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$\overline{\nabla}$–HARMONIC CURVES AND SURFACES IN EUCLIDEAN SPACE E^n

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ABSTRACT. In this study we consider $\overline{\nabla}$-harmonic curves and surfaces in Euclidean n-spaces $\mathbb{E}^n$.We proved that every weak biharmonic curve is $\overline{\nabla}$-harmonic.We also showed that every 1-parallel surface in $\mathbb{E}^4$ is $\overline{\nabla}$-harmonic, but the converse is not true. Finally we give the necessary condition for Vranceanu's surface to become $\overline{\nabla}$-harmonic.

1. INTRODUCTION

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. For all local formulas and computations, we may assume f as an imbedding and thus we shall often identify $p \in M$ with $f(p) \in \widetilde{M}$. The tangent space T_pM is identified with a subspace $f_*(T_pM)$ of $T_p\widetilde{M}$ where f_* is the differential map of f . Letters X, Y and Z (resp. ζ, μ and ξ) vector fields tangent (resp. normal) to M. We denote the tangent bundle of M (resp. $\widetilde{M}$) by TM (resp. $T\widetilde{M}$) , unit tangent bundle of M by UM and the normal bundle by $T^{\perp}M$. Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita connections of $\widetilde{M}$ and M , resp. Then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.1}$$

where h denotes the second fundamental form. The Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.2}$$

where A denotes the shape operator and D the normal connection. Clearly $h(X,Y) = h(Y,X)$ and A is related to h as $\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle$, where $\langle\ ,\rangle$ denotes the Riemannian metrics of M and $\widetilde{M}$ (see [3]).

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Let $\{e_1, e_2, ..., e_n, e_{n+1}, ..., e_m\}$ be an local orthonormal frame field on $\widetilde{M}$ where $\{e_1, e_2, ..., e_n\}$ are tangent vector and $\{e_{n+1}, ..., e_m\}$ are normal vector. The connection form w_i^j are defined by

$$\widetilde{\nabla}_{e_i} = \sum w_i^j e_j \ ; \ w_i^j = -w_j^i \ , 1 \leq i,j \leq m \tag{1.3}$$

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} w_j^k(e_i) e_k \ , \tag{1.4}$$

$$D_{e_i} e_\alpha = \sum_{\beta=n+1}^{m} w_\alpha^\beta(e_i) e_\beta \tag{1.5}$$

The covariant derivations of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{1.6}$$

where X, Y, Z tangent vector fields over M and $\overline{\nabla}$ is the van der Waerden Bortolotti connection. Then we have

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) = (\overline{\nabla}_Z h)(Y,X)$$

which is called *codazzi equations.*

If $\overline{\nabla} h = 0$ then M is said to have parallel second fundamental form (i.e. *1-parallel)* (see [7]).

It is well known that $\overline{\nabla} h$ is a normal bundle valued tensor of type $(0,3)$ .We define the second covariant derivative of h by

$$\begin{aligned}(\overline{\nabla}_W \overline{\nabla}_X h)(Y,Z) &= D_W((\overline{\nabla}_X h)(Y,Z)) - (\overline{\nabla}_X h)(\nabla_W Y, Z) \\ &\quad -(\overline{\nabla}_X h)(Y, \nabla_W Z) - (\overline{\nabla}_{\nabla_W X} h)(Y,Z).\end{aligned} \tag{1.7}$$

If $\overline{\nabla}^2 h = 0$ then M is said to have parallel third fundamental form (i.e. *2-parallel)* [1].

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. For the orthonormal frame $\{e_1, ..., e_n\}$ of T_pM the mean curvature vector H of f is defined by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{1.8}$$

The Laplacian of H associated with D is defined by

$$\Delta^D H = \sum_{i=1}^{n} (D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H) \tag{1.9}$$

where D is the normal connection of M (see [5]).

If $\Delta^D H = 0$ then M is called *D-Harmonic* (or *weak biharmonic*). If $\Delta^D H + cH = 0$ then M is called *harmonic 1-type* (see[6]).

