$ as

$$Tu = \eta$$

for all $u \in X$, where η is a constant. Then T satisfies the inequality (5.1) with $h = \frac{1}{25}$. Therefore T has a unique fixed point $u = \eta$ in X.

Finally, we give an example of an expansive mapping satisfying the conditions of Theorem 6.1.

Example 7.3. Let $X = \mathbb{R}^+ \cup \{0\}$ be the complete parametric N_b-metric space with the parametric N_b-metric defined in Example 7.1. Let us define the self-mapping $T : X \to X$ as

$$Tu = \eta u$$

for all $u \in \mathbb{R}$ with $\eta > 4$. Then T satisfies the inequality (6.1) with $h_1 = \eta$ and $h_2 = h_3 = 0$. Therefore T has a unique fixed point $u = 0$ in X.

8. An application to fixed-circle problem

In this section we present an approach to fixed-point theory on a parametric N_b-metric space.

Definition 8.1. Let (X, N) be a parametric N_b-metric space and $u_0 \in X$, $r \in (0,\infty)$. We define the circle centered at u_0 with radius r as

$$C^{N_b}_{u_0,r} = \{u \in X : N_{u,u_0,t} = r\}.$$

Example 8.2. Let $X = \mathbb{R}^2$, $n = 3$, the function $g : (0,\infty) \to (0,\infty)$ be defined as

$$g(t) = t^2$$

and the function $N : X^3 \times (0,\infty) \to [0,\infty)$ be defined as

$$N(u,v,w,t) = g(t) \sum_{i=1}^{2} \left(|\arctan u_i - \arctan w_i| + |\arctan v_i - \arctan w_i|\right)$$

for each $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbb{R}^2$ and all $t > 0$. Then $\left(\mathbb{R}^2, N\right)$ is a parametric N_b-metric space with $b = 4$. If we choose $u_0 = 0 = (0,0)$ and $r = 10$, then we get

$$\begin{aligned} C^{N_b}_{0,10} &= \left\{u = (u_1,u_2) \in \mathbb{R}^2 : N(u,u,0,t) = 10\right\} \\ &= \left\{u \in \mathbb{R}^2 : |\arctan u_1|^2 + |\arctan u_2|^2 = \frac{5}{t^2}\right\}, \end{aligned}$$

as shown in Figure 1 which is plotted using Mathematica [16] for different $t > 0$.

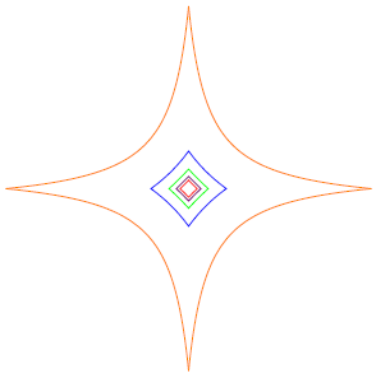

FIGURE 1. The curves of the circle $C_{0,10}^{N_b}$ for $t = 2, 3, 4, 5, 6$.

Definition 8.3. Let (X, N) be a parametric N_b-metric space, $C_{u_0,r}^{N_b}$ be a circle on X and $T : X \to X$ be a self-mapping of X. If $Tu = u$ for all $u \in C_{u_0,r}^{N_b}$, then the circle $C_{u_0,r}^{N_b}$ is called a fixed circle of T.

In the following theorem, we give an existence condition for a self-mapping having a fixed circle.

Theorem 8.4. *Let (X, N) be a parametric N_b-metric space and $C_{u_0,r}^{N_b}$ be any circle on X. Let us define the mapping $\varphi : X \times (0, \infty) \to [0, \infty)$ as*

$$\varphi(u, t) = N_{u,u_0,t}$$

for all $u \in X$ and $t > 0$. If there exists a self-mapping $T : X \to X$ satisfying

$$N_{u,Tu,t} \leq \varphi(u, t) - \varphi(Tu, t) \tag{8.1}$$

and

$$N_{Tu,u_0,t} \geq r \tag{8.2}$$

for all $u \in C_{u_0,r}^{N_b}$, then $C_{u_0,r}^{N_b}$ is a fixed circle of T.

Proof. Let $u \in C_{u_0,r}^{N_b}$. Using the inequality (8.1), we get

$$N_{u,Tu,t} \leq \varphi(u, t) - \varphi(Tu, t) = N_{u,u_0,t} - N_{Tu,u_0,t} = r - N_{Tu,u_0,t}. \tag{8.3}$$

Because of the inequality (8.2), the point Tu should lie on or the exterior of the circle $C_{u_0,r}^{N_b}$. If $N_{Tu,u_0,t} > r$, then using the inequality (8.3) we have a contradiction. Hence it should be $N_{Tu,u_0,t} = r$. Using the inequality (8.3), we obtain

$$N_{u,Tu,t} \leq 0,$$

which implies $Tu = u$ for all $u \in C_{u_0,r}^{N_b}$. Consequently, $C_{u_0,r}^{N_b}$ is a fixed circle of T. □

Notice that the inequality (8.1) guarantees that Tu is not in the exterior of the circle $C_{u_0,r}^{N_b}$ for each $u \in C_{u_0,r}^{N_b}$. Similarly, the inequality (8.2) guarantees that Tu is not in the interior of the circle $C_{u_0,r}^{N_b}$ for each $u \in C_{u_0,r}^{N_b}$. Consequently, we get $Tu \in C_{u_0,r}^{N_b}$ for each $u \in C_{u_0,r}^{N_b}$ and $T\left(C_{u_0,r}^{N_b}\right) \subset C_{u_0,r}^{N_b}$.

