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On the Fixed-Circle Problem and Khan Type Contractions

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Abstract: In this paper, we consider the fixed-circle problem on metric spaces and give new results on this problem. To do this, we present three types of F_C-Khan type contractions. Furthermore, we obtain some solutions to an open problem related to the common fixed-circle problem.

Keywords: fixed circle; common fixed circle; fixed-circle theorem

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1. Introduction

Recently, the fixed-circle problem has been considered for metric and some generalized metric spaces (see [1–6] for more details). For example, in [1], some fixed-circle results were obtained using the Caristi type contraction on a metric space. Using Wardowski's technique and some classical contractive conditions, new fixed-circle theorems were proved in [5,6]. In [2,3], the fixed-circle problem was studied on an S-metric space. In [7], a new fixed-circle theorem was proved using the modified Khan type contractive condition on an S-metric space. Some generalized fixed-circle results with geometric viewpoint were obtained on S_b-metric spaces and parametric N_b-metric spaces (see [8,9] for more details, respectively). Also, it was proposed to investigate some fixed-circle theorems on extended M_b-metric spaces [10]. On the other hand, an application of the obtained fixed-circle results was given to discontinuous activation functions on metric spaces (see [1,4,11]). Hence it is important to study new fixed-circle results using different techniques.

Let (X, d) be a metric space and $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ be any circle on X. In [5], it was given the following open problem.

Open Problem CC**:** What is (are) the condition(s) to make any circle $C_{x_0,r}$ as the common fixed circle for two (or more than two) self-mappings?

In this paper, we give new results to the fixed-circle problem using Khan type contractions and to the above open problem using both of Khan and Ćirić type contractions on a metric space. In Section 2, we introduce three types of F_C-Khan type contractions and obtain new fixed-circle results. In Section 3, we investigate some solutions to the above Open Problem CC. In addition, we construct some examples to support our theoretical results.

2. New Fixed-Circle Theorems

In this section, using Khan type contractions, we give new fixed-circle theorems (see [12–15] for some Khan type contractions used to obtain fixed-point theorems). At first, we recall the following definitions.

Definition 1 ([16]). *Let $\mathbb{F}$ be the family of all functions $F : (0, \infty) \to \mathbb{R}$ such that*

$(F1)$ F is strictly increasing,

$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2 ([16]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, d), if there exist $F \in \mathbb{F}$ and $\tau \in (0, \infty)$ such that*

$$d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

for all $x, y \in X$.

Definition 3 ([15]). *Let $\mathbb{F}_k$ be the family of all increasing functions $F : (0, \infty) \to \mathbb{R}$, that is, for all $x, y \in (0, \infty)$, if $x < y$ then $F(x) \leq F(y)$.*

Definition 4 ([15]). *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be an F-Khan-contraction if there exist $F \in \mathbb{F}_k$ and $t > 0$ such that for all $x, y \in X$ if $\max\{d(Ty, x), d(Tx, y)\} \neq 0$ then $Tx \neq Ty$ and*

$$t + F(d(Tx, Ty)) \leq F\left(\frac{d(Tx, x)d(Ty, x) + d(Ty, y)d(Tx, y)}{\max\{d(Ty, x), d(Tx, y)\}}\right),$$

and if $\max\{d(Ty, x), d(Tx, y)\} = 0$ then $Tx = Ty$.

Now we modify the definition of an F-Khan-contractive condition, which is used to obtain a fixed point theorem in [15], to get new fixed-circle results. Hence, we define the notion of an F_C-Khan type I contractive condition as follows.

Definition 5. *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be an F_C-Khan type I contraction if there exist $F \in \mathbb{F}_k$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ if the following condition holds*

$$\max\{d(Tx_0, x_0), d(Tx, x)\} \neq 0, \tag{1}$$

then

$$t + F(d(Tx, x)) \leq F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{\max\{d(Tx_0, x_0), d(Tx, x)\}}\right),$$

where $h \in \left[0, \frac{1}{2}\right)$ and if $\max\{d(Tx_0, x_0), d(Tx, x)\} = 0$ then $Tx = x$.