We give the following definition

Definition 1.1. The Laplacian of H associated with $\overline{\nabla}$ is defined by

$$\Delta^{\overline{\nabla}} H = \sum_{i=1}^{n} (\overline{\nabla}_{\nabla_{e_i} e_i} H - \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} H) \tag{1.10}$$

where $\overline{\nabla}$ is the van der Waerden Bortolotti connection of M defined by (1.6). If $\Delta^{\overline{\nabla}} H = 0$ then M is called $\overline{\nabla}$-harmonic.

2. $\overline{\nabla}$-Harmonic Curves

Consider an immersed curve $\beta = \beta(s) : I \subset IR \to \mathbb{E}^m$ where s denotes the arclength parameter of β.

$$T = T(s) = \beta'(s)$$

will be the unit tangent vector field of β. Assume that β is not a plane curve (it is not contained in any 2-plane of $\mathbb{E}^m$). So we can define a 2-dimensional subbundle say ν of the normal bundle Λ of β into $\mathbb{E}^m$as

$$\nu(s) = span\{\xi_2, \xi_3\}(s) \tag{2.1}$$

where ξ_2, ξ_3 are unit normal vector fields to β defined by

$$T'(s) = k_1(s)\xi_2(s)$$
$$\xi_2'(s) = -k_1(s)T(s) + k_2(s)\xi_3(s)$$

where $k_1 > 0$ is the curvature (the first curvature if $m > 3$) and k_2 is the torsion (the second curvature with $\tau > 0$ if $m > 3$) of β.

Denote by $\nu^\perp$ the orthogonal complementary subbundle of ν in Λ. Certainly the fibers of $\nu^\perp$ have dimension $m-3$. Therefore the Frenet equations of β can be written as

$$T'(s) = k_1(s)\xi_2(s) \tag{2.2}$$
$$\xi_2'(s) = -k_1(s)T(s) + k_2(s)\xi_3(s) \tag{2.3}$$
$$\xi_3'(s) = -k_2(s)\xi_2(s) + \delta(s) \tag{2.4}$$

where $\delta(s) \in \nu^\perp(s)$, $\delta(s) = k_3(s)\xi_4(s)$ for all $s \in I$.

The curvature vector field of β (the mean curvature vector field of β) is defined by

$$H(s) = T'(s) = k_1(s)\xi_2(s) = h(T,T), \ \nabla_T T = 0 \tag{2.5}$$

Equations (2.3) and (2.4) also give how the normal connection D of β into $\mathbb{E}^m$ behaves on ν

$$D_T\xi_2 = k_2(s)\xi_3(s) \tag{2.6}$$
$$D_T\xi_3 = -k_2(s)\xi_2(s) + \delta(s). \tag{2.7}$$

Let Δ^D be the Laplacian associated with D. One can use the Frenet equations (2.6) and (2.7) to compute $\Delta^D H$ and so one obtains

$$\Delta^D H = (-\kappa_1'' + \kappa_1\kappa_2^2)v_2 + (-2\kappa_1'\kappa_2 - \kappa_1\kappa_2')v_3 - \kappa_1\kappa_2\kappa_3 v_4. \tag{2.8}$$

In [5] it has shown that any immersed curve $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^m$ with the mean curvature vector satisfying $\Delta H = 0$, is a straight line. Recently, in [2] the authors gave a full classification of the immersed curves in an Euclidean space $\mathbb{E}^m$ with the mean curvature vector satisfying $\Delta^D H = 0$.

In [6] we give the following results.

Proposition 1. *Let γ be a Frenet curve of harmonic one type (i.e. $\Delta^D H + cH = 0$) if and only if*

$$-\kappa_1'' + \kappa_1\kappa_2^2 + c\kappa_1 = 0, \ 2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0, \ \kappa_1\kappa_2\kappa_3 = 0. \tag{2.9}$$

By virtue of Proposition 1 one can get the following result.

