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ON TRIGONOMETRIC APPROXIMATION IN WEIGHTED LORENTZ SPACES USING NÖRLUND AND RIESZ SUBMETHODS

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Abstract. In this study, we obtain the degree of approximation by the Nörlund and Riesz submethods of the partial sums of the Fourier series of derivatives of functions in the weighted Lorentz spaces with Muckenhoupt weights. Therefore we generalize the theorems given in [22] to their sharper approximation versions with weaker conditions.

1. Introduction

Let $\mathbf{T} := [-\pi, \pi]$. When $\omega : \mathbf{T} \to [0, \infty]$ is a non-negative measurable 2π-periodic function on $(0, \infty)$ that is not identically zero, we say that ω is a weight.

Given a weight function ω and a measurable set e we put

$$\omega(e) = \int_e \omega(x)dx. \tag{1.1}$$

We define the decreasing rearrangement $f^*_\omega(t)$ of $f : \mathbf{T} \to \mathbb{R}$ with respect to the Borel measure (1.1) by

$$f^*_\omega(t) = \inf \{\tau \geq 0 : \omega (x \in \mathbf{T} : |f(x)| > \tau) \leq t\}$$

and the maximal function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*_\omega(u)du.$$

The weighted Lorentz space $L^{ps}_\omega(\mathbf{T})$ is defined [8, p.20], [3, p.219] as

$$L^{ps}_\omega(\mathbf{T}) = \left\{ f \in \mathbf{M}(\mathbf{T}) : \|f\|_{ps,\omega} = \left(\int_{\mathbf{T}} (f^{**}(t))^s \, t^{\frac{s}{p}} \frac{dt}{t} \right)^{1/s} < \infty \right\},$$

where $\mathbf{M}(\mathbf{T})$ is the set of 2π periodic measurable functions on $\mathbf{T}$, $1 < p, s < \infty$.

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Another thing to notice is that if $p = s$, then $L^{ps}_{\omega}(\mathbf{T})$ turns into the weighted Lebesgue space $L^{p}_{\omega}(\mathbf{T})$ as [8, p. 20].

By $E_n(f)_{L^{ps}_{\omega}}$ we denote the best approximation of $f \in L^{ps}_{\omega}(\mathbf{T})$ by trigonometric polynomials of degree $\leq n$, i.e.,

$$E_n(f)_{L^{ps}_{\omega}} = \inf \|f - T_k\|_{ps,\omega} ,$$

where the infimum is taken with respect to all trigonometric polynomials of degree $k \leq n$.

The weight functions ω used in the paper belong to the Muckenhoupt class $A_p(\mathbf{T})$ [18] which is defined by

$$\sup \frac{1}{|I|} \int\limits_I \omega(x)dx \left(\frac{1}{|I|} \int\limits_I \omega^{1-p'}(x)dx \right)^{p-1} < \infty, \qquad p' = \frac{p}{p-1}, \quad 1 < p < \infty,$$

where the supremum is taken with respect to all the intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I.

The modulus of continuity of the function $f \in L^{ps}_{\omega}(\mathbf{T})$ is defined [14] as

$$\Omega\left(f,\delta\right)_{L^{ps}_{\omega}} = \sup_{|h|\leq\delta} \|A_h f\|_{ps,\omega} , \qquad \delta > 0,$$

where

$$\left(A_h f\right)(x) := \frac{1}{h} \int\limits_0^h |f(x+t) - f(x)|\, dt.$$

In Lebesgue spaces L^p $(1 < p < \infty)$, the traditional modulus of continuity is defined as

$$w_p(f,\delta) = \sup_{0<h\leq\delta} \|f(\cdot + h) - f(\cdot)\|_p , \;\; \delta > 0,$$

It is known that the modulus of continuity $\Omega(f,\delta)_{L^{ps}_{\omega}}$ and traditional modulus of continuity $w_p(f,\delta)$ are equivalent in [14].

