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Generalized Bruck-Reilly ∗-Extension as a New Example of a Monoid with a Non-Finitely Generated Group of Units

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Abstract. We present a new example of a finitely presented monoid, namely Bruck-Reilly extension of generalized Bruck-Reilly ∗-extension of free group with infinite rank, the group of units of which is not finitely generated.

Key words: Bruck-Reilly Extension; Monoid Presentation; Unit.
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1. Introduction and Preliminaries

In combinatorial group and semigroup theory, the relationship between properties of a monoid M and the group of units $U(M)$ has often been subject to research. In this direction in [1], the author studied the properties of finite presentability and solvable word problem for the special monoids and the group of units. After that in [11], the author showed that the conjugacy problem for a special monoid was reducible to the conjugacy problem for its group of units. Then the same author in [12] proved that the group of units of every special monoid was finitely presented. But for any finitely presented monoid it is natural to ask the following question.

Question: Does the group of units of a finitely presented monoid have to be finitely generated?

This question was answered negatively in [3]. In that paper the authors have given an explicit example that has the form of a double Bruck-Reilly extension of the free group with infinite rank. In this short paper, we also answer the question given above with negatively as giving a similar example to [3] by considering the Bruck-Reilly extension of the generalized Bruck-Reilly ∗-extension (its presentation has been firstly given in [4]) of free group with infinite rank.

Definition 1.1. Let M be a monoid and $\theta : M \to M$ be an endomorphism. Then the Bruck-Reilly extension $BR(M,\theta)$ is the set

$$\mathbb{N}^0 \times M \times \mathbb{N}^0 = \{(p,m,q) : p,q \geq 0, m \in M\}$$

with multiplication

$$\begin{aligned}(p_1,m_1,q_1)(p_2,m_2,q_2) &= (p_1-q_1+t,(m_1\theta^{t-q_1})(m_2\theta^{t-p_2}),q_2-p_2+t),\\ t &= \max(q_1,p_2).\end{aligned}$$

$BR(M,\theta)$ is a monoid with identity $(0,1_M,0)$.

If M is defined by the presentation $< A;R >$, then $BR(M,\theta)$ is defined by

$$< A,b,c;\ R,\ bc=1,\ ba=(a\theta)b,\ ac=c(a\theta)(a\in A) > \quad (1)$$

in terms of generators $(0,a,0)$ $(a \in A)$, $(0,1_M,1)$ and $(1,1_M,0)$ [5].
This extension is considered a fundamental construction in the theory of semigroups. Many classes of regular semigroups are characterized by Bruck-Reilly extensions; any bisimple regular w-semigroups is isomorphic to a Reilly extension of a group [9] and any simple regular w-semigroup is isomorphic to a Bruck-Reilly extension of a finite chain of groups [6, 7, 8]. Then in [2], the author have obtained a monoid which is called generalized Bruck-Reilly $*$-extension and then given the structure of the $*$-bisimple type A w-semigroup in which $D^* = \widetilde{D}$ was obtained. After that motivated by this paper, in [10] the authors defined w^2-chain of idempotents and then studied the structure theorem of the $*$-bisimple type A w^2-semigroups as generalized Bruck-Reilly $*$-extension. Therefore, by considering these studies, in [4] the authors have found a presentation for the generalized Bruck-Reilly $*$-extension.

Definition 1.2. [10] *Let T be a monoid with H_1^* and H_1 as the H^*- and H- class which contains the identity 1_T of T, respectively. Then let β,γ be morphisms from T into H_1^*. Let u be an element in H_1 and λ_u the inner automorphism of H_1^* defined by $x \mapsto uxu^{-1}$ such that $\gamma\lambda_u = \beta\gamma$. Now we can make $S = \mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ into a semigroup by defining*

$$(m,n,a,p,q)(m',n',a',p',q') =$$
$$\begin{cases} (m,n-p+max(p,n'),(a\beta^{max(p,n')-p})(a'\beta^{max(p,n')-n'}), \\ \qquad p'-n'+max(p,n'),q'), & \text{if } q=m' \\ (m,n,a((u^{-n'}(a'\gamma)u^{p'})\gamma^{q-m'-1}\beta^{p}),p,q'-m'+q), & \text{if } q>m' \\ (m-q+m',n',((u^{-n}(a\gamma)u^{p})\gamma^{m'-q-1}\beta^{n'})a',p',q'), & \text{if } q<m' \end{cases}$$

where β^0,γ^0 are interpreted as the identity map of T and u^0 is interpreted as the identity 1_T of T. The monoid $S = \mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ constructed above is called generalized Bruck-Reilly $*$-extension of T determined by β,γ,u and will be denoted by $S = GBR^*(T;\beta,\gamma;u)$.