If we set $n = 3$ and $b = 1$ in Theorem 8.4, then we have a fixed-circle theorem on an parametric S-metric space. On the other hand, the metric and S-metric versions of Theorem 8.4 can be found in [7] and [8], respectively.

Now we give an example of a self-mapping which has a fixed circle on a parametric N_b-metric space.

Example 8.5. Let X be any set which contains the interval $(0, \infty)$, (X, N) be a parametric N_b-metric space and the function $g : (0, \infty) \to (0, \infty)$ be defined as $g(t) = t^2$ for all $t > 0$. Let us consider a circle $C_{u_0,r}^{N_b}$ and define the self-mapping $T : X \to X$ as

$$Tu = \begin{cases} u & ; \quad u \in C_{u_0,r}^{N_b} \\ g(u) & ; \quad u \in (0,\infty) \text{ and } u \notin C_{u_0,r}^{N_b} \\ u_0 & ; \quad \text{otherwise} \end{cases}$$

for all $u \in X$. Then a direct computation shows that the inequalities (8.1) and (8.2) are satisfied. Hence T fixes the circle $C_{u_0,r}^{N_b}$.

We give an example of a self-mapping which satisfies the inequality (8.1) and does not satisfy the inequality (8.2).

Example 8.6. Let (X, N) be a parametric N_b-metric space. Let us consider a circle $C_{u_0,r}^{N_b}$ and define the self-mapping $T : X \to X$ as $Tu = u_0$ for all $u \in X$. Then T satisfies the inequality (8.1) but does not satisfy the inequality (8.2). Clearly T does not fix the circle $C_{u_0,r}^{N_b}$.

We give an example of a self-mapping which satisfies the inequality (8.2) and does not satisfy the inequality (8.1).

Example 8.7. Let (X, N) be a parametric N_b-metric space. Let us consider a circle $C_{u_0,r}^{N_b}$ and define the self-mapping $T : X \to X$ as $Tu = c$ for all $u \in X$, where c is an element of X such that

$$N_{c,u_0,t} = 2r.$$

Then T satisfies the inequality (8.2) but does not satisfy the inequality (8.1). Clearly T does not fix the circle $C_{u_0,r}^{N_b}$.

We note that a self-mapping may have more than one fixed circle. For example, let (X, N) be a parametric N_b-metric space and $C_{u_0,r_0}^{N_b}$, $C_{u_1,r_1}^{N_b}$ be two circles on X. Let us define the mappings $\varphi_1, \varphi_2 : X \times (0, \infty) \to [0, \infty)$ as

$$\varphi_1(u, t) = N_{u,u_0,t} \text{ and } \varphi_2(u, t) = N_{u,u_1,t}$$

for all $u \in X$. If we define a self-mapping T as

$$Tu = \begin{cases} u & ; \quad u \in C_{u_0,r}^{N_b} \cup C_{u_1,r_1}^{N_b} \\ u_0 & ; \quad \text{otherwise} \end{cases}$$

for all $u \in X$, then T satisfies the inequalities (8.1) and (8.2) for the circles $C^{N_b}_{u_0,r_0}$ and $C^{N_b}_{u_1,r_1}$. Consequently, these circles are fixed circles of T.

Finally, we investigate the uniqueness conditions for the fixed circles in Theorem 8.4 on a parametric N_b-metric space.

Theorem 8.8. *Let (X, N) be a parametric N_b-metric space and $C^{N_b}_{u_0,r}$ be any circle on X. Let $T : X \to X$ be a self-mapping which fixes the circle $C^{N_b}_{u_0,r}$. If the contractive condition (3.1) is satisfied for all $u \in C^{N_b}_{u_0,r}$, $v \in X \setminus C^{N_b}_{u_0,r}$ by T, then $C^{N_b}_{u_0,r}$ is the unique fixed circle of T.*

Proof. Assume that there exist two fixed circles $C^{N_b}_{u_0,r_0}$ and $C^{N_b}_{u_1,r_1}$ of the self-mapping T. Let $u \in C^{N_b}_{u_0,r_0}$ and $v \in C^{N_b}_{u_1,r_1}$ be arbitrary points with $u \neq v$. Using the contractive condition (3.1) and Lemma 2.6, we obtain

$$N_{Tu,Tv,t} = N_{u,v,t} \leq h \max\{N_{u,v,t}, N_{u,u,t}, N_{v,v,t}, N_{v,u,t}, N_{u,v,t}\} \leq hbN_{u,v,t},$$

which implies $u = v$ since $0 \leq h < \frac{1}{b+b^2(n-1)}$. Consequently, $C^{N_b}_{u_0,r_0}$ is the unique fixed circle of T. □

In Theorem 8.8, if we use the contractive conditions (4.1) or (5.1) instead of the contractive condition (3.1), we get new uniqueness theorems for a fixed circle.

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