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Abstract

In [2], Baik-Howie-Pride defined a set of the generating pictures of $\pi_2(P)$ where P is a presentation of a graph product of the vertex groups. In this paper, as an application of this, we discuss necessary and sufficient conditions for the presentation P to be p-Cockcroft, where p is a prime or 0. In addition we examine some special cases of this result.

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1. Introduction

Let

$$P = < \mathbf{x} \, ; \mathbf{r} > \qquad (1)$$

be a group presentation. Let F denotes the free group on $\mathbf{x}$ and let N denotes the normal closure of $\mathbf{r}$ in F. The quotient $G = F/N$ is the group defined by P.

If we regard P as a 2-complex with one 0-cell, a 1-cell for each $x \in \mathbf{x}$, and a 2-cell for each $R \in \mathbf{r}$ in the standart way, then G is just the fundamental group of P. We then define the second homotopy group $\pi_2(P)$ of P, which is a left $\mathbb{Z}G$-module. The elements of $\pi_2(P)$ can be represented by geometric configurations called *spherical pictures*. These are described in detail in [20] and we refer the reader these for details. Moreover, by [20], there are certain operations on spherical pictures.

Suppose $\mathbf{X}$ is a collection of spherical pictures over P. Then, by [20], one can define the additional operation on spherical pictures. Allowing this additional operation leads to the notion of *eqivalence* (rel $\mathbf{X}$) of spherical pictures. Then, by [20], *the elements* $<\mathbb{P}>$ ($\mathbb{P}\in\mathbf{X}$) *generate* $\pi_2(P)$ *as a module if and only if every spherical picture is equivalent (rel* $\mathbf{X}$) *to the empty picture.* If the elements $<\mathbb{P}>$ ($\mathbb{P}\in\mathbf{X}$) generate $\pi_2(P)$ then we say that $\mathbf{X}$ *generates* $\pi_2(P)$.

For any picture $\mathbb{P}$ over P and for any $R\in\mathbf{r}$, the *exponent sum* of R in $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled by R, minus the number of discs labelled by R^{-1}. It is clear that if pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, then $exp_R(\mathbb{P}_1) = exp_R(\mathbb{P}_2)$ for all $R \in \mathbf{r}$.

Definition 1.1. Let *P be as in* (1), *and let n be a non-negative integer. Then P is said to be n*-Cockcroft *if* $exp_R(\mathbb{P}) \equiv 0$ (*mod n*), (*where congruence* (mod 0) *is taken to be equality*) *for all* $R\in\mathbf{r}$ *and all spherical pictures* $\mathbb{P}$ *over P. A group G is said to be n-Cockcroft if it admits an n-Cockcroft presentation. The n = 0 case is usually just called Cockcroft.*

The reader can find some examples and details about Cockcroft property, for example, in [11], [13], [14], [17] and [19] and about *p*-Cockcroft property, for example, in [8] and [19].

We remark that to verify the *n*-Cockcroft property holds, it is enough to check for pictures $\mathbb{P}\in\mathbf{X}$, where $\mathbf{X}$ is a set of generating pictures.

A *graph* Γ consist of two disjoint set $\mathbf{v} = \mathbf{v}(\Gamma)$ (vertices) and $\mathbf{e} = \mathbf{e}(\Gamma)$ (edges) and three functions

$$\iota : \mathbf{e} \to \mathbf{v}, \qquad \tau : \mathbf{e} \to \mathbf{v} \qquad \text{and} \qquad ^{-1} : \mathbf{e} \to \mathbf{e}$$

satisfying: $\iota(e) = \tau(e^{-1})$, $(e^{-1})^{-1} = e$, $e^{-1} \neq e$ for all $e \in \mathbf{e}$. We call $\iota(e)$ and $\tau(e)$ the *initial* and *terminal* point of $e \in \mathbf{e}$, respectively. An orientation $\mathbf{e}^+$ of Γ consists of a choice of exactly one edge from edge pair e, e^{-1} ($e \in \mathbf{e}$). We will call to pair ($\mathbf{v}, \mathbf{e}^+$) with the functions ι, τ as an *oriented graph* with oriented edge set $\mathbf{e}^+$. A graph Γ is called *simple* if whenever $\iota(e_1) = \iota(e_2)$ and $\tau(e_1) = \tau(e_2)$ then $e_1 = e_2$ for all $e_1, e_2 \in \mathbf{e}$. A simple graph Γ is called *complete* if for any two distinct vertices u and v, there is an edge e with $\iota(e) = u$, $\tau(e) = v$. The details and applications of these can be found, for instance in [3].