One of the consequences of this definition is the following proposition.

Proposition 1. *Let (X, d) be a metric space. If a self-mapping T on X is an F_C-Khan type I contraction with $x_0 \in X$ then we get $Tx_0 = x_0$.*

Proof. Let $Tx_0 \neq x_0$. Then using the hypothesis, we find

$$\max\{d(Tx_0, x_0), d(Tx, x)\} \neq 0$$

and

$$\begin{aligned} t + F(d(Tx_0, x_0)) &\leq F\left(h\frac{d(Tx_0, x_0)d(Tx_0, x_0) + d(Tx_0, x_0)d(Tx_0, x_0)}{d(Tx_0, x_0)}\right) \\ &= F(2hd(Tx_0, x_0)) < F(d(Tx_0, x_0)). \end{aligned}$$

This is a contradiction since $t > 0$ and so it should be $Tx_0 = x_0$. □

Consequently, the condition (1) can be replaced with $d(Tx, x) \neq 0$ and so $Tx \neq x$. Considering this, now we give a new fixed-circle theorem.

Theorem 1. *Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and*

$$r = \inf \{d(Tx, x) : Tx \neq x\}. \tag{2}$$

If T is an F_C-Khan type I contraction with $x_0 \in X$ then $C_{x_0,r}$ is a fixed circle of T.

Proof. Let $x \in C_{x_0,r}$. Assume that $Tx \neq x$. Then we have $d(Tx, x) \neq 0$ and by the F_C-Khan type I contractive condition, we obtain

$$\begin{aligned} t + F(d(Tx, x)) &\leq F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{\max\{d(Tx_0, x_0), d(Tx, x)\}}\right) \\ &= F(hr) \leq F(hd(Tx, x)) < F(d(Tx, x)), \end{aligned}$$

a contradiction since $t > 0$. Therefore, we have $Tx = x$ and so T fixes the circle $C_{x_0,r}$. □

Corollary 1. *Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and r be defined as in (2). If T is an F_C-Khan type I contraction with $x_0 \in X$ then T fixes the disc $D_{x_0,r} = \{x \in X : d(x, x_0) \leq r\}$.*

We recall the following theorem.

Theorem 2 ([12]). *Let (X, d) be a metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leq \begin{cases} k\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} & \text{if } d(x, Ty) + d(y, Tx) \neq 0 \\ 0 & \text{if } d(x, Ty) + d(y, Tx) = 0 \end{cases}, \tag{3}$$

where $k \in [0, 1)$ and $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

We modify the inequality (3) using Wardowski's technique to obtain a new fixed-point theorem. We give the following definition.

Definition 6. *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be an F_C-Khan type II contraction if there exist $F \in \mathbb{F}_k$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ if $d(Tx_0, x_0) + d(Tx, x) \neq 0$ then $Tx \neq x$ and*

$$t + F(d(Tx, x)) \leq F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{d(Tx_0, x_0) + d(Tx, x)}\right),$$

where $h \in \left[0, \frac{1}{2}\right)$ and if $d(Tx_0, x_0) + d(Tx, x) = 0$ then $Tx = x$.

An immediate consequence of this definition is the following result.

Proposition 2. *Let (X, d) be a metric space. If a self-mapping T on X is an F_C-Khan type II contraction then we get $Tx_0 = x_0$.*

Proof. Let $Tx_0 \neq x_0$. Then using the hypothesis, we find

$$d(Tx_0, x_0) + d(Tx, x) \neq 0$$

and

$$\begin{aligned} t + F(d(Tx_0, x_0)) &\leq F\left(h\frac{d(Tx_0, x_0)d(Tx_0, x_0) + d(Tx_0, x_0)d(Tx_0, x_0)}{2d(Tx_0, x_0)}\right) \\ &= F(hd(Tx_0, x_0)) < F(d(Tx_0, x_0)), \end{aligned}$$

which is a contradiction since $t > 0$. Hence it should be $Tx_0 = x_0$. □

Theorem 3. *Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and r be defined as in (2). If T is an F_C-Khan type II contraction with $x_0 \in X$ then $C_{x_0,r}$ is a fixed circle of T.*