Theorem 1.1. [4] *Let T be the monoid defined by the presentation $< X; R >$, and let β, γ be morphisms from T into H_1^* (H^*-class which contains the identity 1_T of T). The monoid $S = GBR^*(T; \beta, \gamma; u)$ is then defined by the presentation*

$$(2) \qquad \begin{aligned} < X, y, z, a, b;\ R,\ yz = 1,\ ba = 1,& \\ yx = (x\gamma)y,\ xz = z(x\gamma),\ bx = (x\beta)b,\ xa = a(x\beta)& \\ yb = uy,\ bz = zu,\ uya = y,\ azu = z >& \end{aligned}$$

The following properties of $GBR^*(T; \beta, \gamma; u)$ are easy to derive from Definition 1.2:

$(GBR1)$ $GBR^*(T; \beta, \gamma; u)$ is a monoid with identity $(0, 0, 1_T, 0, 0)$.

$(GBR2)$ $T \cong \{0\} \times \{0\} \times T \times \{0\} \times \{0\} \leq GBR^*(T; \beta, \gamma; u)$.

$(GBR3)$ $U(GBR^*(T; \beta, \gamma; u)) = \{0\} \times \{0\} \times U(T) \times \{0\} \times \{0\} \cong U(T)$.

In this note since we provide a negative answer to question given above by means of Bruck-Reilly extension of generalized Bruck-Reilly $*$-extension of free group with infinite rank FG_∞, firstly, we define presentation of FG_∞ as a monoid as follows

$$(3) \qquad < q_i^\epsilon; q_i^{-\epsilon} q_i^\epsilon = 1 (\epsilon = \pm 1, i \geq 0) > .$$

2. Main Result

Theorem 2.1. *Let M be the monoid given as Bruck-Reilly extension of generalized Bruck-Reilly $*$-extension of FG_∞ defined by (3). The group of units of the monoid M defined by the finite presentation (16) is not finitely generated.*

Proof: Let us consider generalized Bruck-Reilly $*$-extension of FG_∞ defined by (3), under the homomorphism $\beta, \gamma : FG_\infty \to H_1^*$ (where H_1^* is the H^*-class which contains the identity of FG_∞) such that $q_i^\epsilon \mapsto q_{i+1}^\epsilon$ ($\epsilon = \pm 1$, $i \geq 0$). Thus by considering (2) we get the following presentation

$$(4) \qquad \begin{aligned} < q_i^\epsilon, y, z, a, b;\ q_i^{-\epsilon} q_i^\epsilon = 1,\ yz = 1,\ ba = 1,& \\ yb = uy,\ bz = zu,\ uya = y,\ azu = z,& \\ yq_i^\epsilon = q_{i+1}^\epsilon y,\ q_i^\epsilon z = z q_{i+1}^\epsilon,\ bq_i^\epsilon = q_{i+1}^\epsilon b,\ q_i^\epsilon a = a q_{i+1}^\epsilon >& \end{aligned}$$

for $GBR^*(FG_\infty; \beta, \gamma; u)$. Then by considering $ba = 1$, $bq_i^\epsilon = q_{i+1}^\epsilon b$ we obtain $q_{i+1}^\epsilon = bq_i^\epsilon a$ and $yz = 1$, $yq_i^\epsilon = q_{i+1}^\epsilon y$ we get $q_{i+1}^\epsilon = yq_i^\epsilon z$.
For $i = 0$, we get $q_1^\epsilon = bq_0^\epsilon a$ and $q_1^\epsilon = yq_0^\epsilon z$.
For $i = 1$, we obtain $q_2^\epsilon = bq_1^\epsilon a = b^2 q_0^\epsilon a^2$ and $q_2^\epsilon = yq_1^\epsilon z = y^2 q_0^\epsilon a^2$.
Thus by inductive argument we have

$$(5) \qquad q_i^\epsilon = b^i q_0^\epsilon a^i = y^i q_0^\epsilon z^i (\epsilon = \pm 1, i \geq 0).$$