Corollary 1. *Let γ be a harmonic 1-type curve*
i) If $\kappa_1 = 0$ then γ is a straight line.
ii) If $\kappa_1, \kappa_2 \neq 0, \kappa_3 = 0$ then

$$\kappa_1(s) = \mp \frac{\sqrt{c}}{(4c_1)^{1/4}} \sqrt{\frac{e^{4s-2c_2}+1}{e^{2s-c_2}}} \text{ and } \kappa_2(s) = 2\sqrt{c_1}\left(\frac{e^{2s-c_2}}{e^{4x-2c_2}+1}\right). \tag{2.10}$$

Corollary 2. *Let plane curve γ be a harmonic 1-type curve. Then* $\kappa_1'' \pm c\kappa_1 = 0$. *That is*
a) $\kappa_1 = b_1 Cos(\sqrt{c}s) + b_2\ Sin(\sqrt{c}s)$ for $\kappa_1'' + c\kappa_1 = 0$,
b) $\kappa_1 = b_1 e^{\sqrt{c}s} + b_2 e^{-\sqrt{c}s}$, for $\kappa_1'' - c\kappa_1 = 0$.

Corollary 3. *Every weak biharmonic curve are $\overline{\nabla}$−harmonic.*

Proof. Let $\beta = M$ be a space curve of $\mathbb{E}^m$ with arclength parameter. Then

$T = T(s) = \beta'(s)$ and

$$\beta''(s) = \nabla_T T + h(T,T) = k_1(s)\xi_2(s)$$

which implies that $\nabla_T T = 0$. Therefore the equation (1.6) reduce to

$$(\overline{\nabla}_T h)(T,T) = D_T h(T,T).$$

So the equation (1.9) and (1.10) are equal (i.e. $\Delta^D H = \Delta^{\overline{\nabla}} H$). This complete the proof of the result.

Corollary 4. *Every $\overline{\nabla}$-harmonic curve β is 2-parallel.*

Proof. Let β be a smooth curve in $\mathbb{E}^m$ with arclength parameter. Then differentiating $T = \beta'(s)$ we get

$$H(s) = \beta''(s) = h(T,T)$$

and

$$(\overline{\nabla}_T h)(T,T) = D_T h(T,T)$$

and

$$(\overline{\nabla}_T \overline{\nabla}_T h)(T,T) = D_T D_T h(T,T).$$

Therefore $\Delta^{\overline{\nabla}} H = (\overline{\nabla}_T \overline{\nabla}_T h)(T,T)$. So $\Delta^{\overline{\nabla}} H = 0$ implies that $\overline{\nabla}^2 h = 0$ (i.e. β is 2-parallel curve). The converse of this corollary is also true.

3. $\overline{\nabla}$-Harmonic Surfaces

Let M be a surfaces in $\mathbb{E}^{2+d}$ then the equation (1.10) reduces to

$$\Delta^{\overline{\nabla}} H = \overline{\nabla}_{\nabla_{e_1} e_1} H + \overline{\nabla}_{\nabla_{e_2} e_2} H - \overline{\nabla}_{e_1} \overline{\nabla}_{e_1} H - \overline{\nabla}_{e_2} \overline{\nabla}_{e_2} H. \tag{3.1}$$

In the present section we will consider $\overline{\nabla}$-*harmonic* surfaces M in $\mathbb{E}^{2+d}$. First, we give the following result.

Proposition 2. *Every surface in $\mathbb{E}^3$ is $\overline{\nabla}$-harmonic.*

Proof. Let $\{e_1, e_2\}$ be an orthonormal frame field of T_pM. Then we have

$$\begin{aligned} \nabla_{e_1} e_1 &= \lambda_1 e_2 \\ \nabla_{e_2} e_2 &= -\lambda_2 e_1 \\ \nabla_{e_1} e_2 &= -\lambda_1 e_1 \\ \nabla_{e_2} e_1 &= \lambda_2 e_2 \end{aligned} \tag{3.2}$$

Substituting (1.6-1.8) and (3.2) into (3.1) after some calculations we get $\Delta^{\overline{\nabla}} H = 0$.This completes the proof of the proposition.

Theorem 3.1. *Let $M \subset \mathbb{E}^{2+d}$ be smooth surfaces in $\mathbb{E}^{2+d}$. Then*

$$\Delta^{\overline{\nabla}} H = \Delta^D H + \frac{1}{2}\sum_{i=1}^{2} D_{e_i} D_{e_i} H$$

where $\{e_1, e_2\}$ is the orthonormal frame field of T_pM and H is the mean curvature vector of M.