Whenever $\omega \in A_p(\mathbf{T})$, $1 < p, s < \infty$, the Hardy-Littlewood maximal function of every $f \in L^{ps}_{\omega}(\mathbf{T})$. The existence of the modulus $\Omega(f,\delta)_{L^{ps}_{\omega}}$ follows from the boundedness of the Hardy-Littlewood maximal function in the space $L^{ps}_{\omega}(\mathbf{T})$ [4, Theorem 3]. That is the modulus of continuity $\Omega\left(f,\delta\right)_{L^{ps}_{\omega}}$ is well defined for every $\omega \in A_p(\mathbf{T})$.

The modulus of continuity $\Omega\left(f,\delta\right)_{L^{ps}_{\omega}}$ is non-decreasing, non-negative, continuous function such that

$$\lim_{\delta\to 0} \Omega\left(f,\delta\right)_{L^{ps}_{\omega}} = 0, \quad \Omega\left(f_1 + f_2,\delta\right)_{L^{ps}_{\omega}} \leq \Omega\left(f_1,\delta\right)_{L^{ps}_{\omega}} + \Omega\left(f_2,\delta\right)_{L^{ps}_{\omega}} .$$

The modulus of continuity $\Omega\left(f,\delta\right)_{L^{ps}_{\omega}}$ is defined in this way, since the space $L^{ps}_{\omega}(\mathbf{T})$ is non-invariant, in general, under the usual shift $f\left(x\right) \to f\left(x+h\right), \left(h > 0\right).$

For $0 < \alpha \leq 1$, it is defined the Lipschitz class $Lip\left(\alpha, L^{ps}_{\omega}\right)$ [9] as

$$Lip\left(\alpha, L^{ps}_{\omega}\right) = \left\{ f \in L^{ps}_{\omega}(\mathbf{T}) : \Omega\left(f,\delta\right)_{L^{ps}_{\omega}} = O\left(\delta^{\alpha}\right), \;\delta > 0 \right\}$$

and for $r = 1, 2, ...$ the classes $W^{r}_{ps,\omega}$, $W^{r,\alpha}_{ps,\omega}$ as

$$\begin{aligned} W^{r}_{ps,\omega} &= \left\{ f \in L^{ps}_{\omega}(\mathbf{T}) : f^{(r-1)} \text{ is absolutely continuous and } f^{(r)} \in L^{ps}_{\omega} \right\} \\ W^{r,\alpha}_{ps,\omega} &= \left\{ f \in W^{r}_{ps,\omega} : f^{(r)} \in Lip\left(\alpha, L^{ps}_{\omega}\right) \right\}. \end{aligned}$$

Since $L_{\omega}^{ps}(\mathbf{T}) \subset L^{1}(\mathbf{T})$ when $\omega \in A_p(\mathbf{T})$, $1 < p, s < \infty$ (see [8, the proof of Prop. 3.3]), we can define the Fourier series of $f \in L_{\omega}^{ps}(\mathbf{T})$

$$f(x) \backsim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) \tag{1.2}$$

and the conjugate Fourier series

$$\tilde{f}(x) \backsim \sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx).$$

Here $a_0(f), a_k(f), b_k(f)$, $k = 1, ...$, are Fourier coefficients of f. Let $S_n(x, f)$, $(n = 0, 1, 2, ...)$ be the nth partial sums of the series (1.2) at the point x, that is,

$$S_n(x, f) := \sum_{k=0}^{n} A_k(f)(x),$$

where

$$A_0(f)(x) = \frac{a_0}{2}, \quad A_k(f)(x) = a_k(f) \cos kx + b_k(f) \sin kx, \quad k = 1, 2, ...$$

Let $(\lambda_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. For a sequence (x_k) of the real or complex numbers, the Cesàro submethod C_λ is defined by

$$(C_\lambda x)_n := \frac{1}{\lambda_n} \sum_{k=1}^{\lambda_n} x_k, \ (n = 1, 2, ...).$$

Particularly, when $\lambda_n = n$ we note that $(C_\lambda x)_n$ is the classical Cesàro method $(C, 1)$ of (x_k). Thus, the Cesàro submethod C_λ yields a subsequence of the Cesàro method $(C, 1)$. The detailed information about Cesàro submethod C_λ can be found in the papers [2, 19].

