mathematics

MDPI

Article

New Types of F_c-Contractions and the Fixed-Circle Problem

Nihal Taş [1], Nihal Yılmaz Özgür [1] and Nabil Mlaiki [2,*]

[1] Department of Mathematics, Balıkesir University, 10145 Balıkesir, Turkey; nihaltas@balikesir.edu.tr (N.T.); nihal@balikesir.edu.tr (N.Y.Ö.)
[2] Department of Mathematical Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
* Correspondence: nmlaiki@psu.edu.sa

Received: 2 September 2018; Accepted: 27 September 2018; Published: 2 October 2018

Abstract: In this paper we investigate some fixed-circle theorems using Ćirić's technique (resp. Hardy-Rogers' technique, Reich's technique and Chatterjea's technique) on a metric space. To do this, we define new types of F_c-contractions such as Ćirić type, Hardy-Rogers type, Reich type and Chatterjea type. Two illustrative examples are presented to show the effectiveness of our results. Also, it is given an application of a Ćirić type F_c-contraction to discontinuous self-mappings which have fixed circles.

Keywords: fixed circle; Ćirić type F_c-contraction; Hardy–Rogers type F_c-contraction; Reich type F_c-contraction; Chatterjea type F_c-contraction

Classification: primary 54H25; secondary 47H10

---

1. Introduction

Fixed point theory has become the focus of many researchers lately (see [1–4]). One of the main important results of fixed point theory is when we show that a self mapping on a metric space under some specific conditions has a unique fixed point. In some cases when we do not have uniqueness of the fixed point, such a map fixes a circle which we call a fixed circle, the fixed-circle problem arises naturally in practice. There exist a lot of examples of self-mappings that map a circle onto itself and fixes all the points of the circle, whereas the circle is not fixed by the self-mapping. For example, let $(\mathbb{C}, d)$ be the usual metric space and $C_{0,1}$ be the unit circle. Let us consider the self-mappings $T_1 : \mathbb{C} \to \mathbb{C}$ and $T_2 : \mathbb{C} \to \mathbb{C}$ defined by

$$T_1 z = \begin{cases} \frac{1}{\overline{z}} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$$

and

$$T_2 z = \begin{cases} \frac{1}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases},$$

for all $z \in \mathbb{C}$ where $\overline{z}$ is the complex conjugate of the complex number z. Then, we have $T_i(C_{0,1}) = C_{0,1}$ $(i = 1, 2)$, but $C_{0,1}$ is the fixed circle of T_1 while it is not the fixed circle of T_2 (especially T_2 fixes only two points of the unit circle). Thus, a natural question arises as follows:

What is (are) the necessary and sufficient condition(s) for a self-mapping T that make a given circle as the fixed circle of T? Therefore, it is important to investigate new fixed-circle results.

Various fixed-circle theorems have been obtained using different approaches on metric and some generalized metric spaces (see [5–9] for more details). For example, in [5], fixed-circle results were proved using the Caristi's inequality on metric spaces. In [8], it was given a fixed-circle theorem for a self-mapping that maps a given circle onto itself. In [9], it was extended known fixed-circle results in many directions and introduced a new notion called as an F_c-contraction. In addition, some generalized fixed-circle theorems were investigated on an S-metric space (see [6,7]).

Motivated by the above studies, we present some new fixed-circle theorems using the ideas given in [10,11]. In [10], it was proved some fixed-point results using an F-contraction of the Hardy-Rogers-type and in [11], it was obtained a fixed-point theorem using a Ćirić type generalized F-contraction. We generate some fixed-circle results from these types of contractions using Wardowski's technique. For some fixed-point results obtained by this technique, one can consult the references [10–13]. In Section 2, we define the notions of a Ćirić type F_c-contraction, Hardy-Rogers type F_c-contraction, Reich type F_c-contraction and Chatterjea type F_c-contraction. Using these concepts, we prove some results related to the fixed-circle problem. In Section 3, we present an application of our obtained results to a discontinuous self-mapping that has a fixed circle.

2. New Fixed-Circle Results via Some Classical Techniques

Let (X, d) be a metric space and $T : X \to X$ be a self-mapping in the whole paper. Now we investigate some new fixed-circle theorems using the ideas of some classical fixed-point theorems.

