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On A Class of Para-Sakakian Manifolds

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Abstract

In this study, we investigate Weyl-pseudosymmetric Para-Sasakian manifolds and Para-Sasakian manifolds satisfying the condition $C \cdot S = 0$.

Key Words: Para-Sasakian manifold, Weyl-pseudosymmetric manifold.

1. Introduction

Let (M, g) be an n-dimensional, $n \geq 3$, differentiable manifold of class C^∞. We denote by ∇ its Levi-Civita connection. We define endomorphisms $\mathcal{R}(X,Y)$ and $X \wedge Y$ by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \tag{1}$$

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{2}$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on M. The Riemannian Christoffel curvature tensor R is defined by $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W)$, $W \in \chi(M)$. Let S and κ denote the Ricci tensor and the scalar curvature of M, respectively. The Ricci operator $\mathcal{S}$ and the (0,2)-tensor S^2 are defined by

$$g(\mathcal{S}X, Y) = S(X,Y), \tag{3}$$

and

$$S^2(X,Y) = S(\mathcal{S}X, Y). \tag{4}$$

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The *Weyl conformal curvature operator* $\mathcal{C}$ is defined by

$$\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y), \tag{5}$$

and the *Weyl conformal curvature tensor* C is defined by $C(X,Y,Z,W) = g(\mathcal{C}(X,Y)Z,W)$. If $C = 0$, $n \geq 4$, then M is called *conformally flat*.

For a $(0,k)$-tensor field T, $k \geq 1$, on (M,g) we define the tensors $R \cdot T$ and $Q(g,T)$ by

$$\begin{aligned} (R(X,Y) \cdot T)(X_1,...,X_k) &= -T(\mathcal{R}(X,Y)X_1, X_2,...,X_k) \\ &\quad -...-T(X_1,...,X_{k-1}, \mathcal{R}(X,Y)X_k), \end{aligned} \tag{6}$$

$$\begin{aligned} Q(g,T)(X_1,...,X_k;X,Y) &= -T((X \wedge Y)X_1, X_2,...,X_k) \\ &\quad -...-T(X_1,...,X_{k-1}, (X \wedge Y)X_k), \end{aligned} \tag{7}$$

respectively [8].

If the tensors $R \cdot C$ and $Q(g,C)$ are linearly dependent then M is called *Weyl-pseudosymmetric*. This is equivalent to

$$R \cdot C = L_C Q(g,C), \tag{8}$$

holding on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. If $R \cdot C = 0$ then M is called *Weyl-semisymmetric* (see [7], [9], [8]). If $\nabla C = 0$ then M is called *conformally symmetric* (see [4]). It is obvious that a conformally symmetric manifold is Weyl-semisymmetric.

Furthermore we define the tensor $C \cdot S$ on (M,g) by

$$(C(X,Y) \cdot S)(Z,W) = -S(\mathcal{C}(X,Y)Z,W) - S(Z,\mathcal{C}(X,Y)W). \tag{9}$$

In [1], T. Adati and K. Matsumoto defined para-Sasakian and special para-Sasakian manifolds which are considered as special cases of an almost paracontact manifold introduced by I. Satō [11]. In the same paper, the authors studied conformally symmetric para-Sasakian manifolds and they proved that an n-dimensional conformally symmetric para-Sasakian manifold is conformally flat and SP-Sasakian ($n > 3$). In [5], the authors studied Weyl-semisymmetric para-Sasakian manifolds and they showed that an n-dimensional Weyl-semisymmetric para-Sasakian manifold is conformally flat. In this study, our aim is to obtain the characterizations of the Weyl-pseudosymmetric para-Sasakian manifolds which are the extended class of Weyl-semisymmetric para-Sasakian manifolds and some further characterization of para-Sasakian manifolds satisfying the condition $C \cdot S = 0$.

2. Sasakian and Para-Sasakian Manifolds

Let M be a n-dimensional contact manifold with contact form η, i.e., $\eta \wedge (d\eta)^n \neq 0$. It is well known that a contact manifold admits a vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for every $X \in \chi(M)$. Moreover, M admits a Riemannian metric g and a tensor field ϕ of type (1,1) such that

$$\phi^2 = I - \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \quad g(X, \phi Y) = d\eta(X, Y).$$

We then say that (ϕ, ξ, η, g) is a contact metric structure. A contact metric manifold is said to be a *Sasakian* if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

in which case

$$\nabla_X \xi = -\phi X, \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Now we give a structure similar to Sasakian but not hawing contact.

An n-dimensional differentiable manifold M is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X),$$

$$\phi^2 X = X - \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X and Y on M. The equation $\eta(\xi) = 1$ is equivalent to $|\eta| \equiv 1$, and then ξ is just the metric dual of η. If (ϕ, ξ, η, g) satisfy the equations

$$d\eta = 0, \qquad \nabla_X \xi = \phi X,$$

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

then M is called a *Para-Sasakian* manifold or, briefly, a *P-Sasakian manifold*. Especially, a P-Sasakian manifold M is called a *special para-Sasakian manifold* or briefly a *SP-Sasakian manifold* if M admits a 1-form η satisfying

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y).$$

It is known that in a P-Sasakian manifold the following relations hold:

$$S(X, \xi) = (1 - n)\eta(X), \tag{10}$$

$$\eta(\mathcal{R}(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{11}$$

for any vector fields $X, Y, Z \in \chi(M)$, (see [2], [11] and [12]).