Let (p_n) be a positive sequence of real numbers. We define Nörlund and Riesz submethods means as

$$N_n^{\lambda}(f; x) := \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n - m} S_m(f; x)$$

and

$$R_n^{\lambda}(f; x) := \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_m S_m(f; x)$$

where

$$P_{\lambda_n} = p_0 + p_1 + p_2 + ... + p_{\lambda_n} \neq 0 \quad (n \geqslant 0)$$

and by convention, $p_{-1} = P_{-1} = 0$.

In the case $p_n = 1$, $n \geq 0$, $\lambda_n = n$, both of $N_n^{\lambda}(f)(x)$ and $R_n^{\lambda}(f)(x)$ are equal to the Cesàro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f; x).$$

2. Historical Background

The basic properties of the Cesàro submethod C_λ were investigated firstly by Armitage and Maddox in [2] and Osikiewicz [19]. In these works, the relations between the classical Cesàro method σ_n and Cesàro submethod C_λ were obtained.

Deger et al. [6] obtained theorems of trigonometric approximation using trigonometric polynomials obtained by Cesàro submethod C_λ in Lebesgue spaces. Deger and Kaya [5] investigated the degree of approximation of functions in Lebesgue spaces by trigonometric polynomials $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$. In [17], Mittal and Singh improved the results given by Deger et al. [6] by dropped monotonicity conditions on the elements of matrix rows. In [16], Mittal and Singh examined the approximation rate of functions using matrix submethods obtained by means of Cesàro Submethod in Lebesgue spaces. In [20], the results given by Deger and Kaya [5] were improved using more general summability methods. In [13], the some results obtained in variable exponent Lebesgue spaces were extended using a wider class of numerical sequences, a sharper degrees of approximation and Nörlund submethod $N_n^\lambda(f;x)$ instead of Nörlund method $N_n(f;x)$. In [7], in the variable exponent Lebesgue spaces the results on the degree of approximation by the Nörlund and the Riesz submethods of the partial sums Fourier series of functions were given. In [10], the approximation properties of the matrix method τ_n of partial sums of Fourier series of functions in the weighted variable exponent Lebesgue spaces were investigated.

Lebesgue space may be generalized in different ways. One of the important generalizations of this space is Lorentz space. Lorentz space was firstly introduced by G. G. Lorentz in [15]. By means of the weight functions satisfying Muckenhoupt condition, the weighted Lorentz spaces were defined in [3, 8].

In weighted Lorentz spaces, some researchers obtained results about approximation theory using different methods [1, 12, 21, 22]. But in these papers degree of approximation using Cesàro submethod were not examined for derivatives of functions in the weighted Lorentz spaces.

In this paper, we generalize the results obtained by Mittal and Singh [16] to the weighted Lorentz spaces for derivatives of functions in these spaces. Also, we obtain the similars of the results in [22] using Cesàro submethod under weaker conditions.

3. Auxiliary Results

We need some helpful lemmas.

Lemma 3.1. *Let $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $r \in \mathbb{N}$. If $f \in Lip(1, L_\omega^{ps})$, then $f^{(r)}$ is absolutely continuous and $f^{(r+1)} \in L_\omega^{ps}(\mathbf{T})$.*

Proof. We follow the method in [14, Th. 3]. If $f^{(r)} \in L_\omega^{ps}(\mathbb{T})$, then there exists $p_0 > 1$ such that $f^{(r)} \in L^{p_0}$ and

$$\left\| f^{(r)} \right\|_{L^{p_0}} \preceq \left\| f^{(r)} \right\|_{L_\omega^{ps}} \tag{3.1}$$