At first, we recall some necessary definitions and a theorem related to fixed circle. A circle and a disc are defined on a metric space as follows, respectively:

$$C_{u_0,r} = \{u \in X : d(u, u_0) = r\}$$

and

$$D_{u_0,r} = \{u \in X : d(u, u_0) \leq r\}.$$

Definition 1 ([5]). *Let $C_{u_0,r}$ be a circle on X. If $Tu = u$ for every $u \in C_{u_0,r}$ then the circle $C_{u_0,r}$ is said to be a fixed circle of T.*

Definition 2 ([13]). *Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that*

(F_1) *F is strictly increasing,*

(F_2) *For each sequence $\{\alpha_n\}$ in $(0, \infty)$ the following holds*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F_3) *There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

Definition 3 ([9]). *If there exist $t > 0$, $F \in \mathbb{F}$ and $u_0 \in X$ such that for all $u \in X$ the following holds:*

$$d(u, Tu) > 0 \Rightarrow t + F(d(u, Tu)) \leq F(d(u_0, u)),$$

then T is said to be an F_c-contraction on X.

Theorem 1 ([9]). *Let T be an F_c-contractive self-mapping with $u_0 \in X$ and*

$$r = \min \{d(u, Tu) : u \neq Tu\}. \tag{1}$$

Then $C_{u_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}$ where $\rho < r$.

Now we define new contractive conditions and give some fixed-circle results.

Definition 4. *If there exist $t > 0$, $F \in \mathbb{F}$ and $u_0 \in X$ such that for all $u \in X$ the following holds*:

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F(m(u, u_0)), \tag{2}$$

where

$$m(u, v) = \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{1}{2}\left[d(u, Tv) + d(v, Tu)\right]\right\},$$

then T is said to be a Ćirić type F_c-contraction on X.

Proposition 1. *If T is a Ćirić type F_c-contraction with $u_0 \in X$ then we have $Tu_0 = u_0$.*

Proof. Assume that $Tu_0 \neq u_0$. From the definition of a Ćirić type F_c-contraction, we get

$$\begin{aligned} d(u_0, Tu_0) &> 0 \Longrightarrow t + F(d(u_0, Tu_0)) \leq F(m(u_0, u_0)) \\ &= F\left(\max\left\{\begin{array}{c} d(u_0, u_0), d(u_0, Tu_0), d(u_0, Tu_0), \\ \frac{1}{2}\left[d(u_0, Tu_0) + d(u_0, Tu_0)\right] \end{array}\right\}\right) \\ &= F(d(u_0, Tu_0)), \end{aligned}$$

a contradiction because of $t > 0$. Then we have $Tu_0 = u_0$. □

Theorem 2. *Let T be a Ćirić type F_c-contraction with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then $C_{u_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}$ with $\rho < r$.*

Proof. Let $u \in C_{u_0,r}$. Since $d(u_0, Tu) = r$, the self-mapping T maps $C_{u_0,r}$ into (or onto) itself. If $Tu \neq u$, by the definition of r, we have $d(u, Tu) \geq r$. So using the Ćirić type F_c-contractive property, Proposition 1 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(d(u, Tu)) \leq F(m(u, u_0)) - t < F(m(u, u_0)) \\ &= F\left(\max\left\{d(u, u_0), d(u, Tu), d(u_0, Tu_0), \frac{1}{2}\left[d(u, Tu_0) + d(u_0, Tu)\right]\right\}\right) \\ &= F\left(\max\left\{r, d(u, Tu), 0, r\right\}\right) = F(d(u, Tu)), \end{aligned}$$

a contradiction. Therefore, $d(u, Tu) = 0$ and so $Tu = u$. Consequently, $C_{u_0,r}$ is a fixed circle of T.

Now we show that T also fixes any circle $C_{u_0,\rho}$ with $\rho < r$. Let $u \in C_{u_0,\rho}$ and assume that $d(u, Tu) > 0$. By the Ćirić type F_c-contractive property, we have

$$F(d(u, Tu)) \leq F(m(u, u_0)) - t < F(m(u, u_0)) = F(d(u, Tu)),$$

a contradiction. Thus we obtain $d(u, Tu) = 0$ and $Tu = u$. So, $C_{u_0,\rho}$ is a fixed circle of T. □

Corollary 1. *Let T be a Ćirić type F_c-contractive self-mapping with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then T fixes the disc $D_{u_0,r}$.*

Definition 5. *If there exist $t > 0$, $F \in \mathbb{F}$ and $u_0 \in X$ such that for all $u \in X$ the following holds*:

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F\left(\begin{array}{c} \alpha d(u, u_0) + \beta d(u, Tu) + \gamma d(u_0, Tu_0) \\ +\delta d(u, Tu_0) + \eta d(u_0, Tu) \end{array}\right), \tag{3}$$

where

$$\alpha + \beta + \gamma + \delta + \eta = 1, \alpha, \beta, \gamma, \delta, \eta \geq 0 \text{ and } \alpha \neq 0,$$

then T is said to be a Hardy-Rogers type F_c-contraction on X.