1.1. Graph Product

Let Γ be a simple oriented graph with a vertex set $\mathbf{v}$ and edge $\mathbf{e}$ (thus $\mathbf{e}$ is a collection of 2-element subsets of $\mathbf{v}$). For each $v \in \mathbf{v}$, let G_v be a *vertex group* given by a presentation $P_v = <\mathbf{x}_v; \mathbf{s}_v>$ where the elements of $\mathbf{s}_v$ are cyclically reduced words on $\mathbf{x}_v$. For each $e \in \mathbf{e}$ with $\iota(e) = u$ and $\tau(e) = v$, let G_e be an *edge group* given by a presentation $P_e = <\mathbf{x}_u, \mathbf{x}_v ; \mathbf{s}_u, \mathbf{s}_v, \mathbf{r}_e>$ where the elements of $\mathbf{r}_e$ are cyclically reduced words on $\mathbf{x}_u \cup \mathbf{x}_v$ each involving at least one $\mathbf{x}_u$-symbol and $\mathbf{x}_v$- symbol and each $\mathbf{r}_e$ ($e \in \mathbf{e}$) consists of all words $[x,y] = xyx^{-1}y^{-1}$ ($x \in \mathbf{x}_{\iota(e)}$, $y \in \mathbf{x}_{\tau(e)}$).

Let

$$P = < \mathbf{x} ; \mathbf{s}, \mathbf{r} > \qquad (2)$$

be a presentation where $\mathbf{x} = \bigcup_{v \in \mathbf{v}} \mathbf{x}_v$, $\mathbf{s} = \bigcup_{v \in \mathbf{v}} \mathbf{s}_v$, $\mathbf{r} = \bigcup_{e \in \mathbf{e}} \mathbf{r}_e$. The group $G = G(\Gamma)$ defined by P is called a *graph product* of the vertex groups G_v for all $v \in \mathbf{v}$ ([2], [5], [15], [16]).

A graph product has two extreme cases:

- If the edge set $\mathbf{e}$ is empty then G is the *free product* of the groups G_v ($v \in \mathbf{v}$).
- If Γ is complete and each G_v is finite then G is the *direct product* of the groups G_v ($v \in \mathbf{v}$).

If all the vertex groups G_v ($v \in \mathbf{v}$) are infinite cyclic then G is called a *graph group* (see [2], [9], [10], [22], [23]).

The main result of this paper is the following:

Theorem 1.2. (Main Theorem) *Let p be a prime or* 0 *and let P be a presentation as in* (2). *Then P is p-Cockcroft if and only if*

i) each P_v ($v \in \mathbf{v}$) *is p-Cockcroft,*

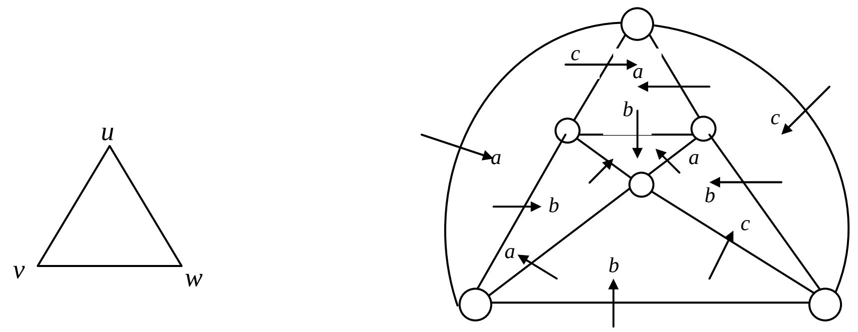
ii) for each $v \in \mathbf{v}$, $exp_x(S) \equiv 0 \pmod p$ *where* $x \in \mathbf{x}_v$, $S \in \mathbf{s}_v$.