[12, Prop. 3.3]. Using (3.1) and the equivalence of traditional modulus of continuity $w(f,\delta)_{L^p}$ and $\Omega(f,\delta)_{L_\omega^{ps}}$, we get

$$w(f^{(r)},\delta)_{L^{p_0}} \preceq \Omega(f^{(r)},\delta)_{L_\omega^{ps}}.$$

Since $\Omega(f^{(r)},\delta)_{L^{ps}_\omega} = O(\delta)$, the same estimate holds for $w(f^{(r)},\delta)_{L^{p_0}}$, too. From here, we obtain that $f^{(r)}$ is absolutely continuous in $[-\pi,\pi]$ and for almost every x

$$\frac{f^{(r)}(x+t)-f^{(r)}(x)}{t} \to f^{(r+1)}(x), \quad (t\to 0). \tag{3.2}$$

From (3.2) we obtain for almost every x

$$\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta} \frac{\left|f^{(r)}(x+t)-f^{(r)}(x)\right|}{t}dt \to \left|f^{(r+1)}(x)\right|, \quad (\delta\to 0_+).$$

Using Fatou Lemma, we get

$$\begin{aligned}
\left\|f^{(r+1)}\right\|_{L^{ps}_\omega} &\le \liminf_{\delta\to 0_+}\left\|\frac{2}{\delta}\int_{\frac{\delta}{2}}^{\delta}\frac{\left|f^{(r)}(x+t)-f^{(r)}(x)\right|}{t}dt\right\|_{L^{ps}_\omega} \\
&\le \limsup_{\delta\to 0_+}\frac{4}{\delta}\left\|\frac{1}{\delta}\int_0^{\delta}\left|f^{(r)}(x+t)-f^{(r)}(x)\right|dt\right\|_{L^{ps}_\omega} \\
&\le \limsup_{\delta\to 0_+} 4\frac{\Omega(f^{(r)},\delta)}{\delta} < \infty.
\end{aligned}$$

This proves the lemma. □

Lemma 3.2. [22]*Let $1<p,s<\infty$, $\omega\in A_p(\mathbf{T})$, $0<\alpha\le 1$, $r\in\mathbb{N}$. Then, the estimate*

$$\left\|f^{(r)}-S_n\left(f^{(r)}\right)\right\|_{ps,\omega} = O\left(n^{-\alpha}\right) \tag{3.3}$$

holds for every $f\in W^{r,\alpha}_{ps,\omega}$ and $n=1,2,\ldots$.

Lemma 3.3. *Let $1<p,s<\infty$, $\omega\in A_p(\mathbf{T})$, $r\in\mathbb{N}$. Then, the estimate*

$$\left\|S_n\left(f^{(r)}\right)-\sigma_n(f^{(r)})\right\|_{ps,\omega} = O\left(n^{-1}\right) \tag{3.4}$$

holds for every $f\in W^{r,1}_{ps,\omega}$ and $n=1,2,\ldots$.

Proof. If $f^{(r)}$ has the Fourier series

$$f^{(r)}(x) \sim \sum_{k=0}^{\infty} A_k(f^{(r)})(x),$$

then the Fourier series of the conjugate function $\tilde{f}^{(r+1)}(x)$ is

$$\tilde{f}^{(r+1)}(x) \sim \sum_{k=0}^{\infty} kA_k(f^{(r)})(x).$$

On the other hand,

$$\begin{aligned}
S_n\left(f^{(r)}\right)(x)-\sigma_n(f^{(r)})(x) &= \sum_{k=1}^{n}\frac{k}{n+1}A_k(f^{(r)})(x) \\
&= \frac{1}{n+1}S_n\left(\tilde{f}^{(r+1)}\right)(x).
\end{aligned}$$

Since the partial sums and the conjugate operator is uniform bounded in the space $L^{ps}_{\omega}(\mathbf{T})$ (see [12], [11, Th. 6.6.2], [23, Chap. VI]), we get from Lemma 3.1

$$\begin{aligned}\left\|S_n\left(f^{(r)}\right)-\sigma_n(f^{(r)})\right\|_{ps,\omega} &= \frac{1}{n+1}\left\|S_n\left(\tilde{f}^{(r+1)}\right)\right\|_{ps,\omega}\le C\frac{1}{n+1}\left\|\tilde{f}^{(r+1)}\right\|_{ps,\omega}\\ &\le C\frac{1}{n+1}\left\|f^{(r+1)}\right\|_{ps,\omega}=O\left(n^{-1}\right)\end{aligned}$$

for $n=1,2,....$ □

Lemma 3.4. *Let (p_n) be a non-increasing sequence of positive numbers. Then,*

$$\sum_{m=1}^{\lambda_n} m^{-\alpha}p_{\lambda_n-m}=O\left(\lambda_n^{-\alpha}P_{\lambda_n}\right)$$

for $0<\alpha<1$.