Proposition 2. *If T is a Hardy-Rogers type F_c-contraction with $u_0 \in X$ then we have $Tu_0 = u_0$.*

Proof. Assume that $Tu_0 \neq u_0$. From the definition of a Hardy-Rogers type F_c-contraction, we get

$$\begin{aligned} d(u_0, Tu_0) &> 0 \Longrightarrow t + F(d(u_0, Tu_0)) \\ &\leq F\left(\begin{array}{c} \alpha d(u_0, u_0) + \beta d(u_0, Tu_0) + \gamma d(u_0, Tu_0) \\ +\delta d(u_0, Tu_0) + \eta d(u_0, Tu_0) \end{array} \right) \\ &= F\left((\beta + \gamma + \delta + \eta)\, d(u_0, Tu_0)\right) \\ &< F(d(u_0, Tu_0)), \end{aligned}$$

a contradiction because of $t > 0$. Then we have $Tu_0 = u_0$. □

Using Proposition 2, we rewrite the condition (3) as follows:

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F\left(\begin{array}{c} \alpha d(u, u_0) + \beta d(u, Tu) \\ +\delta d(u, Tu_0) + \eta d(u_0, Tu) \end{array} \right),$$

where

$$\alpha + \beta + \delta + \eta \leq 1, \alpha, \beta, \delta, \eta \geq 0 \text{ and } \alpha \neq 0.$$

Using this inequality, we obtain the following fixed-circle result.

Theorem 3. *Let T be a Hardy-Rogers type F_c-contraction with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then $C_{u_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}$ with $\rho < r$.*

Proof. Let $u \in C_{u_0,r}$. Using the Hardy-Rogers type F_c-contractive property, Proposition 2 and the fact that F is increasing, we get

$$\begin{aligned} F(r) &\leq F(d(u, Tu)) \\ &\leq F\left(\alpha d(u, u_0) + \beta d(u, Tu) + \delta d(u, Tu_0) + \eta d(u_0, Tu)\right) - t \\ &< F(\alpha r + \beta d(u, Tu) + \delta r + \eta r) \\ &\leq F((\alpha + \beta + \delta + \eta) d(u, Tu)) \leq F(d(u, Tu)), \end{aligned}$$

a contradiction. Therefore, $d(u, Tu) = 0$ and so $Tu = u$. Consequently, $C_{u_0,r}$ is a fixed circle of T. By the similar arguments used in the proof of Theorem 2, T also fixes any circle $C_{u_0,\rho}$ with $\rho < r$. □

Corollary 2. *Let T be a Hardy-Rogers type F_c-contractive self-mapping with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then T fixes the disc $D_{u_0,r}$.*

Remark 1. *If we consider $\alpha = 1$ and $\beta = \gamma = \delta = \eta = 0$ in Definition 5, then we get the notion of an F_c-contractive mapping.*

In Definition 5, if we choose $\delta = \eta = 0$, then we obtain the following definition.

Definition 6. *If there exist $t > 0$, $F \in \mathbb{F}$ and $u_0 \in X$ such that for all $u \in X$ the following holds:*

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F\left(\alpha d(u, u_0) + \beta d(u, Tu) + \gamma d(u_0, Tu_0)\right), \tag{4}$$

where

$$\alpha + \beta + \gamma < 1 \text{ and } \alpha, \beta, \gamma \geq 0,$$

then T is said to be a Reich type F_c-contraction on X.

Proposition 3. *If a self-mapping T on X is a Reich type F_c-contraction with $u_0 \in X$ then we have $Tu_0 = u_0$.*

Proof. From the similar arguments used in the proof of Proposition 2, the proof follows easily since $\beta + \gamma < 1$. □

Using Proposition 3, we rewrite the condition (4) as follows:

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F\left(\alpha d(u, u_0) + \beta d(u, Tu)\right),$$

where

$$\alpha + \beta < 1 \text{ and } \alpha, \beta \geq 0.$$

Using this inequality, we obtain the following fixed-circle result.