Proof. Let $\{e_1, e_2\}$ be a orthonormal frame field of T_pM. By (1.10) we get

$$\Delta^{\overline{\nabla}} H = \sum_{i=1}^{2} (\overline{\nabla}_{\nabla_{e_i} e_i} H - \overline{\nabla}_{e_i} \overline{\nabla}_{e_i} H). \tag{3.3}$$

Substituting $H = \frac{1}{2}\sum\limits_{i=1}^{2} h(e_i, e_i)$ into (3.3) we obtain

$$\begin{aligned} 2\Delta^{\overline{\nabla}} H \;\; = \;\; & (\overline{\nabla}_{\nabla_{e_1} e_1} h)(e_1, e_1) + (\overline{\nabla}_{\nabla_{e_1} e_1} h)(e_2, e_2) + (\overline{\nabla}_{\nabla_{e_2} e_2} h)(e_1, e_1) + \\ & + (\overline{\nabla}_{\nabla_{e_2} e_2} h)(e_2, e_2) - (\overline{\nabla}_{e_1} \overline{\nabla}_{e_1} h)(e_1, e_1) - (\overline{\nabla}_{e_1} \overline{\nabla}_{e_1} h)(e_2, e_2) - \\ & - (\overline{\nabla}_{e_2} \overline{\nabla}_{e_2} h)(e_1, e_1) - (\overline{\nabla}_{e_2} \overline{\nabla}_{e_2} h)(e_2, e_2). \end{aligned} \tag{3.4}$$

Substituting (1.4) , (1.5) and (3.2) into (3.4) and using (1.9) we get

$$2\Delta^{\overline{\nabla}} H = \Delta^D H + \sum_{i=1}^{2} D_{\nabla_{e_i} e_i} H \tag{3.5}$$

or similarly

$$\begin{aligned} 2\Delta^{\overline{\nabla}} H \;\; = \;\; & D_{\nabla_{e_1} e_1} H - D_{e_1} D_{e_1} H + D_{\nabla_{e_2} e_2} H \\ & - D_{e_2} D_{e_2} H + D_{\nabla_{e_1} e_1} H + D_{\nabla_{e_2} e_2} H. \end{aligned} \tag{3.6}$$

Adding and subtracting the terms $D_{e_1} D_{e_1} H$ and $D_{e_2} D_{e_2} H$ into the equations (3.6) we get

$$-2\Delta^{\overline{\nabla}} H + 2\Delta^D H + \sum_{i=1}^{2} D_{e_i} D_{e_i} H = 0.$$

This completes the proof of the theorem.□

Proposition 3. [4] *Let M be a connected normally flat surfaces in $\mathbb{E}^5$.e_3 is parallel to the mean curvature vector H of M such that*

$$A_{e_3} = \begin{bmatrix} \lambda & 0 \\ 0 & \eta \end{bmatrix}, \; A_{e_4} = \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}, \; A_{e_5} = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}. \tag{3.7}$$

Using (3.3), (3.7), (1.6), (1.7) and codazzi equations we get

$$\begin{aligned}\Delta^{\overline{\nabla}}H &= \{-e_1e_1(\lambda+\eta)-e_2e_2(\lambda+\eta)+2e_2(\lambda+\eta)w_1^2(e_1)-2e_1(\lambda+\eta)w_1^2(e_2)\\ &+(\lambda+\eta)[(w_3^4(e_1))^2+(w_3^4(e_2))^2+(w_3^5(e_1))^2+(w_3^5(e_2))^2]\}e_3\\ &+\{2w_3^4(e_2)[w_1^2(e_1)(\lambda+\eta)-e_2(\lambda+\eta)]-2e_1(\lambda+\eta)w_3^4(e_1)\\ &-2w_1^2(e_2)[w_3^4(e_1)(\lambda+2\eta)-2\beta w_1^2(e_2)-e_1(\beta)]\\ &+(\lambda+\eta)[-e_1(w_3^4(e_1))-e_2(w_3^4(e_2))-w_3^5(e_1)w_5^4(e_1)\\ &-w_3^5(e_2)w_5^4(e_2)]\}e_4+\{2w_3^5(e_2)[w_1^2(e_1)(\lambda+\eta)-e_2(\lambda+\eta)]\\ &-2w_3^5(e_1)[w_1^2(e_2)(\lambda+\eta)+e_1(\lambda+\eta)]\\ &+(\lambda+\eta)[-e_1(w_3^5(e_1))-e_2(w_3^5(e_2))-w_3^4(e_1)w_4^5(e_1)-w_3^4(e_2)w_4^5(e_2)]\}e_5.\end{aligned} \tag{3.8}$$