Proof. Due to (p_n) is non-increasing sequence, we have

$$\begin{aligned}\sum_{m=1}^{\lambda_n} m^{-\alpha}p_{\lambda_n-m} &= \sum_{m=1}^{r} m^{-\alpha}p_{\lambda_n-m}+\sum_{m=r+1}^{\lambda_n} m^{-\alpha}p_{\lambda_n-m}\\ &\le p_{\lambda_n-r}\sum_{m=1}^{r} m^{-\alpha}+(r+1)^{-\alpha}\sum_{m=0}^{\lambda_n}p_{\lambda_n-m}\\ &= O(\lambda_n^{1-\alpha})p_{\lambda_n-r}+O(\lambda_n^{-\alpha})P_{\lambda_n}\\ &= O(\lambda_n^{-\alpha})P_{\lambda_n}\end{aligned}$$

where r integer part of $\lambda_n/2$. The proof is completed. □

4. Main Results

Theorem 4.1. *Let $1<p,s<\infty$, $\omega\in A_p(\mathbf{T})$, $0<\alpha\le 1$, $r\in\mathbb{N}$ and let $(p_n)_0^{\infty}$ be a monotonic sequence of positive numbers such that*

$$(\lambda_n+1)p_{\lambda_n}=O(P_{\lambda_n}). \tag{4.1}$$

If $f\in W^{r,\alpha}_{ps,\omega}$ then we have

$$\left\|f^{(r)}-N_n^{\lambda}\left(f^{\ (r)}\right)\right\|_{ps,\omega}=O\left(\lambda_n^{-\alpha}\right).$$

Theorem 4.2. *Let $1<p,s<\infty$, $\omega\in A_p(\mathbf{T})$, $0<\alpha\le 1$, $r\in\mathbb{N}$ and let $(p_n)_0^{\infty}$ be a sequence of positive real numbers satisfying the relation*

$$\sum_{\lambda_m=0}^{\lambda_n-1}\left|\frac{P_{\lambda_m}}{\lambda_m+1}-\frac{P_{\lambda_m+1}}{\lambda_m+2}\right|=O\left(\frac{P_{\lambda_n}}{\lambda_n+1}\right). \tag{4.2}$$

If $f\in W^{r,\alpha}_{ps,\omega}$ then the estimate

$$\left\|f^{(r)}-R_n^{\lambda}\left(f^{\ (r)}\right)\right\|_{ps,\omega}=O\left(\lambda_n^{-\alpha}\right)$$

holds.

If we take $p_{\lambda_n} = A_{\lambda_n}^{\beta-1}$ $(\beta > 0)$, where

$$A_0^{\beta} = 1, \quad A_k^{\beta} = \frac{\beta(\beta+1)...(\beta+k)}{k!}, \quad k \geq 1,$$

we have

$$N_n^{\lambda}\left(f^{\,(r)}\right)(x) = \sigma_{\lambda_n}^{\beta}(f^{(r)})(x) = \frac{1}{A_{\lambda_n}^{\beta}} \sum_{m=0}^{\lambda_n} A_{\lambda_n - m}^{\beta-1} S_m(x, f^{(r)}).$$

We can estimate the deviation of $f \in W_{ps,\omega}^{r,\alpha}$ using the Cesàro submethods $\sigma_{\lambda_n}^{\beta}(f^{(r)})$. We formulate this estimate in the following corollary.