Proof. Let $x \in C_{x_0,r}$. Assume that $Tx \neq x$. Then using Proposition 2, we get

$$d(Tx_0, x_0) + d(Tx, x) = d(Tx, x) \neq 0.$$

By the F_C-Khan type II contractive condition, we obtain

$$\begin{aligned} t + F(d(Tx, x)) &\leq F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{d(Tx_0, x_0) + d(Tx, x)}\right) \\ &= F(hr) \leq F(hd(Tx, x)) < F(d(Tx, x)), \end{aligned}$$

a contradiction since $t > 0$. Therefore, we have $Tx = x$ and T fixes the circle $C_{x_0,r}$. □

Corollary 2. *Let (X, d) be a metric space, $T : X \to X$ be a self-mapping and r be defined as in (2). If T is an F_C-Khan type II contraction with $x_0 \in X$ then T fixes the disc $D_{x_0,r}$.*

In the following theorem, we see that the F_C-Khan type I and F_C-Khan type II contractive conditions are equivalent.

Theorem 4. *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T satisfies the F_C-Khan type I contractive condition if and only if T satisfies the F_C-Khan type II contractive condition.*

Proof. Let the F_C-Khan type I contractive condition be satisfied by T. Using Proposition 1 and Proposition 2, we get

$$\begin{aligned} t + F(d(Tx, x)) &\leq F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{\max\{d(Tx_0, x_0), d(Tx, x)\}}\right) \\ &= F\left(h\frac{d(Tx, x)d(Tx_0, x)}{d(Tx, x)}\right) \\ &= F(hd(Tx_0, x)) \\ &= F\left(h\frac{d(Tx, x)d(Tx_0, x) + d(Tx_0, x_0)d(Tx, x_0)}{d(Tx_0, x_0) + d(Tx, x)}\right). \end{aligned}$$

Using the similar arguments, the converse statement is clear. Consequently, the F_C-Khan type I contractive and the F_C-Khan type II contractive conditions are equivalent. □

Remark 1. *By Theorem 4, we see that Theorem 1 and Theorem 3 are equivalent.*

Now we give an example.

Example 1. *Let $X = \mathbb{R}$ be the metric space with the usual metric $d(x,y) = |x-y|$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tx = \begin{cases} x & \text{if } |x| < 6 \\ x+1 & \text{if } |x| \geq 6 \end{cases},$$

for all $x \in \mathbb{R}$. The self-mapping T is both of an F_C-Khan type I and an F_C-Khan type II contraction with $F = \ln x$, $t = \ln 2$, $x_0 = 0$ and $h = \frac{1}{3}$. Indeed, we get

$$d(Tx,x) = 1 \neq 0,$$

for all $x \in \mathbb{R}$ such that $|x| \geq 6$. Then we have

$$\begin{aligned} \ln 2 &\leq \ln\left(\frac{1}{3}|x|\right) \\ &\Longrightarrow \ln 2 + \ln 1 \leq \ln(hd(x,0)) = \ln(hd(x,x_0)) \\ &\Longrightarrow t + F(d(Tx,x)) \leq F\left(h\frac{d(Tx,x)d(Tx_0,x) + d(Tx_0,x_0)d(Tx,x_0)}{\max\{d(Tx_0,x_0), d(Tx,x)\}}\right) \end{aligned}$$

and

$$\begin{aligned} \ln 2 &\leq \ln\left(\frac{1}{3}|x|\right) \\ &\Longrightarrow \ln 2 + \ln 1 \leq \ln(hd(x,0)) = \ln(hd(x,x_0)) \\ &\Longrightarrow t + F(d(Tx,x)) \leq F\left(h\frac{d(Tx,x)d(Tx_0,x) + d(Tx_0,x_0)d(Tx,x_0)}{d(Tx_0,x_0) + d(Tx,x)}\right). \end{aligned}$$