Now we can use the equation (5) to eliminate all generators q_i^ϵ from (4). For facility in working, we rename q_0^ϵ as q^ϵ, thus we get the following finitely generated (but not finitely presented) presentation for $GBR^*(FG_\infty;\beta,\gamma;u)$:

$$(6)\qquad \begin{aligned} &< q, q^{-1}, y, z, a, b;\ b^i q^\epsilon a^i b^i q^{-\epsilon} a^i = 1,\ ba = 1,\ yz = 1,\\ &\quad yb = uy,\ bz = zu,\ uya = y,\ azu = z,\\ &\quad b^i q^\epsilon a^i = y^i q^\epsilon z^i,\ b^{i+1} q^\epsilon a^i = b^{i+1} q^\epsilon a^{i+1} b,\ b^i q^\epsilon a^{i+1} = ab^{i+1} q^\epsilon a^{i+1},\\ &\quad y^{i+1} q^\epsilon z^i = y^{i+1} q^\epsilon z^{i+1} y,\ y^i q^\epsilon z^{i+1} = zy^{i+1} q^\epsilon z^{i+1} > . \end{aligned}$$

We note that it is not possible to obtain a finitely presented presentation for $GBR^*(FG_\infty;\beta,\gamma;u)$ even if we apply some reductions on relations. So we define second endomorphism $\phi : GBR^*(FG_\infty;\beta,\gamma;u) \to GBR^*(FG_\infty;\beta,\gamma;u)$ by:

$$\begin{aligned} \phi \quad : \quad & q^\epsilon \mapsto bq^\epsilon a = yq^\epsilon z,\\ & b \mapsto b, \qquad a \mapsto a,\\ & y \mapsto y, \qquad z \mapsto z. \end{aligned}$$

Now we check ϕ defines an endomorphism from $GBR^*(FG_\infty;\beta,\gamma;u)$ to itself. To do that we must control that ϕ maps the defining relations in (6) into relations that are valid in $GBR^*(FG_\infty;\beta,\gamma;u)$:

$$\begin{aligned} (b^i q^\epsilon a^i b^i q^{-\epsilon} a^i)\phi &= b^i.bq^\epsilon a.a^i.b^i.bq^{-\epsilon}a.a^i = b^{i+1} q^\epsilon a^{i+1} b^{i+1} q^{-\epsilon} a^{i+1} = 1 = 1\phi,\\ (b^i q^\epsilon a^i)\phi &= b^i.bq^\epsilon a.a^i = b^{i+1} q^\epsilon a^{i+1} = y^{i+1} q^\epsilon z^{i+1} = (y^i q^\epsilon z^i)\phi,\\ (y^{i+1} q^\epsilon z^i)\phi &= y^{i+1}.yq^\epsilon z.z^i = y^{i+2} q^\epsilon z^{i+1} = y^{i+2} q^\epsilon z^{i+2} y = (y^{i+1} q^\epsilon z^{i+1} y)\phi,\\ (y^i q^\epsilon z^{i+1})\phi &= y^i.yq^\epsilon z.z^{i+1} = y^{i+1} q^\epsilon z^{i+2} = zy^{i+2} q^\epsilon z^{i+2} = (zy^{i+1} q^\epsilon z^{i+1})\phi,\\ (b^{i+1} q^\epsilon a^i)\phi &= b^{i+1}.bq^\epsilon a.a^i = b^{i+2} q^\epsilon a^{i+1} = b^{i+2} q^\epsilon a^{i+2} b = (b^{i+1} q^\epsilon a^{i+1} b)\phi,\\ (b^i q^\epsilon a^{i+1})\phi &= b^i.bq^\epsilon a.a^{i+1} = b^{i+1} q^\epsilon a^{i+2} = ab^{i+2} q^\epsilon a^{i+2} = (ab^{i+1} q^\epsilon a^{i+1})\phi,\\ (ba)\phi &= b.a = 1 = 1\phi, \qquad (yz)\phi = y.z = 1 = 1\phi. \end{aligned}$$