Theorem 4. *Let T be a Reich type F_c-contraction with $u_0 \in X$ and r be defined as in (1). Then $C_{u_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}$ with $\rho < r$.*

Proof. It can be easily seen since

$$F(r) \leq F(d(u, Tu)) \leq F((\alpha + \beta)d(u, Tu)) < F(d(u, Tu)).$$

□

Corollary 3. *Let T be a Reich type F_c-contractive self-mapping with $u_0 \in X$ and r be defined as in (1). Then T fixes the disc $D_{u_0,r}$.*

In Definition 5, if we choose $\alpha = \beta = \gamma = 0$ and $\delta = \eta$, then we obtain the following definition.

Definition 7. *If there exist $t > 0$, $F \in \mathbb{F}$ and $u_0 \in X$ such that for all $u \in X$ the following holds:*

$$d(u, Tu) > 0 \Longrightarrow t + F(d(u, Tu)) \leq F\left(\eta(d(u, Tu_0) + d(u_0, Tu))\right), \tag{5}$$

where

$$\eta \in \left(0, \frac{1}{2}\right),$$

then T is said to be a Chatterjea type F_c-contraction on X.

Proposition 4. *If a self-mapping T on X is a Chatterjea type F_c-contraction with $u_0 \in X$ then we have $Tu_0 = u_0$.*

Proof. From the similar arguments used in the proof of Proposition 2, it can be easily proved. □

Theorem 5. *Let T be a Chatterjea type F_c-contraction with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then $C_{u_0,r}$ is a fixed circle of T. Especially, T fixes every circle $C_{u_0,\rho}$ with $\rho < r$.*

Proof. By the similar arguments used in the proof of Theorem 3 and Definition 7, it can be easily checked. □

Corollary 4. *Let T be a Chatterjea type F_c-contractive self-mapping with $u_0 \in X$ and r be defined as in (1). If $d(u_0, Tu) = r$ for all $u \in C_{u_0,r}$ then T fixes the disc $D_{u_0,r}$.*

Now we give two illustrative examples of our obtained results.

Example 1. *Let* $X = \{1, 2, e^3 - 1, e^3, e^3 + 1\}$ *be the metric space with the usual metric. Let us define the self-mapping* $T : X \to X$ *as*

$$Tu = \begin{cases} 2 & if \quad u = 1 \\ u & \quad otherwise \end{cases},$$

for all $u \in X$.

The Ćirić type F_c*-contractive self-mapping T: The self-mapping T is a Ćirić type* F_c*-contractive self-mapping with* $F = \ln u$, $t = \ln(e^3 - 1)$ *and* $u_0 = e^3$. *Indeed, we get*

$$d(u, Tu) = d(1, T1) = d(1, 2) = 1 > 0$$

for $u = 1$ *and*

$$\begin{aligned} m(u, u_0) &= m(1, e^3) = \max\left\{d(1, e^3), d(1, 2), \frac{1}{2}\left[d(1, e^3) + d(e^3, 2)\right]\right\} \\ &= \max\left\{e^3 - 1, 1, e^3 - \frac{3}{2}\right\} = e^3 - 1. \end{aligned}$$

Then, we have

$$\begin{aligned} t + F(d(u, Tu)) &= \ln(e^3 - 1) + \ln(d(1, 2)) = \ln(e^3 - 1) \\ &\leq \ln(d(m(u, u_0))) = \ln(e^3 - 1). \end{aligned}$$

The Hardy-Rogers type F_c*-contractive self-mapping T: The self-mapping T is a Hardy-Rogers type* F_c*-contractive self-mapping with* $F = \ln u$, $t = \ln(e^3) - \ln 3$, $\alpha = \beta = \frac{1}{3}$, $\delta = \eta = 0$ *and* $u_0 = e^3$. *Indeed, we get*

$$d(u, Tu) = d(1, T1) = d(1, 2) = 1 > 0$$

for $u = 1$ *and*

$$\begin{aligned} \alpha d(u, u_0) + \beta d(u, Tu) + \delta d(u, Tu_0) + \eta d(u_0, Tu) &= \frac{1}{3}\left[d(1, e^3) + d(1, 2)\right] \\ &= \frac{1}{3}\left[e^3 - 1 + 1\right] = \frac{e^3}{3}. \end{aligned}$$

Then, we have

$$\begin{aligned} t + F(d(u, Tu)) &= \ln(e^3) - \ln 3 + \ln(d(1, 2)) = \ln(e^3) - \ln 3 \\ &\leq \ln(d(\alpha d(u, u_0) + \beta d(u, Tu) + \delta d(u, Tu_0) + \eta d(u_0, Tu))) \\ &= \ln(e^3) - \ln 3. \end{aligned}$$