Also we obtain

$$r = \min\{d(Tx,x) : Tx \neq x\} = 1.$$

Consequently, T fixes the circle $C_{0,1} = \{-1,1\}$ and the disc $D_{0,1} = \{x \in X : |x| \leq 1\}$. Notice that the self-mapping T has other fixed circles. The above results give us only one of these circles. Also, T has infinitely many fixed circles.

Now we consider the case if $T : X \to X$ is a self-mapping, then for all $x, y \in X$,

$$x \neq y \Longrightarrow d(Ty,x) + d(Tx,y) \neq 0.$$

Definition 7. *Let (X,d) be a metric space and $T : X \to X$ be a self-mapping. Then T is called a C-Khan type contraction if there exists $x_0 \in X$ such that*

$$d(Tx,x) \leq h\frac{d(Tx,x)d(Tx_0,x) + d(Tx_0,x_0)d(Tx,x_0)}{d(Tx_0,x) + d(Tx,x_0)}, \tag{4}$$

where $h \in [0,1)$ for all $x \in X - \{x_0\}$.

We can give the following fixed-circle result.

Theorem 5. *Let (X,d) be a metric space, $T : X \to X$ be a self-mapping and $C_{x_0,r}$ be a circle on X. If T satisfies the C-Khan type contractive condition (4) for all $x \in C_{x_0,r}$ with $Tx_0 = x_0$, then T fixes the circle $C_{x_0,r}$.*

Proof. Let $x \in C_{x_0,r}$. Suppose that $Tx \neq x$. Using the C-Khan type contractive condition with $Tx_0 = x_0$, we find

$$\begin{aligned} d(Tx,x) &\leq h\frac{d(Tx,x)d(Tx_0,x)+d(Tx_0,x_0)d(Tx,x_0)}{d(Tx_0,x)+d(Tx,x_0)} \\ &= \frac{hrd(Tx,x)}{r+d(Tx,x_0)} \\ &\leq \frac{hrd(Tx,x)}{r} = hd(Tx,x), \end{aligned}$$

which is a contradiction since $h < 1$. Consequently, T fixes the circle $C_{x_0,r}$. □

Theorem 6. *Let (X,d) be a metric space, $x_0 \in X$ and $T : X \to X$ be a self-mapping. If T is a C-Khan type contraction for all $x \in X - \{x_0\}$ with $Tx_0 = x_0$, then T is the identity map I_X on X.*

Proof. Let $x \in X - \{x_0\}$ be any point. If $Tx \neq x$ then using the C-Khan type contractive condition (4) with $Tx_0 = x_0$, we find

$$\begin{aligned} d(Tx,x) &\leq h\frac{d(Tx,x)d(Tx_0,x)+d(Tx_0,x_0)d(Tx,x_0)}{d(Tx_0,x)+d(Tx,x_0)} \\ &= h\frac{d(Tx,x)d(x_0,x)}{d(x_0,x)+d(Tx,x_0)} \\ &\leq h\frac{d(Tx,x)d(x_0,x)+d(Tx,x)d(Tx,x_0)}{d(x_0,x)+d(Tx,x_0)} \\ &= h\frac{d(Tx,x)\left[d(x_0,x)+d(Tx,x_0)\right]}{d(x_0,x)+d(Tx,x_0)} \\ &= hd(Tx,x), \end{aligned}$$

which is a contradiction since $h < 1$. Consequently, we have $Tx = x$ and hence T is the identity map I_X on X. □

Example 2. *Let $X = \mathbb{R}$ be the usual metric space and consider the circle $C_{0,3} = \{-3,3\}$. Let us define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as*

$$Tx = \begin{cases} \frac{-9x+8}{2x-9} & \text{if} \quad x \in \{-3,3\} \\ 0 & \text{if} \quad x \in \mathbb{R} - \{-3,3\} \end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping T satisfies the C-Khan type contractive condition for all $x \in C_{0,3}$ and $T0 = 0$. Consequently, $C_{0,3}$ is a fixed circle of T.