The check for the remaining relations is trivial/analogous.
Thus we have monoid $BR(GBR^*(FG_\infty;\beta,\gamma;u),\phi)$ and the following presentation

$$(7)\qquad < q, q^{-1}, y, z, a, b, \overline{a}, \overline{b}\ ;\ b^i q^\epsilon a^i b^i q^{-\epsilon} a^i = 1,$$

$$(8)\qquad ba = 1,\ yz = 1,$$

$$(9)\qquad yb = uy,\ bz = zu,\ uya = y,\ azu = z,$$

$$(10)\qquad b^i q^\epsilon a^i = y^i q^\epsilon z^i,$$

$$y^{i+1}q^{\epsilon}z^i = y^{i+1}q^{\epsilon}z^{i+1}y,\ y^iq^{\epsilon}z^{i+1} = zy^{i+1}q^{\epsilon}z^{i+1}, \tag{11}$$

$$b^{i+1}q^{\epsilon}a^i = b^{i+1}q^{\epsilon}a^{i+1}b,\ b^iq^{\epsilon}a^{i+1} = ab^{i+1}q^{\epsilon}a^{i+1}, \tag{12}$$

$$\overline{a}\overline{b} = 1,\ \overline{a}q^{\epsilon} = bq^{\epsilon}a\overline{a},\ q^{\epsilon}\overline{b} = \overline{b}bq^{\epsilon}a, \tag{13}$$

$$\overline{a}y = y\overline{a},\ \overline{a}z = z\overline{a},\ \overline{a}a = a\overline{a},\ \overline{a}b = b\overline{a}, \tag{14}$$

$$y\overline{b} = \overline{b}y,\ z\overline{b} = \overline{b}z,\ a\overline{b} = a,\ b\overline{b} = \overline{b}b > \tag{15}$$

where $\epsilon = \pm 1$ and $i \geq 0$.
Now we consider a relation (7) and multiply it by $\overline{a}$ from the left and by $\overline{b}$ from the right, and then use relations $(13) - (15)$. Thus we get

$$\begin{aligned} \overline{a}b^iq^{\epsilon}a^ib^iq^{-\epsilon}a^i\overline{b} = \overline{a}\overline{b} \quad &\Rightarrow \quad b^i.bq^{\epsilon}a.a^ib^i.bq^{-\epsilon}aa^i\overline{a}\overline{b} = 1 \\ &\Rightarrow \quad b^{i+1}q^{\epsilon}a^{i+1}b^{i+1}q^{-\epsilon}a^{i+1} = 1. \end{aligned}$$

So it is easily seen that all relations (7) are consequences of $q^{\epsilon}q^{-\epsilon} = 1$ and relations $(13) - (15)$. A similar argument gives that all relations (10) are consequences of the relations (10) for $i = 1$ and relations $(13) - (15)$. Analogously all relations of the form (11) and (12) are the consequences of these relations for $i = 0$ and $(13) - (15)$. Therefore we conclude that our monoid $BR(GBR^*(FG_{\infty};\beta,\gamma;u),\phi)$ is defined by

$$\begin{aligned} < q, q^{-1}, y, z, a, b, \overline{a}, \overline{b};\ & qq^{-1} = q^{-1}q = ba = yz = \overline{a}\overline{b} = 1, \\ & yb = uy,\ bz = zu,\ uya = y,\ azu = z, \\ & bq^{\epsilon}a = yq^{\epsilon}z, \\ & yq^{\epsilon} = yq^{\epsilon}zy,\ q^{\epsilon}z = zyq^{\epsilon}z, \\ & bq^{\epsilon} = bq^{\epsilon}ab,\ q^{\epsilon}a = abq^{\epsilon}a, \\ & aq^{\epsilon} = bq^{\epsilon}a\overline{a},\ q^{\epsilon}\overline{b} = \overline{b}bq^{\epsilon}a, \\ & \overline{a}y = y\overline{a},\ \overline{a}z = z\overline{a},\ \overline{a}a = a\overline{a}, \overline{a}b = b\overline{a}, \\ & y\overline{b} = \overline{b}y,\ z\overline{b} = \overline{b}z,\ a\overline{b} = a,\ b\overline{b} = \overline{b}b\ (\epsilon = \pm 1) >, \end{aligned} \tag{16}$$

which is finitely presented. By the property $(GBR3)$ it is seen that

$$\begin{aligned} U(M) = U(BR(GBR^*(FG_{\infty};\beta,\gamma;u),\phi)) \quad &\cong \quad U(GBR^*(FG_{\infty};\beta,\gamma;u)) \\ &= \quad \{0\} \times \{0\} \times U(FG_{\infty}) \times \{0\} \times \{0\} \\ &\cong \quad U(FG_{\infty}) = FG_{\infty}, \end{aligned}$$

and so the group of units of M is not finitely generated.
Hence the result.

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