Corollary 4.3. *Let* $1 < p, s < \infty$, $\omega \in A_p(\mathbf{T})$, $0 < \alpha \leq 1$, $r \in \mathbb{N}$. *If* $f \in W_{ps,\omega}^{r,\alpha}$, *we have*

$$\left\| f^{(r)} - \sigma_{\lambda_n}^{\beta}(f^{(r)}) \right\|_{ps,\omega} = O\left(\lambda_n^{-\alpha}\right).$$

Note that the our main submethod results are sharper than the results obtained using Cesàro method, because $\lambda_n^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$.

5. Proof of Main Results

Proof of Theorem 4.1. Let $0 < \alpha < 1$. We can write

$$f^{(r)}(x) = \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n - m} f^{(r)}(x),$$

then we get

$$f^{(r)}(x) - N_n^{\lambda}\left(f^{\,(r)}\right)(x) = \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n - m} \left[f^{(r)}(x) - S_m\left(f^{(r)}\right)(x) \right].$$

By considering Lemma 3.2, Lemma 3.4 and condition (4.1) we get

$$\begin{aligned}
\left\| f^{(r)} - N_n^{\lambda}\left(f^{\,(r)}\right) \right\|_{ps,\omega} &\leq \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n - m} \left\| f^{(r)} - S_m\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&= \frac{1}{P_{\lambda_n}} \sum_{m=1}^{\lambda_n} p_{\lambda_n - m} O\left(m^{-\alpha}\right) + \frac{p_{\lambda_n}}{P_{\lambda_n}} \left\| f^{(r)} - S_0\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&= \frac{1}{P_{\lambda_n}} O\left(\lambda_n^{-\alpha} P_{\lambda_n}\right) + O\left(\frac{1}{\lambda_n}\right) \\
&= O\left(\lambda_n^{-\alpha}\right).
\end{aligned}$$

Let $\alpha = 1$. It is easily seen that

$$N_n^{\lambda}\left(f^{\,(r)}\right)(x) = \frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n} p_{\lambda_n - m} A_m(f^{(r)})(x).$$

Using Abel transform,

$$S_n\left(f^{(r)}\right)(x) - N_n^{\lambda}\left(f^{\,(r)}\right)(x) = \frac{1}{P_{\lambda_n}} \sum_{m=1}^{\lambda_n} \left(P_{\lambda_n} - P_{\lambda_n - m}\right) A_m(f^{(r)})(x)$$

$$
\begin{aligned}
&= \frac{1}{P_{\lambda_n}} \sum_{m=1}^{\lambda_n} \left(\frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} \right) \left(\sum_{k=1}^{m} k A_k (f^{(r)})(x) \right) \\
&\quad + \frac{1}{\lambda_n + 1} \sum_{k=1}^{\lambda_n} k A_k (f^{(r)})(x),
\end{aligned}
$$

and so

$$
\begin{aligned}
\left\| S_n \left(f^{(r)} \right) - N_n^{\lambda} \left(f^{\ (r)} \right) \right\|_{ps,\omega} &\leq \frac{1}{P_{\lambda_n}} \sum_{m=1}^{\lambda_n} \left| \frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} \right| \\
&\quad \times \left\| \sum_{k=1}^{m} k A_k (f^{(r)}) \right\|_{ps,\omega} + \frac{1}{\lambda_n + 1} \left\| \sum_{k=1}^{\lambda_n} k A_k (f^{(r)}) \right\|_{ps,\omega}.
\end{aligned}
$$

Since

$$
S_n \left(f^{(r)} \right)(x) - \sigma_n \left(f^{(r)} \right)(x) = \frac{1}{\lambda_n + 1} \sum_{k=1}^{\lambda_n} k A_k (f^{(r)})(x),
$$

using Lemma 3.3 we obtain

$$
\left\| \sum_{k=1}^{\lambda_n} k A_k (f^{(r)}) \right\|_{ps,\omega} = (\lambda_n + 1) \left\| S_n \left(f^{(r)} \right) - \sigma_n \left(f^{(r)} \right) \right\|_{ps,\omega} = O(1).
$$