3. Common Fixed-Circle Results

Recently, it was obtained some coincidence and common fixed-point theorems using Wardowski's technique and the Ćirić type contractions (see [17] for more details). In this section, we extend the notion of a Khan type F_C-contraction to a pair of maps to obtain a solution to the Open Problem CC. At first, we give the following definition.

Definition 8. *Let (X,d) be a metric space and $T, S : X \to X$ be two self-mappings. A pair of self-mappings (T,S) is called a Khan type $F_{T,S}$-contraction if there exist $F \in \mathbb{F}_k$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$ if the following condition holds*

$$\max\{d(Tx_0,x_0), d(Sx_0,x_0)\} \neq 0,$$

then

$$t + F(d(Tx,Sx)) \leq F\left(h\frac{d(Tx,Sx)d(Tx,x_0)+d(Tx_0,Sx_0)d(Sx,x_0)}{\max\{d(Tx_0,x_0), d(Sx_0,x_0)\}}\right),$$

where $h \in \left[0,\frac{1}{2}\right)$ and if $\max\{d(Tx_0,x_0), d(Sx_0,x_0)\} = 0$ then $Tx = Sx$.

An immediate consequence of this definition is the following proposition.

Proposition 3. *Let (X,d) be a metric space and $T,S: X \to X$ be two self-mappings. If the pair of self-mappings (T,S) is a Khan type $F_{T,S}$-contraction with $x_0 \in X$ then x_0 is a coincidence point of T and S, that is, $Tx_0 = Sx_0$.*

Proof. We prove this proposition under the following cases:

Case 1: Let $Tx_0 = x_0$ and $Sx_0 \neq x_0$. Then using the hypothesis, we get

$$\max\{d(Tx_0,x_0), d(Sx_0,x_0)\} = d(Sx_0,x_0) \neq 0$$

and so

$$\begin{aligned} t + F(d(Tx_0,Sx_0)) &\leq F\left(h\frac{d(Tx_0,Sx_0)d(Tx_0,x_0) + d(Tx_0,Sx_0)d(Sx_0,x_0)}{d(Sx_0,x_0)}\right) \\ &= F(hd(Tx_0,Sx_0)), \end{aligned}$$

which is a contradiction since $h \in \left[0,\frac{1}{2}\right)$ and $t > 0$.

Case 2: Let $Tx_0 \neq x_0$ and $Sx_0 = x_0$. By the similar arguments used in the proof of Case 1, we get a contradiction.

Case 3: Let $Tx_0 = x_0$ and $Sx_0 = x_0$. Then we get $Tx_0 = Sx_0$.

Case 4: Let $Tx_0 \neq x_0$, $Sx_0 \neq x_0$ and $Tx_0 \neq Sx_0$. Using the hypothesis, we obtain

$$\max\{d(Tx_0,x_0), d(Sx_0,x_0)\} \neq 0$$

and so

$$t + F(d(Tx_0,Sx_0)) \leq F\left(h\frac{d(Tx_0,Sx_0)d(Tx_0,x_0) + d(Tx_0,Sx_0)d(Sx_0,x_0)}{\max\{d(Tx_0,x_0), d(Sx_0,x_0)\}}\right). \tag{5}$$

