On Canal Surfaces in E^3

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Abstract. In this paper we deal with the geometric properties of canal surfaces in $\mathbb{E}^3$. Further, the first and second fundamental form of canal surfaces are presented. By the use of the second fundamental form, the Gaussian and mean curvature of canal surfaces are obtained. Finally, the visualization of canal surfaces which their spine curves are unit circle and a straight line are presented.

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1. Introduction

A canal surface is defined as envelope of a one-parameter set of spheres, centered at a spine curve $\gamma(t)$ with radius $r(t)$: When $r(t)$ is a constant function, the canal surface is the envelope of a moving sphere and is called a pipe surface. Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, robotic path planning, etc. (see, [1], [2], [3]). Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve and rational radius function is rational, and it is therefore natural to ask for methods which allow one to construct a rational parametrization of canal surfaces from its spine curve and radius function. The developable surface plays a important role in CAGD. It is well known that, at regular points, the Gaussian curvature of a developable surface is identically zero. In [3] it has been proved that developable canal surface is either a cylinder or a cone.

The paper is organized as follows. In section 2, we shall discuss geometric properties of canal surfaces. The first and second fundamental forms of canal surfaces are presented. By the second fundamental form, the Gaussian and mean curvature of canal surfaces are given. In section 3, the visualization of canal surfaces which their spine curves are unit circle and a straight line are presented. All the figures in this paper were created using Maple programme.

2. Canal Surfaces

Recall definitions and results of [5]. In the present section we will consider canal surfaces in $\mathbb{E}^3$. Canal surface is a surface formed as the envelope of a family of spheres whose centers lie on a space curve.

Let $\gamma(u) = (f_1(u), f_2(u), 0)$ be a plane curve parametrized by arclength. The corresponding Frenet formulas have the following form:

$$
\begin{aligned}
\gamma'(u) &= e_1(u), \\
e_1'(u) &= \kappa(u)e_2(u), \\
e_2'(u) &= -\kappa(u)e_1(u),
\end{aligned}
$$
(2.1)

where $e_1(u)$ and $e_2(u)$ are the tangent and normal vectors of the curve γ respectively. We choose a constant vector $e_3(u)$ of $\mathbb{E}^3$. The canal surface of the planar curve γ has the following parametrization (see, [6]):

$$
(2.2) \qquad M : X(u,v) = \gamma(u) + r(u)\left(e_2(u)\cos v + e_3(u)\sin v\right),
$$

where $r(u)$ is the radius of the spheres from the definition of the canal surface. The tangent space to M at an arbitrary point $p = X(u,v)$ of M is spanned by

$$
\begin{aligned}
X_u(u,v) &= (1 - rk\cos v)e_1 + (r'\cos v)e_2 + (r'\sin v)e_3, \\
X_v(u,v) &= -(r\sin v)e_2 + (r\cos v)e_3.
\end{aligned}
$$
(2.3)

Further, the coefficients of the first fundamental form become

$$
\begin{aligned}
E &= \langle X_u, X_u \rangle = (r')^2 + (1 - rk\cos v)^2, \\
F &= \langle X_u, X_v \rangle = 0, \\
G &= \langle X_v, X_v \rangle = r^2.
\end{aligned}
$$
(2.4)

The unit normal vector field of M is

$$
\begin{aligned}
N &= \frac{X_u \times X_v}{\|X_u \times X_v\|} \\
&= \frac{1}{W}\{rr'e_1 - r\cos v(1 - rk\cos v)e_2 - r\sin v(1 - rk\cos v)e_3\}.
\end{aligned}
$$
(2.5)

where $W^2 = EG - F^2$.

The second partial derivatives of $X(u,v)$ are expressed as follows:

$$
\begin{aligned}
X_{uu} &= -\cos v(2r'k + rk')e_1 + (\kappa - rk^2\cos v + r''\cos v)e_2 + (r''\sin v)e_3, \\
X_{uv} &= (rk\sin v)e_1 - (r'\sin v)e_2 + (r'\cos v)e_3, \\
X_{vv} &= -(r\cos v)e_2 - (r\sin v)e_3.
\end{aligned}
$$
(2.6)

Further, the coefficients of the second fundamental form become,

$$
\begin{aligned}
e &=< X_{uu}, N > \\
&= \frac{1}{W}\{\cos v(-2r(r')^2 k - r^2 r'k' - rk + r^2 r''k\sin^2 v) \\
&\quad + \cos^2 v(2r^2 k^2) + \cos^3 v(r^2 r''k - r^3 k^3) - rr''\}, \\
f &=< X_{uv}, N > \\
&= \frac{r^2 r'k\sin v}{W}, \\
g &=< X_{vv}, N > \\
&= \frac{r^2 - r^3 k\cos v}{W}.
\end{aligned}
$$

(2.7)

The Gaussian and mean curvature of a regular patch are given by

(2.8)
$$
K = \frac{eg - f^2}{EG - F^2},
$$

(2.9)
$$
H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)},
$$

respectively (see, [6]).

Summing up the following results are proved.