Substuting (3.8) into (1.10) we get the following result.

Proposition 4. *Let M be a connected normally flat surfaces in $\mathbb{E}^5$ with e_3 is parallel to the mean curvature vector H of M. If M is $\overline{\nabla}$-harmonic surfaces then*

$$\begin{aligned}0 &= -e_1e_1(\lambda+\eta)-e_2e_2(\lambda+\eta)+2e_2(\lambda+\eta)w_1^2(e_1)-2e_1(\lambda+\eta)w_1^2(e_2)\\ &+(\lambda+\eta)[(w_3^4(e_1))^2+(w_3^4(e_2))^2+(w_3^5(e_1))^2+(w_3^5(e_2))^2],\end{aligned}$$

$$\begin{aligned}0 &= 2w_3^4(e_2)[w_1^2(e_1)(\lambda+\eta)-e_2(\lambda+\eta)]-2e_1(\lambda+\eta)w_3^4(e_1)\\ &-2w_1^2(e_2)[w_3^4(e_1)(\lambda+2\eta)-2\beta w_1^2(e_2)-e_1(\beta)]\\ &+(\lambda+\eta)[-e_1(w_3^4(e_1))-e_2(w_3^4(e_2))-w_3^5(e_1)w_5^4(e_1)-w_3^5(e_2)w_5^4(e_2)],\end{aligned}$$

$$\begin{aligned}0 &= 2w_3^5(e_2)[w_1^2(e_1)(\lambda+\eta)-e_2(\lambda+\eta)]\\ &-2w_3^5(e_1)[w_1^2(e_2)(\lambda+\eta)+e_1(\lambda+\eta)]\\ &+(\lambda+\eta)[-e_1(w_3^5(e_1))-e_2(w_3^5(e_2))-w_3^4(e_1)w_4^5(e_1)-w_3^4(e_2)w_4^5(e_2)].\end{aligned}$$

Example 3.2. We give some examples;

1) The torus $\mathbb{T}^2$ embedded in $\mathbb{E}^4$ by

$$\mathbb{T}^2=\{(\cos\theta,\sin\theta,\cos\varphi,\sin\varphi):\theta,\varphi\in IR\}$$

is $\overline{\nabla}$-*harmonic.*

2) The helical cylinder $\mathbb{H}^2$ embedded in $\mathbb{E}^4$ by

$$\mathbb{H}^2=\{(\theta,c\cos\varphi,c\sin\varphi,d\varphi):\theta,\varphi\in IR\}$$

is $\overline{\nabla}$-*harmonic.*

3) The Klein Bottle $\mathbb{K}^2$ embeded in $\mathbb{E}^4$ by

$$\mathbb{K}^2=\{(\cos\theta\cos\varphi,\sin\theta\cos\varphi,\cos 2\theta\sin\varphi,\sin 2\theta\sin\varphi):\theta,\varphi\in IR\}$$

is $\overline{\nabla}$-*harmonic.*

4)Möbius band $\mathbb{M}^2$ embedded in $\mathbb{E}^4$ by

$$\mathbb{M}^2=\{(\cos\theta,\sin\theta,\varphi\cos\frac{\theta}{2},\varphi\sin\frac{\theta}{2}):\theta,\varphi\in IR\}$$

has

$$\begin{aligned}\Delta^{\overline{\nabla}}H &= \left\{e_1^2(\frac{1}{4+\varphi^2})+e_2^2(\frac{1}{4+\varphi^2})+\frac{4\varphi}{4+\varphi^2}e_2(\frac{1}{4+\varphi^2})\right\}e_3\\ &\left\{\frac{3\varphi}{4+\varphi^2}e_1(\frac{1}{4+\varphi^2})+\frac{2}{4+\varphi^2}e_1(\frac{\varphi}{4+\varphi^2})\right\}e_4.\end{aligned}$$