Assume that $d(Tx_0,x_0) > d(Sx_0,x_0)$. Using the inequality (5), we get

$$\begin{aligned} t + F(d(Tx_0,Sx_0)) &\leq F\left(h\frac{d(Tx_0,Sx_0)d(Tx_0,x_0) + d(Tx_0,Sx_0)d(Sx_0,x_0)}{d(Tx_0,x_0)}\right) \\ &= F\left(hd(Tx_0,Sx_0) + h\frac{d(Tx_0,Sx_0)d(Sx_0,x_0)}{d(Tx_0,x_0)}\right) \\ &< F(2hd(Tx_0,Sx_0)) < F(d(Tx_0,Sx_0)), \end{aligned}$$

which is a contradiction. Suppose that $d(Tx_0,x_0) < d(Sx_0,x_0)$. Using the inequality (5), we find

$$t + F(d(Tx_0,Sx_0)) < F(d(Tx_0,Sx_0)),$$

which is a contradiction. Consequently, x_0 is a coincidence point of T and S, that is, $Tx_0 = Sx_0$. □

Now we use the following number given in [17] (see Definition 3.1 on page 183):

$$M(x,y) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{d(Sx,Ty) + d(Sy,Tx)}{2}\right\}. \tag{6}$$

We give the following definition.

Definition 9. *Let (X,d) be a metric space and $T,S: X \to X$ be two self-mappings. A pair of self-mappings (T,S) is called a Ćirić type $F_{T,S}$-contraction if there exist $F \in \mathbb{F}_k$, $t > 0$ and $x_0 \in X$ such that for all $x \in X$*

$$d(Tx,x) > 0 \Longrightarrow t + F(d(Tx,x)) \leq F(M(x,x_0)).$$

We get the following proposition.

Proposition 4. *Let (X, d) be a metric space and $T, S : X \to X$ be two self-mappings. If the pair of self-mappings (T, S) is both a Khan type $F_{T,S}$-contraction and a Ćirić type $F_{T,S}$-contraction with $x_0 \in X$ then x_0 is a common fixed point of T and S, that is, $Tx_0 = Sx_0 = x_0$.*

Proof. By the Khan type $F_{T,S}$-contractive property and Proposition 3, we know that x_0 is a coincidence point of T and S, that is, $Tx_0 = Sx_0$. Now we prove that x_0 is a common fixed point of T and S. Let $Tx_0 \neq x_0$. Then using the Ćirić type $F_{T,S}$-contractive condition, we get

$$\begin{aligned} t + F(d(Tx_0, x_0)) &\leq F(M(x_0, x_0)) \\ &= F\left(\max\left\{ \begin{array}{c} d(Sx_0, Sx_0), d(Sx_0, Tx_0), d(Sx_0, Tx_0), \\ \frac{d(Sx_0,Tx_0)+d(Sx_0,Tx_0)}{2} \end{array} \right\}\right) \\ &= F(d(Sx_0, Tx_0)) = F(0), \end{aligned}$$

which is a contradiction because of the definition of F. Therefore it should be $Tx_0 = x_0$. Consequently, x_0 is a common fixed point of T and S, that is, $Tx_0 = Sx_0 = x_0$. □

Notice that we get a coincidence point result for a pair of self-mappings using the Khan type $F_{T,S}$-contractive condition by Proposition 3. We obtain a common fixed-point result for a pair of self-mappings using the both of Khan type $F_{T,S}$-contractive condition and the Ćirić type $F_{T,S}$-contractive condition by Proposition 4.

We prove the following common fixed-circle theorem as a solution to the Open Problem *CC*.

Theorem 7. *Let (X, d) be a metric space, $T, S : X \to X$ be two self-mappings and r be defined as in (2). If $d(Tx, x_0) = d(Sx, x_0) = r$ for all $x \in C_{x_0,r}$ and the pair of self-mappings (T, S) is both a Khan type $F_{T,S}$-contraction and a Ćirić type $F_{T,S}$-contraction with $x_0 \in X$ then $C_{x_0,r}$ is a common fixed circle of T and S, that is, $Tx = Sx = x$ for all $x \in C_{x_0,r}$.*