The Reich type F_c*-contractive self-mapping T: The self-mapping T is a Reich type* F_c*-contractive self-mapping with* $F = \ln u$, $t = \ln(e^3) - \ln 4$, $\alpha = \beta = \frac{1}{4}$ *and* $u_0 = e^3$. *Indeed, we get*

$$d(u, Tu) = d(1, T1) = d(1, 2) = 1 > 0$$

for $u = 1$ *and*

$$\alpha d(u, u_0) + \beta d(u, Tu) = \frac{1}{4}\left[d(1, e^3) + d(1, 2)\right] = \frac{1}{4}\left[e^3 - 1 + 1\right] = \frac{e^3}{4}.$$

Then, we have

$$\begin{aligned} t + F(d(u,Tu)) &= \ln(e^3) - \ln 4 + \ln(d(1,2)) = \ln(e^3) - \ln 4 \\ &\leq \ln(d(\alpha d(u,u_0) + \beta d(u,Tu))) = \ln(e^3) - \ln 4. \end{aligned}$$

The Chatterjea type F_c-contractive self-mapping T: The self-mapping T is a Chatterjea type F_c-contractive self-mapping with $F = \ln u$, $t = \ln\left(\frac{2}{3}e^3 - 1\right)$, $\eta = \frac{1}{3}$ and $u_0 = e^3$. Indeed, we get

$$d(u,Tu) = d(1,T1) = d(1,2) = 1 > 0$$

for $u = 1$ and

$$\begin{aligned} \eta(d(u,Tu_0) + d(u_0,Tu)) &= \frac{1}{3}\left[d(1,e^3) + d(e^3,2)\right] \\ &= \frac{1}{3}\left[e^3 - 1 + e^3 - 2\right] = \frac{2e^3}{3} - 1. \end{aligned}$$

Then, we have

$$\begin{aligned} t + F(d(u,Tu)) &= \ln\left(\frac{2}{3}e^3 - 1\right) + \ln(d(1,2)) = \ln\left(\frac{2}{3}e^3 - 1\right) \\ &\leq \ln(\eta(d(u,Tu_0) + d(u_0,Tu))) = \ln\left(\frac{2}{3}e^3 - 1\right). \end{aligned}$$

Also, we obtain

$$r = \min\{d(u,Tu) : u \neq Tu\} = \{d(1,2)\} = 1.$$

Consequently, T fixes the circle $C_{e^3,1} = \{e^3 - 1, e^3 + 1\}$ and the disc $D_{e^3,1} = \{e^3 - 1, e^3, e^3 + 1\}$.

In the following example, we see that the converse statements of Theorems 2–5 are not always true.

Example 2. *Let $x_0 \in X$ be any point and the self-mapping $T : X \to X$ be defined as*

$$Tu = \begin{cases} u & \text{if} \quad u \in D_{u_0,\mu} \\ u_0 & \text{if} \quad u \notin D_{u_0,\mu} \end{cases},$$

for all $u \in X$ with $\mu > 0$. Then T is not a Ćirić type F_c-contractive self-mapping (resp. Hardy-Rogers type F_c-contractive self-mapping, Reich type F_c-contractive self-mapping and Chatterjea type F_c-contractive self-mapping). But T fixes every circle $C_{x_0,\rho}$ where $\rho \leq \mu$.

3. An Application to Discontinuity Problem

In this section, we give some examples of discontinuous functions and obtain a discontinuity result related to fixed circle.

Example 3. *Let $X = \{1, 2, e^3 - 1, e^3, e^3 + 1\}$ be the metric space with the usual metric. Let us define the self-mapping $T : X \to X$ as*

$$Tu = \begin{cases} 2 & \text{if} \quad u < e^3 - 1 \\ u & \text{if} \quad u \geq e^3 - 1 \end{cases},$$

for all $u \in X$. As in Example 1, it is easily verified that the self-mapping T is a Ćirić type F_c-contractive self-mapping and $C_{e^3,1} = \{e^3 - 1, e^3 + 1\}$ is a fixed circle of T. We note that the self-mapping T is continuous at the point $e^3 + 1$ while the self-mapping T is discontinuous at the point $e^3 - 1$.