2. Preliminaries

In this section we exhibit the generating pictures of $\pi_2(P)$ where P is a presentation as in (2), in order to prove the main theorem. *We may refer to* [2] *for the details of this material.*

Let Γ be a oriented graph. For each triangle $\{u, v, w\}$ (that is a 3-element subset of **v** for which $\{u, v\}$, $\{v, w\}$, $\{w, u\}$ $\in$ **e**) in Γ (see Figure 1-(*a*)), we have a collection of spherical pictures of the form depicted in Figure 1-(*b*) where $a \in \mathbf{x}_u$, $b \in \mathbf{x}_v$, $c \in \mathbf{x}_w$. Let **Z** be the union of all these collections over all triangles of Γ.

For each $e \in \mathbf{e}$ with $\iota(e) = u$, $\tau(e) = v$, let $S = x_1x_2 .....x_n$ ($x_i \in \mathbf{x}_u$, $i = 1, 2, ...,$ n where $n \in \mathbb{Z}^+$) be a relator in $\mathbf{s}_u$. Then for each $y \in \mathbf{x}_v$, we have a spherical picture $\mathbb{P}_{S,y}$ over P of the form as depicted in Figure 2-(*a*).



(a) *(b)*

FIGURE 1

Similarly, for each $x \in \mathbf{x}_u$, we get a spherical picture $\mathbb{P}_{T,x}$ over P where $T = y_1y_2 .....y_m$ ($y_i \in \mathbf{x}_v$, j = 1,2,..., m where $m \in \mathbb{Z}^+$) is a relator in $\mathbf{s}_v$ (Figure 2-(*b*)).

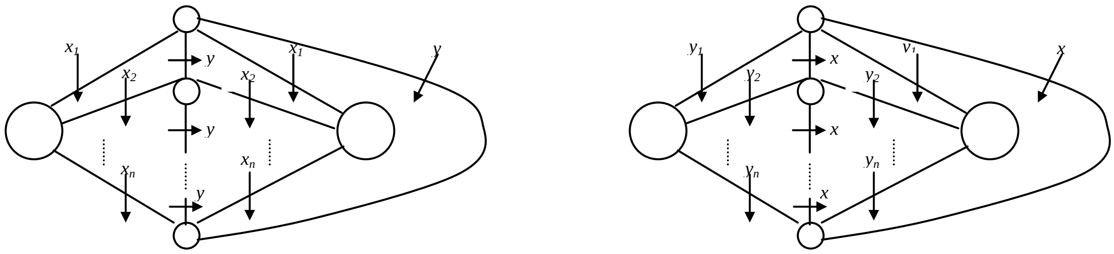
Let $\mathrm{Y}_{e,u} = \{\mathbb{P}_{S,y} : S \in \mathbf{s}_u, y \in \mathbf{x}_v\}$ and $\mathrm{Y}_{e,v} = \{\mathbb{P}_{T,x} : T \in \mathbf{s}_v, x \in \mathbf{x}_u\}$ be the sets of these spherical pictures. Also for each $e \in \mathbf{e}$ in Γ, let us define $Y_e = Y_{e,u} \cup Y_{e,v}$ and $\mathbf{Y} = \bigcup_{e \in \mathbf{e}} Y_e$.

Let X_v be a collection of spherical pictures over P_v such that $\pi_2(P_v)$ is generated by X_v and let $\mathbf{X} = \bigcup_{v \in \mathbf{v}} X_v$.

The proof of the following result can be found in [2] and [3].

Theorem 2.1. *Let P be a presentation as in* (2). *Then $\pi_2(P)$ is generated by* $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$.

$\mathbb{P}_{S,y}$ $\mathbb{P}_{T,x}$



(a) *(b)*

FIGURE 2

3. Proof of the Main Theorem

Throughout this section the notations will be the same as in the previous ones.