Therefore we have

$$
\begin{aligned}
&\left\| S_n \left(f^{(r)} \right) - N_n^{\lambda} \left(f^{\ (r)} \right) \right\|_{ps,\omega} \\
&\leq \frac{1}{P_{\lambda_n}} \sum_{m=1}^{\lambda_n} \left| \frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} \right| O(1) + O\left(\lambda_n^{-1} \right) \\
&= O\left(\frac{1}{P_{\lambda_n}} \right) \sum_{m=1}^{\lambda_n} \left| \frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} \right| + O\left(\lambda_n^{-1} \right). \qquad (5.1)
\end{aligned}
$$

then it can be easily seen that

$$
\frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} = \frac{1}{m(m+1)} \left(\sum_{k=\lambda_n - m + 1}^{\lambda_n} p_k - m p_{\lambda_n - m} \right).
$$

This equality implies that

$$
\left(\frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} \right)_{m=1}^{\lambda_n + 1}
$$

is non-increasing whenever (p_n) is non-decreasing and non-decreasing whenever (p_n) is non-increasing. This shows the following equality

$$
\sum_{m=1}^{\lambda_n} \left| \frac{P_{\lambda_n} - P_{\lambda_n - m}}{m} - \frac{P_{\lambda_n} - P_{\lambda_n-(m+1)}}{m+1} \right| = \left| p_{\lambda_n} - \frac{P_{\lambda_n}}{\lambda_n + 1} \right| = \frac{1}{\lambda_n + 1} O(P_{\lambda_n}).
$$

From this and the inequality (5.1), we have

$$
\left\| S_n \left(f^{(r)} \right) - N_n^{\lambda} \left(f^{\ (r)} \right) \right\|_{ps,\omega} = O\left(\lambda_n^{-1} \right).
$$

Combining the last estimate with (3.3) we get

$$\left\| f^{(r)} - N_n^{\lambda}\left(f^{\ (r)}\right)\right\|_{ps,\omega} = O\left(\lambda_n^{-1}\right).$$

□

Proof of Theorem 4.2. Let $0 < \alpha < 1$. Using definition of $R_n^{\lambda}\left(f^{\ (r)}\right)(x)$, we can write

$$f^{(r)}(x) - R_n^{\lambda}\left(f^{\ (r)}\right)(x) = \frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n} p_m\left[f^{(r)}(x) - S_m\left(f^{(r)}\right)(x)\right],$$

then using Lemma 3.2, we get

$$\left\| f^{(r)} - R_n^{\lambda}\left(f^{\ (r)}\right)\right\|_{ps,\omega} \le \frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n} p_m\left\| f^{(r)} - S_m\left(f^{(r)}\right)\right\|_{ps,\omega}$$

$$= O\left(\frac{1}{P_{\lambda_n}}\right)\sum_{m=1}^{\lambda_n} p_m m^{-\alpha} + \frac{p_0}{P_{\lambda_n}}\left\| f^{(r)} - S_0\left(f^{(r)}\right)\right\|_{ps,\omega}$$

$$= O\left(\frac{1}{P_{\lambda_n}}\right)\sum_{m=1}^{\lambda_n} p_m m^{-\alpha}. \tag{5.2}$$

Using Abel transform, we get

$$\begin{aligned}\sum_{m=1}^{\lambda_n} p_m m^{-\alpha} &= \sum_{m=1}^{\lambda_n-1} P_m\left[m^{-\alpha} - (m+1)^{-\alpha}\right] + \lambda_n^{-\alpha}P_{\lambda_n} \\ &\le \sum_{m=1}^{\lambda_n-1} m^{-\alpha}\frac{P_m}{m+1} + \lambda_n^{-\alpha}P_{\lambda_n},\end{aligned}$$

and using condition (4.2) we can write

$$\begin{aligned}\sum_{m=1}^{\lambda_n-1} m^{-\alpha}\frac{P_m}{m+1} &= \sum_{m=1}^{\lambda_n-1}\left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_{\lambda_n}}{\lambda_n+1}\sum_{m=1}^{\lambda_n-1} m^{-\alpha} \\ &= O\left(\lambda_n^{-\alpha}P_{\lambda_n}\right)\end{aligned}$$