A para-Sasakian manifold M is said to be *η-Einstein* if

$$\mathcal{S} = aI_d + b\eta \otimes \xi, \tag{12}$$

where $\mathcal{S}$ is the Ricci operator and a, b are smooth functions on M [2].

3. Main Results

In the present section our aim is to find the characterization of P-Sasakian manifolds satisfying the conditions $C \cdot S = 0$ and $R \cdot C = L_C Q(g, C)$.

Firstly we give the following proposition.

Proposition 3.1 *Let M be an n-dimensional, $n \geq 4$, P-Sasakian manifold. If the condition $C \cdot S = 0$ holds on M then*

$$S^2(X, Y) = \left[\frac{\kappa}{n-1} - n + 2\right] S(X, Y) + [\kappa + n - 1]\, g(X, Y) \tag{13}$$

is satisfied on M.

Proof. Assume that M is an n-dimensional, $n \geq 4$, P-Sasakian manifold satisfying the condition $C \cdot S = 0$. From (9) we have

$$S(\mathcal{C}(U, X)Y, Z) + S(Y, \mathcal{C}(U, X)Z) = 0, \tag{14}$$

where $U, X, Y, Z \in \chi(M)$. Taking $U = \xi$ in (14) we have

$$S(\mathcal{C}(\xi, X)Y, Z) + S(Y, \mathcal{C}(\xi, X)Z) = 0. \tag{15}$$

So using (5), (10) and (11) we get

$$\begin{gathered}0=\eta(Y)S(X,Z)-g(X,Y)S(\xi,Z)+\eta(Z)S(X,Y)-g(X,Z)S(\xi,Y)\\ -\tfrac{1}{n-2}\{S(X,Y)S(\xi,Z)-S(\xi,Y)S(X,Z)+g(X,Y)S^{2}(\xi,Z)\\ -\eta(Y)S^{2}(X,Z)+S(X,Z)S(\xi,Y)-S(\xi,Z)S(X,Y)\\ +g(X,Z)S^{2}(\xi,Y)-\eta(Z)S^{2}(X,Y)\}+\tfrac{\kappa}{(n-1)(n-2)}\{g(X,Y)S(\xi,Z)\\ -\eta(Y)S(X,Z)+g(X,Z)S(\xi,Y)-\eta(Z)S(X,Y)\}.\end{gathered}$$

Hence by the use of (4), (10) we find

$$\begin{gathered}0=\eta(Y)S(X,Z)-(1-n)g(X,Y)\eta(Z)+\eta(Z)S(X,Y)\\ -(1-n)g(X,Z)\eta(Y)-\tfrac{1}{n-2}[-\eta(Y)S^{2}(X,Z)-\eta(Z)S^{2}(X,Y)\\ +(1-n)^{2}\eta(Z)g(X,Y)+(1-n)^{2}\eta(Y)g(X,Z)]\\ +\tfrac{\kappa}{(n-1)(n-2)}[-\eta(Y)S(X,Z)-\eta(Z)S(X,Y)\\ +(1-n)\eta(Z)g(X,Y)+(1-n)\eta(Y)g(X,Z)].\end{gathered} \tag{16}$$

Thus replacing Z with ξ in (16) and using (4), (10) we obtain

$$\begin{aligned}\frac{1}{n-2}S^{2}(X,Y) &= \left[\frac{\kappa}{(n-1)(n-2)}-1\right]S(X,Y)\\ &\quad+\left[\frac{\kappa}{n-2}+\frac{(n-1)^{2}}{n-2}-(n-1)\right]g(X,Y),\end{aligned}$$

since $n \geq 4$, we get (13). □

Let us consider an η-Einstein P-Sasakian manifold. Then we can write

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{17}$$

where X, Y are any vector fields and a, b are smooth functions on M.

Contracting (17), we have

$$\kappa = na + b. \tag{18}$$

On the other hand, putting $X = Y = \xi$ in (17) and using (10) we also have

$$1-n=a+b. \tag{19}$$

Hence it follows from (18) and (19) that

$$a = 1 - \tfrac{\kappa}{1-n} \quad , \quad b = \tfrac{\kappa}{1-n} - n.$$

So the Ricci tensor S of an η-Einstein P-Sasakian manifold is given by

$$S(Y,Z) = (1 - \frac{\kappa}{1-n})g(Y,Z) + (\frac{\kappa}{1-n} - n)\eta(Y)\eta(Z), \tag{20}$$

(For more details see [2]).