Let p be a prime or 0. In this part of the proof, let us suppose that the presentation P, as given in (2), is p-Cockcroft for any prime p. By Theorem 2.1, since $\pi_2(P)$ is generated by $\mathbf{X} \cup \mathbf{Y} \cup \mathbf{Z}$ and, by the assumption, since the exponent sum of the discs of each spherical picture defined in the sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ is equivalent to zero by *mod p* then, by Definition 1.1, this gives that each P_v ($v \in \mathbf{v}$) is p-Cockcroft and then since each of the relators S is defined in the presentation P_v , so we get

$$exp_x(S) \equiv 0 \ (mod\ p),$$

for all $x \in \mathbf{x}_v$, $S \in \mathbf{s}_v$, as required.

For the other part of the proof let us assume that conditions *i)* and *ii)* hold for the presentation P. By Theorem 2.1, for a presentation P as in (2), since $\pi_2(P)$ is generated by the union of the sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ then we need to calculate the exponent sum of the pictures in the sets $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ under these assumptions. Now by the definition of $\mathbf{X}$, the exponent sum of the discs $\mathbf{s}_v$ ($v \in \mathbf{v}$) of each spherical picture is equivalent to zero by *mod p*. Additionaly, since the exponent sum of the discs $\mathbf{r}_e$ ($e \in \mathbf{e}$) of the spherical pictures in the set $\mathbf{Z}$ is equal to zero (see Figure 1-(b)) then it is enough to check that the exponent sum of the discs of spherical pictures in the set $\mathbf{Y}$.

By Figure 2, the spherical pictures $\mathbb{P}_{S,y}$ and $\mathbb{P}_{T,x}$ ($S \in \mathbf{s}_u$, $T \in \mathbf{s}_v$, $y \in \mathbf{x}_v$, $x \in \mathbf{x}_u$) in the set **Y** consist of the discs S, T, $[y,x_i]$ ($x_i \in \mathbf{x}_u$, $i = 1, 2, \dots, n$ where $n \in \mathbb{Z}^+$), $[x,y_j]$ ($y_j \in \mathbf{x}_v$, $j = 1, 2, \dots, m$ where $m \in \mathbb{Z}^+$). Since the conditions *i), ii)* hold for P and each of the relators S, T is defined in the presentation P_v then we have

$$exp_{x_i}(S) \equiv 0 \ (mod\ p), \quad \forall x_i \in \mathbf{x_u} \ \text{ for } \ i = 1,2, \dots, n$$

and

$$exp_{y_j}(T) \equiv 0 \ (mod\ p), \quad \forall y_j \in \mathbf{x_v} \ \text{ for } \ j = 1,2, \dots, m.$$

Also it is easy to see that

$$\exp_{[y,x_i]}(\mathbb{P}_{S,y}) = \exp_{x_i}(S) \quad \text{and} \quad \exp_{[x,y_j]}(\mathbb{P}_{T,x}) = \exp_{y_j}(T)$$

where $\forall x_i \in \mathbf{x}_u$, $\forall y_j \in \mathbf{x}_v$, $i=1, 2, \dots, n$, $j=1, 2, \dots, m$ and $n, m \in \mathbb{Z}^+$. Therefore we get

$$\exp_{[y,x_i]}(\mathbb{P}_{S,y}) \equiv 0 \ (mod\ p) \quad \text{and} \quad \exp_{[x,y_j]}(\mathbb{P}_{T,x}) \equiv 0 \ (mod\ p).$$

Moreover, for the discs S, T in the spherical pictures $\mathbb{P}_{S,y}$ and $\mathbb{P}_{T,x}$, it is clear that

$$exp_S(\mathbb{P}_{S,y}) = 1 - 1 = exp_T(\mathbb{P}_{T,x}) = 0.$$

Hence, since the above processing can be made for all the spherical pictures in the set **Y** then we get P is p-Cockcroft where p is a prime or 0, as required.

Hence the result. ◊

4. Applications of the Main Theorem

Our aim in this section is to investigate what we get by changing some situations in the main theorem and then trying to get some consequences of it. In fact we will obtain some well-known results by applying these variations.