Proof. Let $x \in C_{x_0,r}$. We show that x is a coincidence point of T and S. Using Proposition 4, we get

$$\max\{d(Tx_0, x_0), d(Sx_0, x_0)\} = 0$$

and so by the definition of the Khan type $F_{T,S}$-contraction we obtain

$$Tx = Sx.$$

Now we prove that $C_{x_0,r}$ is a common fixed circle of T and S. Assume that $Tx \neq x$. Using Proposition 4 and the hypothesis Ćirić type $F_{T,S}$-contractive condition, we find

$$\begin{aligned} t + F(d(Tx, x)) &\leq F(M(x, x_0)) \\ &= F\left(\max\left\{ \begin{array}{c} d(Sx, Sx_0), d(Sx, Tx), d(Sx_0, Tx_0), \\ \frac{d(Sx,Tx_0)+d(Sx_0,Tx)}{2} \end{array} \right\}\right) \\ &= F\left(\max\left\{d(Sx, x_0), d(Sx, Tx), \frac{d(Sx, x_0) + d(x_0, Tx)}{2}\right\}\right) \\ &= F(\max\{r, d(Sx, Tx), r\}) = F(r), \end{aligned}$$

which contradicts with the definition of r. Consequently, we have $Tx = x$ and so $C_{x_0,r}$ is a common fixed circle of T and S. □

Corollary 3. *Let (X, d) be a metric space, $T, S : X \to X$ be two self-mappings and r be defined as in (2). If $d(Tx, x_0) = d(Sx, x_0) = r$ for all $x \in C_{x_0,r}$ and the pair of self-mappings (T, S) is both a Khan type*

$F_{T,S}$-contraction and a Ćirić type $F_{T,S}$-contraction with $x_0 \in X$ then T and S fix the disc $D_{x_0,r}$, that is, $Tx = Sx = x$ for all $x \in D_{x_0,r}$.

We give an illustrative example.

Example 3. *Let $X = [1,\infty) \cup \{-1,0\}$ be the metric space with the usual metric. Let us define the self-mappings $T : X \to X$ and $S : X \to X$ as*

$$Tx = \begin{cases} x^2 & \text{if} \quad x \in \{0,1,3\} \\ -1 & \text{if} \quad x = -1 \\ x+1 & \text{otherwise} \end{cases}$$

and

$$Sx = \begin{cases} \frac{1}{x} & \text{if} \quad x \in \{-1,1\} \\ 3x & \text{if} \quad x \in \{0,3\} \\ x+1 & \text{otherwise} \end{cases},$$

for all $x \in X$. The pair of the self-mappings (T,S) is both a Khan type $F_{T,S}$-contraction and a Ćirić type $F_{T,S}$-contraction with $F = \ln x$, $t = \ln \frac{3}{2}$ and $x_0 = 0$. Indeed, we get

$$\max\{d(T0,0), d(S0,0)\} = 0$$

and so $Tx = Sx$. Therefore, the pair (T,S) is a Khan type $F_{T,S}$-contraction. Also we get

$$d(T3,3) = 6 \neq 0,$$

for $x = 3$ and

$$d(Tx,x) = 1 \neq 0,$$

for all $x \in X \setminus \{-1,0,1,3\}$. Then we have

$$\begin{aligned} \ln \frac{3}{2} &\leq \ln 9 \\ &\Longrightarrow \ln \frac{3}{2} + \ln 6 \leq \ln 9 \\ &\Longrightarrow \ln \frac{3}{2} + \ln(d(T3,3)) \leq \ln(M(3,0)) \end{aligned}$$

and

$$\begin{aligned} \ln \frac{3}{2} &\leq \ln|x+1| \\ &\Longrightarrow \ln \frac{3}{2} + \ln 1 \leq \ln|x+1| \\ &\Longrightarrow \ln \frac{3}{2} + \ln(d(Tx,x)) \leq \ln(M(x,0)). \end{aligned}$$

Hence the pair (T,S) is a Ćirić type $F_{T,S}$-contraction. Also we obtain

$$r = \min\{d(Tx,x) : Tx \neq x\} = \min\{1,6\} = 1.$$

Consequently, T fixes the circle $C_{0,1} = \{-1,1\}$ and the disc $D_{0,1}$.