Proposition 5. [9]*Let $f : M \to \mathbb{E}^n$ be isometric immersion. If M is 1-parallel (i.e. $\overline{\nabla}h = 0$) then $f(M)$ is one of the following surfaces*

i)$\mathbb{E}^2$
ii) $S^2 \subset \mathbb{E}^3$
iii) $IR^1 \times S^1 \subset \mathbb{E}^3$
iv) $S^1(a) \times S^1(b) \subset \mathbb{E}^4$
v) $V^2 \subset \mathbb{E}^5$.

Comparing above proposition with the Examples we have the following result.

Corollary 5. *Every 1-parallel surface in $\mathbb{E}^4$ is $\overline{\nabla}$-harmonic. But the converse is not true.*

Proposition 6. *Vranceanu surfaces is given by*

$$x(s,t) = (u(s)\cos s\cos t, u(s)\cos s\sin t, u(s)\sin s\cos t, u(s)\sin s\sin t)$$

is $\overline{\nabla}$-harmonic surfaces if and only if the equation

$$\alpha_s A(-4\alpha\kappa A_s - 1) + \beta_s A(-2\alpha\kappa A_s - 1) - A_s(\alpha_{ss} + \beta_{ss}) - 3\kappa^2 A_s(\alpha - \beta) = 0 \quad (3.9)$$

is satisfied, where $u = u(s)$ is a smooth function and

$$\alpha = \frac{1}{A}, A = \sqrt{u^2 + (u')^2}, \kappa = \frac{u'}{u}, \beta = \frac{2(u')^2 - uu'' + u^2}{(u^2 + (u')^2)^{\frac{3}{2}}}.$$

Proof. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as given by the following

$$e_1 = (-\cos s\sin t, \cos s\cos t, -\sin s\sin t, \sin s\cos t)$$
$$e_2 = \frac{1}{A}(B\cos t, B\sin t, C\cos t, C\sin t)$$
$$e_3 = \frac{1}{A}(-C\cos t, -C\sin t, B\cos t, B\sin t)$$
$$e_4 = (-\sin s\sin t, \sin s\cos t, \cos s\sin t, -\cos s\cos t)$$

where we put $A = \sqrt{u^2 + (u')^2}$, $B = u'\cos s - u\sin s$, $C = u'\sin s + u\cos s$.Then we have

$$e_1 = \frac{1}{u}\frac{\partial}{\partial t},\ e_2 = \frac{1}{A}\frac{\partial}{\partial s}.$$

Using (1.1) we get

$$\begin{aligned} \nabla_{e_1} e_1 &= -\alpha\kappa e_2, \\ \nabla_{e_1} e_2 &= \alpha\kappa e_1, \\ \nabla_{e_2} e_1 &= 0, \nabla_{e_2} e_2 = 0 \end{aligned} \quad (3.10)$$

$$h(e_1, e_1) = \alpha e_3,\ h(e_2, e_2) = \beta e_3,\ h(e_1, e_2) = -\alpha e_4. \quad (3.11)$$

Substituting (1.4) , (1.5), (3.10) and (3.11) into (1.10) we get the result. □

ÖZET: Bu çalışmada, n-boyutlu E^n Öklid uzayında $\overline{\nabla}$-harmonik eğriler ve yüzeyler gözönünde bulunduruldu. Her zayıf biharmonik eğrinin $\overline{\nabla}$-harmonik olduğu ispatlandı. E^4 deki her 1-paralel yüzeyin $\overline{\nabla}$-harmonik olduğu fakat tersinin doğru olmadığı gösterildi. Sonuçta, Vranceanu yüzeyinin $\overline{\nabla}$-harmonik olması için gerekli koşul verildi.

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