Let Γ be a graph with vertex set **v** and edge set **e** and let $G = G(\Gamma)$ be a graph product.

First let us suppose that all the vertex groups G_v ($v \in \mathbf{v}$) are infinite cyclic. Then a presentation of a group G_v is given by $P_v = \langle \mathbf{x_v} \ ; \quad \rangle$. Thus the presentation P, as in (2), becomes

$$P = <\mathbf{x}\ ;\quad >. \tag{3}$$

Corollary 4.1. *Let p be a prime or* 0 *and let P be a presentation as in* (3). *Then P is p-Cockcroft.*

Proof. By the definition of **X** and **Y**, it is easy to see that they are equal to the empty sets. Then by Theorem 2.1, $\pi_2(P)$ is generated by only the set **Z**. Additionally, since all the spherical pictures are Cockcroft in the set **Z** (see Figure 1-(*b*)) then *P* is *p*-Cockcroft, as required.◊

Now assume that the edge set **e** is empty in Γ. Then the set $\mathbf{r}_e$ ($e\in\mathbf{e}$) in the presentations P_e ($e\in\mathbf{e}$) will be empty. Therefore $\mathbf{r} = \varnothing$ and the group *G* becomes the free product of the groups G_v ($v\in\mathbf{v}$). Thus, by [18], *G* is given by a presentation

$$P = <\mathbf{x}\ ;\ \mathbf{s}>. \tag{4}$$

Corollary 4.2. *Let p be a prime or* 0 *and let P be a presentation as in* (4). *Then P is p-Cockcroft if and only if each* P_v ($v\in\mathbf{v}$) *is p-Cockcroft.*

Proof. It is clear that $\mathbf{Y}=\varnothing$ and $\mathbf{Z}=\varnothing$. Then $\pi_2(P)$ is generated by only the set $\mathbf{X} = \bigcup_{v\in\mathbf{v}} X_v$. First assume that *P* is *p*-Cockcroft for some prime or 0. Therefore, since $\pi_2(P)$ is generated by the union of the sets X_v ($v\in\mathbf{v}$) then we get each P_v is *p*-Cockcroft.

Now suppose that each P_v ($v\in\mathbf{v}$) is *p*-Cockcroft. Since $\mathbf{X} = \bigcup_{v\in\mathbf{v}} X_v$ and $\pi_2(P)$ is generated by the set **X** then *P* is *p*-Cockcroft, as required. ◊

Finally let us suppose that the oriented edge set is $\{e_1, e_2, e_3\}$ and the vertex set is $\{u, v, w\}$ in Γ as shown in Figure 3. Also let $P_u = <x\ ;\ x^{p_1}>$, $P_v = <y\ ;\ y^{p_2}>$, $P_w = <z\ ;\ z^{p_3}>$ be the presentations of the vertex groups G_u, G_v and G_w, respectively where p_1, p_2, p_3 are distinct primes. Thus

$$P = <x,y,z\ ;\ x^{p_1}, y^{p_2}, z^{p_3}, [x,y], [y,z], [z,x]> \tag{5}$$

is a presentation of the graph product of the groups G_u, G_v, G_w.

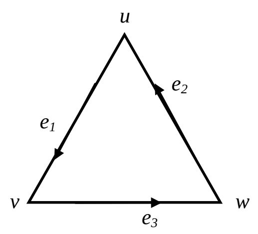


FIGURE 3 : **The graph Γ**

Remark 4.3. *It is well known that the presentation P, as in* (5)*, is actually presenting the direct product of the cyclic groups of order* p_1*,* p_2 *and* p_3*, respectively where* p_1*,* p_2*,* p_3 *are distinct primes.*

Let P be a presentation as in (5). As a consequence of Section 2, we can define the generating pictures of $\pi_2(P)$ as depicted in Figure 4. It is easy to see that set **Z** is empty for this group (we may refer [1] for the details of this).

It is clear that the proof of the following lemma is a quick consequence of Theorem 2.1.