This implies

$$\sum_{m=1}^{\lambda_n} p_m m^{-\alpha} = O\left(\lambda_n^{-\alpha}P_{\lambda_n}\right).$$

Combining last inequality and and the condition (5.2), we obtain that

$$\left\| f^{(r)} - R_n^{\lambda}\left(f^{\ (r)}\right)\right\|_{ps,\omega} = O\left(\lambda_n^{-\alpha}\right).$$

Let $\alpha = 1$. From Abel transform,

$$\begin{aligned}R_n^{\lambda}\left(f^{\ (r)}\right)(x) &= \frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n-1}\left[P_m\left(S_m\left(f^{(r)}\right)(x) - S_{m+1}\left(f^{(r)}\right)(x)\right) + P_{\lambda_n}S_n\left(f^{(r)}\right)(x)\right] \\ &= \frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n-1} P_m\left(-A_{m+1}\left(f^{(r)}\right)(x)\right) + S_n\left(f^{(r)}\right)(x),\end{aligned}$$

and so

$$R_n^{\lambda}\left(f^{\ (r)}\right)(x) - S_n f^{(r)}(x) = -\frac{1}{P_{\lambda_n}} \sum_{m=0}^{\lambda_n - 1} P_m A_{m+1}\left(f^{(r)}\right)(x).$$

Using Abel transform again, we get

$$\begin{aligned}
\sum_{m=0}^{\lambda_n-1} P_m A_{m+1}\left(f^{(r)}\right)(x) &= \sum_{m=0}^{\lambda_n-1} \frac{P_m}{m+1}(m+1) A_{m+1}\left(f^{(r)}\right)(x) \\
&= \sum_{m=0}^{\lambda_n-1}\left(\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right)\left(\sum_{k=0}^{m}(k+1) A_{k+1}\left(f^{(r)}\right)(x)\right) \\
&\quad + \frac{P_{\lambda_n}}{\lambda_n+1} \sum_{k=0}^{\lambda_n-1}(k+1) A_{k+1}\left(f^{(r)}\right)(x).
\end{aligned}$$

By considering (3.3) and (4.2) we get

$$\begin{aligned}
\left\| \sum_{m=0}^{\lambda_n-1} P_m A_{m+1}\left(f^{(r)}\right) \right\|_{ps,\omega} &\leq \sum_{m=0}^{\lambda_n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| \left\| \sum_{k=0}^{m}(k+1) A_{k+1}\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&\quad + \frac{P_{\lambda_n}}{\lambda_n+1} \left\| \sum_{k=0}^{\lambda_n-1}(k+1) A_{k+1}\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&= \sum_{m=0}^{\lambda_n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| (m+2) \left\| S_{m+1}\left(f^{(r)}\right) - \sigma_{m+1}\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&\quad + P_{\lambda_n} \left\| S_n\left(f^{(r)}\right) - \sigma_n\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&= O(1) \sum_{m=0}^{\lambda_n-1}\left|\frac{P_m}{m+1} - \frac{P_{m+1}}{m+2}\right| + O\left(\frac{P_{\lambda_n}}{\lambda_n}\right).
\end{aligned}$$

This yields

$$\begin{aligned}
\left\| R_n^{\lambda}\left(f^{\ (r)}\right) - S_n\left(f^{(r)}\right) \right\|_{L_w^{ps}} &= \frac{1}{P_{\lambda_n}} \left\| \sum_{m=0}^{\lambda_n-1} P_m A_{m+1}\left(f^{(r)}\right) \right\|_{ps,\omega} \\
&= \frac{1}{P_{\lambda_n}} O\left(\frac{P_{\lambda_n}}{\lambda_n}\right) = O\left(\frac{1}{\lambda_n}\right).
\end{aligned}$$

Combining this estimate with (3.3), we get

$$\left\| f^{(r)} - R_n^{\lambda}\left(f^{\ (r)}\right) \right\|_{ps,\omega} = O\left(\lambda_n^{-1}\right).$$

□

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