In closing, we want to bring to the reader attention the following question, under what conditions we can prove the results in [18–20] in fixed circle?

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References

1. Özgür, N.Y.; Taş, N. Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **2017**. [CrossRef]
2. Özgür, N.Y.; Taş, N.; Çelik, U. New fixed-circle results on S-metric spaces. *Bull. Math. Anal. Appl.* **2017**, *9*, 10–23.
3. Özgür, N.Y.; Taş, N. Fixed-circle problem on S-metric spaces with a geometric viewpoint. *arXiv* **2017**, arXiv:1704.08838.
4. Özgür, N.Y.; Taş, N. Some fixed-circle theorems and discontinuity at fixed circle. *AIP Conf. Proc.* **2018**. [CrossRef]
5. Taş, N.; Özgür, N.Y.; Mlaiki, N. New fixed-circle results related to F_c-contractive and F_c-expanding mappings on metric spaces. **2018**, submitted for publication.
6. Taş, N.; Özgür, N.Y.; Mlaiki, N. New types of F_c-contractions and the fixed-circle problem. *Mathematics* **2018**, *6*, 188. [CrossRef]
7. Taş, N. Various types of fixed-point theorems on S-metric spaces. *J. BAUN Inst. Sci. Technol.* **2018**. [CrossRef]
8. Özgür, N.Y.; Taş, N. Generalizations of metric spaces: From the fixed-point theory to the fixed-circle theory. In *Applications of Nonlinear Analysis*; Rassias T., Ed.; Springer: Cham, Switzerland, 2018; Volume 134.
9. Taş, N.; Özgür, N.Y. Some fixed-point results on parametric N_b-metric spaces. *Commun. Korean Math. Soc.* **2018**, *33*, 943–960.
10. Mlaiki, N.; Özgür, N.Y.; Mukheimer, A.; Taş, N. A new extension of the M_b-metric spaces. *J. Math. Anal.* **2018**, *9*, 118–133.
11. Taş, N.; Özgür, N.Y. A new contribution to discontinuity at fixed point. *arXiv* **2017**, arXiv:1705.03699.
12. Fisher, B. On a theorem of Khan. *Riv. Math. Univ. Parma* **1978**, *4*, 135–137.
13. Khan, M.S. A fixed point theorem for metric spaces. *Rend. Inst. Math. Univ. Trieste* **1976**, *8*, 69–72. [CrossRef]
14. Piri, H.; Rahrovi, S.; Kumam, P. Generalization of Khan fixed point theorem. *J. Math. Comput. Sci.* **2017**, *17*, 76–83. [CrossRef]
15. Piri, H.; Rahrovi, S.; Marasi, H.; Kumam, P. A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations. *J. Nonlinear Sci. Appl.* **2017**, *10*, 4564–4573. [CrossRef]
16. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
17. Tomar, A.; Sharma, R. Some coincidence and common fixed point theorems concerning F-contraction and applications. *J. Int. Math. Virtual Inst.* **2018**, *8*, 181–198.
18. Kadelburg, Z.; Radenovic, S. Notes on some recent papers concerning Fcontractions in b-metric spaces. *Constr. Math. Anal.* **2018**, *1*, 108–112.
19. Satish, S.; Stojan, R.; Zoran, K. Some fixed point theorems for F-generalized contractions in 0-orbitally complete partial metric spaces. *Theory Appl. Math. Comput. Sci.* **2014**, *4*, 87–98.
20. Lukacs, A.; Kajanto, S. Fixed point therorems for various types of F-contractions in complete b-metric spaces. *Fixed Point Theory* **2018**, *19*, 321–334 [CrossRef]