Lemma 4.4. *Let P be a presentation as in* (5)*. Then* $\pi_2(P)$ *is generated by* $\mathbf{X} \cup \mathbf{Y}$**.**

Now as an application of Theorem 1.2 we can obtain the following result.

Corollary 4.5. *Let P be a presentation as in* (5) *and let p be a prime. Then P is p-Cockcroft if each prime* p_i *is equal to p (i=1,2,3).*

Proof. Let us label the relations in the presentations P_u , P_v , P_w as follows:

$$\begin{array}{c} x^{p_1} = R_1, y^{p_2} = R_2, z^{p_3} = R_3, \\ [x,y] = S_1, [y,z] = S_2, [z,x] = S_3 \end{array}.$$

By Lemma 4.4, we must check the exponent sum of the discs in the sets **X** and **Y**. By Figure 4, the spherical pictures $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ in the set **X** consist of discs R_1, R_2, R_3 and the spherical pictures $\mathbb{P}_4$, $\mathbb{P}_5$, $\mathbb{P}_6$, $\mathbb{P}_7$, $\mathbb{P}_8$, $\mathbb{P}_9$ in the set **Y** consist of discs R_1, R_2, R_3, S_1, S_2 and S_3. Then the exponent sum of these discs are

$$exp_{R_1}(\mathbb{P}_1) = exp_{R_2}(\mathbb{P}_2) = exp_{R_3}(\mathbb{P}_3) = exp_{R_1}(\mathbb{P}_4) = exp_{R_2}(\mathbb{P}_5) = 1\text{-}1 = 0,$$

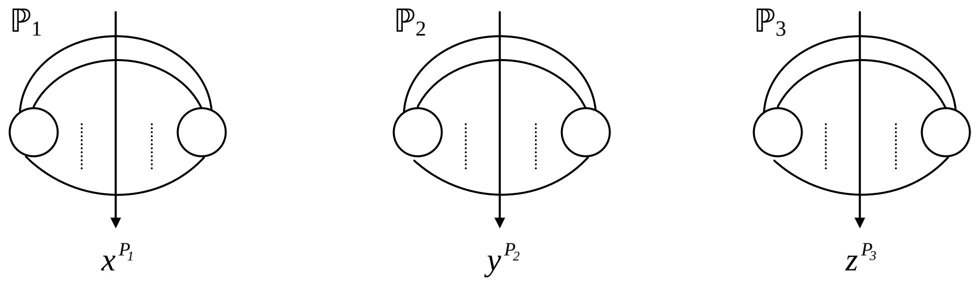
$$exp_{R_2}(\mathbb{P}_6) = exp_{R_3}(\mathbb{P}_7) = exp_{R_3}(\mathbb{P}_8) = exp_{R_1}(\mathbb{P}_9) = 1\text{-}1 = 0,$$

$$exp_{s_1}(\mathbb{P}_4) = p_1,\ exp_{s_1}(\mathbb{P}_5) = p_2,\ exp_{s_2}(\mathbb{P}_6) = p_2,$$

$$exp_{s_2}(\mathbb{P}_7) = p_3 = exp_{s_3}(\mathbb{P}_8),\ exp_{s_3}(\mathbb{P}_9) = p_1.$$

Therefore, by Theorem 1.2, P is p-Cockcroft if each of the presentation P_u , P_v , P_w is p-Cockcroft for the same prime p. Thus the primes p_1, p_2, p_3 must be equal to p, as required. ◊