Example 4. *Let* $X = \{1, 2, e^3 - 1, e^3, e^3 + 1\}$ *be the metric space with the usual metric. Let us define the self-mapping* $T : X \to X$ *as*

$$Tu = \begin{cases} 2 & \text{if} \quad u < e^3 - 1 \\ e^3 - 1 & \text{if} \quad e^3 - 1 \leq u < e^3 \\ u & \text{if} \quad e^3 \leq u \leq e^3 + 1 \\ u - 1 & \text{if} \quad u > e^3 + 1 \end{cases},$$

for all $u \in X$*. As in Example 1, it is easily checked that the self-mapping* T *is a Ćirić type* F_c*-contractive self-mapping and* $C_{e^3,1} = \{e^3 - 1, e^3 + 1\}$ *is a fixed circle of* T*. We note that the self-mapping* T *is discontinuous at the center* e^3 *and on the circle* $C_{e^3,1}$*.*

Consider the above examples, we give the following theorem.

Theorem 6. *Let* T *be a Ćirić type* F_c*-contraction with* $u_0 \in X$ *and* r *be defined as in (1). If* $d(u_0, Tu) = r$ *for all* $u \in C_{u_0,r}$ *then* $C_{u_0,r}$ *is a fixed circle of* T*. Also* T *is discontinuous at* $u \in C_{u_0,r}$ *if and only if* $\lim_{v \to u} m(u, v) \neq 0$*.*

Proof. From Theorem 2, we see that $C_{u_0,r}$ is a fixed circle of T. Used the idea given in Theorem 2.1 on page 1240 in [14], we see that T is discontinuous at $u \in C_{u_0,r}$ if and only if $\lim_{v \to u} m(u, v) \neq 0$. □

4. Conclusions

We have presented new generalized fixed-circle results using new types of contractive conditions on metric spaces. The obtained results can be also considered as fixed-disc results. By means of some known techniques which are used to obtain some fixed-point results, we have generated useful fixed-circle theorems. As we have seen in the last section, our main results can be applied to other research areas.

Author Contributions: All authors contributed equally in writing this article. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Acknowledgments: The third author would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Abdeljawad, T.; Alzabut, J.O.; Mukheimer, A.; Zaidan, Y. Best Proximity Points For Cyclical Contraction MappingsWith 0—Boundedly Compact Decompositions. *J. Comput. Anal. Appl.* **2013**, *15*, 678–685.
2. Abdeljawad, T.; Alzabut, J.O.; Mukheimer, A.; Zaidan, Y. Banach contraction principle for cyclical mappings on partial metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 154. [CrossRef]
3. Shatanawi, W.; Pitea, A.; Lazovic, R. Contraction conditions using comparison functions on b-metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 135. [CrossRef]
4. Shatanawi, W. Fixed and Common Fixed Point for Mapping Satisfying Some Nonlinear Contraction in *b*-metric Spaces. *J. Math. Anal.* **2016**, *7*, 1–12.
5. Özgür, N.Y.; Taş, N. Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **2017**. [CrossRef]
6. Özgür, N.Y.; Taş, N.; Çelik, U. New fixed-circle results on *S*-metric spaces. *Bull. Math. Anal. Appl.* **2017**, *9*, 10–23.
7. Özgür, N.Y.; Taş, N. Some fixed-circle theorems on *S*-metric spaces with a geometric viewpoint. *arXiv* **2017**, arXiv:1704.08838.
8. Özgür, N.Y.; Taş, N. Some fixed-circle theorems and discontinuity at fixed circle. *AIP Conf. Proc.* **2018**, 020048. [CrossRef]

9. Taş, N.; Özgür, N.Y.; Mlaiki, N. New fixed-circle results related to F_c-contractive and F_c-expanding mappings on metric spaces. *Math. Notes* **2018**, submitted.
10. Cosentino, M.; Vetro, P. Fixed point results for F-contractive mappings of Hardy–Rogers-Type. *Filomat* **2014**, *28*, 715–722. [CrossRef]
11. Mınak, G.; Helvacı, A.; Altun, İ. Ćirić type generalized F-contractions on complete metric spaces and fixed point results. *Filomat* **2014**, *28*, 1143–1151. [CrossRef]
12. Al-Rawashdeh, A.; Aydi, H.; Felhi, A.; Sahmim, S.; Shatanawi, W. On common fixed points for α-F-contractions and applications. *J. Nonlinear Sci. Appl*. **2016**, *9*, 3445–3458. [CrossRef]
13. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
14. Bisht, R.K.; Pant, R.P. A remark on discontinuity at fixed point. *J. Math. Anal. Appl.* **2017**, *445*, 1239–1242. [CrossRef]