Proposition 3.2 *Let M be an n-dimensional, $n \geq 4$, η-Einstein P-Sasakian manifold. Then the condition $C \cdot S = 0$ holds on M.*

Proof. Let M be an η-Einstein P-Sasakian manifold. Since the Weyl tensor C has all symmetries of a curvature tensor, then from (9) it is easy to show that

$$(C(U,X) \cdot S)(Y,Z) = (\frac{\kappa}{n-1} + n)\left[\eta(C(U,X)Y)\eta(Z) + \eta(C(U,X)Z)\eta(Y)\right],$$

for all vector fields U, X, Y, Z on M. So using (5), (10), (11) and (20), by a straightforward calculation, we get $(C(U,X) \cdot S)(Y,Z) = 0$, which proves the proposition. □

Theorem 3.3 *Let M be an n-dimensional, $n \geq 4$, P-Sasakian manifold. If M is Weyl-pseudosymmetric then M is either conformally flat, in which case M is a SP-Sasakian manifold, or $L_C = -1$ holds on M.*

Proof. Assume that M, $(n \geq 4)$, is a Weyl pseudosymmetric P-Sasakian manifold and $X, Y, U, V, W \in \chi(M)$. So we have

$$(\mathcal{R}(X,Y) \cdot \mathcal{C})(U,V,W) = L_C Q(g, \mathcal{C})(U,V,W;X,Y).$$

Then from (6) and (7) we can write

$$\begin{gathered}\mathcal{R}(X,Y)\mathcal{C}(U,V)W - \mathcal{C}(\mathcal{R}(X,Y)U,V)W - \mathcal{C}(U,\mathcal{R}(X,Y)V)W \\ -\mathcal{C}(U,V)\mathcal{R}(X,Y)W = L_C[(X \wedge Y)\mathcal{C}(U,V)W - \mathcal{C}((X \wedge Y)U,V)W \\ -\mathcal{C}(U,(X \wedge Y)V)W - \mathcal{C}(U,V)(X \wedge Y)W].\end{gathered} \tag{21}$$

Therefore replacing X with ξ in (21) we have

$$\begin{aligned}\mathcal{R}(\xi,Y)\mathcal{C}(U,V)W-\mathcal{C}(\mathcal{R}(\xi,Y)U,V)W-\mathcal{C}(U,\mathcal{R}(\xi,Y)V)W\\ -\mathcal{C}(U,V)\mathcal{R}(\xi,Y)W=L_C[(\xi\wedge Y)\mathcal{C}(U,V)W-\mathcal{C}((\xi\wedge Y)U,V)W\\ -\mathcal{C}(U,(\xi\wedge Y)V)W-\mathcal{C}(U,V)(\xi\wedge Y)W].\end{aligned} \tag{22}$$

So using (11), (2) and taking the inner product of (22) with ξ we get

$$\begin{aligned}[1+L_C][-\eta(Y)\eta(\mathcal{C}(U,V)W)+C(U,V,W,Y)+\eta(U)\eta(\mathcal{C}(Y,V)W)\\ -g(Y,U)\eta(\mathcal{C}(\xi,V)W)+\eta(V)\eta(\mathcal{C}(U,Y)W)-g(Y,V)\eta(\mathcal{C}(U,\xi)W)\\ +\eta(W)\eta(\mathcal{C}(U,V)Y)-g(Y,W)\eta(\mathcal{C}(U,V)\xi)]=0.\end{aligned} \tag{23}$$

Putting $Y=U$ in (23) we have

$$\begin{aligned}[1+L_C][C(U,V,W,U)+\eta(W)\eta(\mathcal{C}(U,V)U)\\ -g(U,U)\eta(\mathcal{C}(\xi,V)W)-g(U,V)\eta(\mathcal{C}(U,\xi)W)]=0.\end{aligned} \tag{24}$$

So a contraction of (24) with respect to U gives us

$$[1+L_C]\eta(\mathcal{C}(\xi,V)W)=0. \tag{25}$$

If $L_C=0$ then M is Weyl-semisymmetric and so the equation (25) is reduced to

$$\eta(\mathcal{C}(\xi,V)W)=0, \tag{26}$$

which gives

$$S(V,W)=\left(1+\frac{\kappa}{n-1}\right)g(V,W)-\left(n+\frac{\kappa}{n-1}\right)\eta(V)\eta(W). \tag{27}$$

Therefore M is an η-Einstein manifold. So using (26) and (27) the equation (23) takes the form

$$C(U,V,W,Y)=0,$$

which means that M is conformally flat. So by [2], M is a SP-Sasakian manifold.

If $L_C\neq 0$ and $\eta(\mathcal{C}(\xi,V)W)\neq 0$ then $1+L_C=0$, which gives $L_C=-1$. This completes the proof of the theorem. □

So we have the following corollary.

Corollary 3.4 *Every n-dimensional* ($n\geq 4$) *para-Sasakian is a Weyl-pseudosymmetric manifold of the form* $R\cdot C=-Q(g,C)$.

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