The set **X:**



The set **Y:**

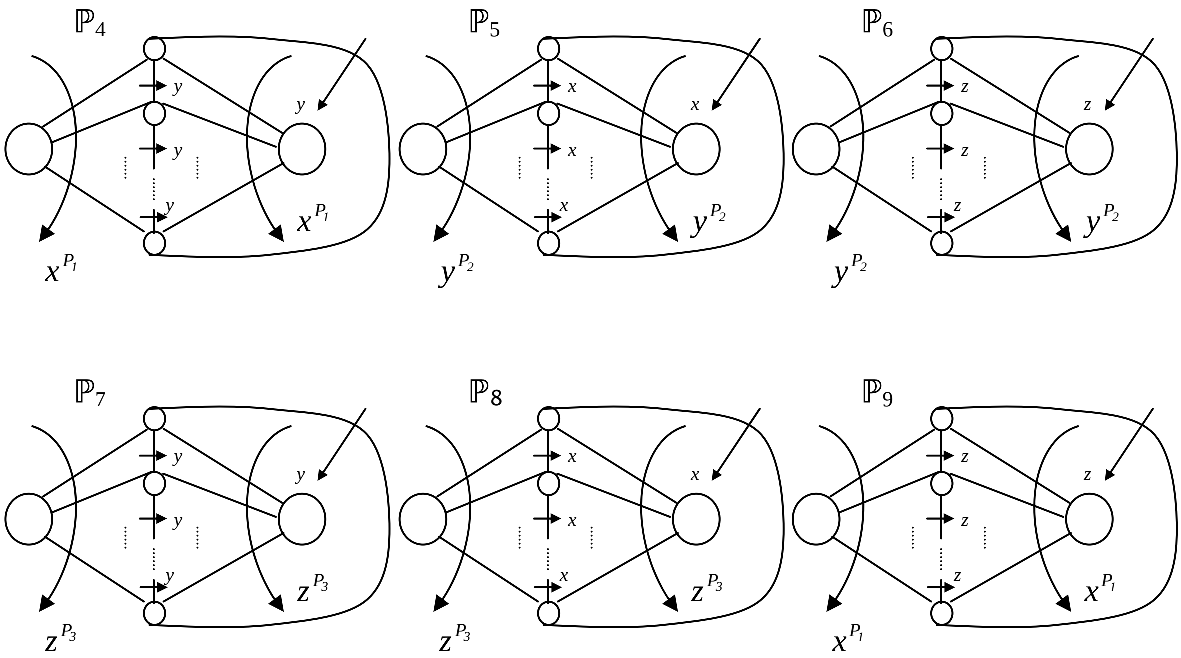


FIGURE 4

Remark 4.6. *Although Corollary* 4.5 *states the p-Cockcroft property of a sphecial case of the graph product, in fact, by Remark* 4.3*, it states that the p-Cockcroft property of the direct product of the cyclic groups of order* p_1, p_2 , p_3 *respectively which was studied and taken too much attention by the authors* (*see*, *for example*, *in* [21]).

One can find the definition of *efficiency* for a presentation P as in (1), for instance, in [4], [6], [7] and [24]. In [12], Epstein (and later on Kilgour-Pride in [19]) showed that a presentation P, as given in (1), is *efficient* if and only if it is *p-Cockcroft* for some prime p.

As an application of Theorem 1.2, we can also give the following example which is used the term efficiency instead of the p-Cockcroft property for the presentations of the vertex groups.

Example 4.7. *Let* Γ *be a graph with the oriented edge set is* $\{e_1\}$ *and the vertex set is* $\{v_1,v_2\}$. *Let*

$$P_{v_1} =< a,b;a^n ,aba^{-m}b^{-1} > \text{ and } P_{v_2} =< c,d;c^t ,cdc^{-k}d^{-1} >$$

be presentations of the vertex groups G_{v_1} *and* G_{v_2}, *respectively where* $(n, m\text{-}1) \neq 1$, $(t ,k\text{-}1) \neq 1$ $(n, m, t, k \in \mathbb{Z}^+)$ *and* p *is any prime with* $p \mid n$, $p \mid t$. *Then*

$$P=< a,b,c,d ; a^n, aba^{-m}b^{-1}, c^t, cdc^{-k}d^{-1}, [a,c], [a,d], [b,c], [b,d] >$$

is a presentation of the graph product of the groups G_{v_1} *and* G_{v_2}. *By* [3], *each of the presentations* P_{v_1} *and* P_{v_2} *is efficient (and so is p-Cockcroft by the last paragraph before example). Moreover the exponent sum of the each letter in the relators of the presentations* P_{v_1} *and* P_{v_2} *is congruent to zero by mod p. Then, by Theorem* 1.2, *we have that P is p-Cockcroft.*

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