Theorem 2.1. Let M be a canal surface in $\mathbb{E}^3$ given with the parametrization (2.2). Then the Gaussian and mean curvatures of M at point p are

(2.10)
$$
\begin{aligned}
K =\frac{r^3}{W^4}\{ &\cos v(2rr''k - rr'k' - 2(r')^2 k - k) \\
&+ \cos^2 v(3r(r')^2 k^2 + r^2 r'kk' + 3rk^2 - r^2 r''k^2) \\
&+ \cos^3 v(-3r^2 k^3) + \cos^4 v(r^3 k^4) - r'' - r(r')^2 k^2\},
\end{aligned}
$$

and

(2.11)
$$
\begin{aligned}
H = \frac{r^2}{2W^3}\{&\cos v(r^2 r''k - 3r(r')^2 k - r^2 r'k' - 4kr) \\
&+ \cos^2 v(5r^2 k^2) + \cos^3 v(-2r^3 k^3) - rr'' + 1 + (r')^2\}.
\end{aligned}
$$

Corollary 2.1. Let M be a pipe surface (i.e. $r(u)$ is a constant function) in $\mathbb{E}^3$ given with the parametrization (2.2). Then the Gauss and mean curvatures of M at a point p are

$$
\begin{aligned}
K &= \frac{\cos v}{r(1 - rk\cos v)^4}\{-k + (3rk^2)\cos v - (3r^2 k^3)\cos^2 v + (r^3 k^4)\cos^3 v\}, \\
H &= \frac{1}{2r(1 - rk\cos v)^3}\{1 - (4kr)\cos v + (5r^2 k^2)\cos^2 v - (2r^3 k^3)\cos^3 v\}.
\end{aligned}
$$

Corollary 2.2. Let M be a canal surface in $\mathbb{E}^3$ given with the parametrization (2.2). If the spine curve γ is a straight line then the Gauss and mean curvatures of M at a point p are

$$(2.12) \qquad\qquad K = \frac{-r''}{r(1+(r')^2)^2},$$

(see, [6]) and

$$(2.13) \qquad\qquad H = \frac{-rr'' + (r')^2 + 1}{2r(1+(r')^2)^{3/2}}.$$

Example 2.1. Consider the unit circle $\gamma(u) = (\cos u, \sin u, 0)$ in $\mathbb{E}^3$. Then the canal surface of the spine curve γ in $\mathbb{E}^3$ has the following parametrization

(2.14)
$$X(u,v) = (\,\cos(u) - r(u)\cos(u)cos(v), \sin(u) - r(u)\sin(u)\cos(v), r(u)\sin(v)).$$

Example 2.2. Consider the straight line $\gamma(u) = (a_1 u + b_1, a_2 u + b_2, 0)$ in $\mathbb{E}^3$ where $a_1^2 + a_2^2 = 1$. Then the canal surface of the spine curve γ in $\mathbb{E}^3$ has the following parametrization

(2.15)
$$X(u,v) = (a_1 u + b_1 - a_2 r(u)u\cos(v), a_2 u + b_2 + a_1 r(u)u\cos(v), r(u)\sin(v)),$$

where a_1, b_1, a_2, b_2 are real constants.

3. Visualization

We visualize the surfaces given with the patch $X(u,v) = (x(u,v), y(u,v), z(u,v))$ making use of Maple. Furthermore, we plot the graph of the given surface by using maple plotting command

$$(3.1) \qquad\qquad plot3d([x,y,z], u = a..b, v = c..d);$$

In the sequel we construct some 3D geometric shape models of canal surfaces. First, we construct the geometric model of the canal surface given in Example 2.1. We visualize some surface models of the patch (2.14) using the following values respectively (see Figure 1) ;

$$(3.2) \qquad\qquad \begin{array}{ll} a) & r(u) = e^u, \\ b) & r(u) = u^2, \\ c) & r(u) = \sin u. \end{array}$$
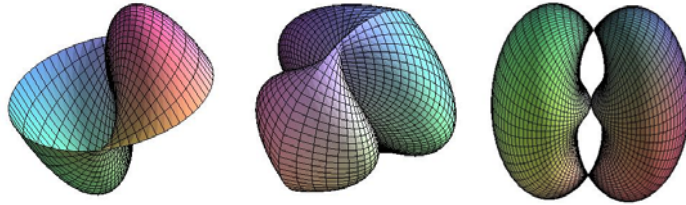
Figure 1. The canal surfaces of a unit circle

Secondly, we construct the geometric model of the canal surface given in Example 2.2. We visualize some surface models of the patch (2.15) using the values $a_1 = \frac{4}{5}$; $a_2 = \frac{3}{5}$; $b_1 = 4$; $b_2 = 5$ and $r(u)$ given in (3.2) respectively (see Figure 2);
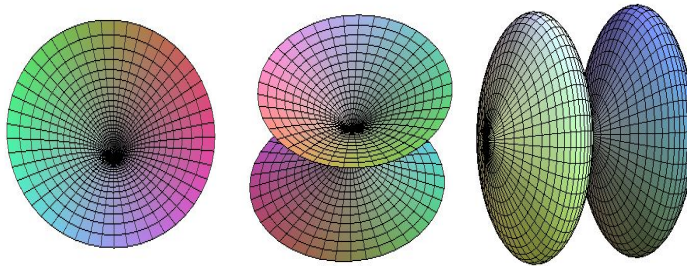

Figure 2. The canal surfaces of a straight line

Finally, we visualize the graph of the flat and minimal canal surfaces given by (2.15) using the following values of $r(u)$ respectively, (see Figure 3);

$$d) \quad r(u) \quad = \quad 3u + 5,$$
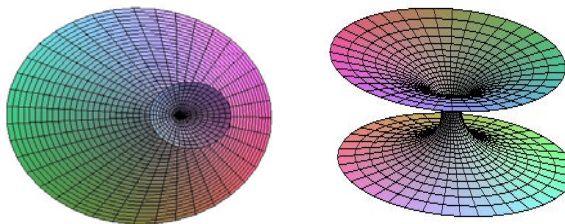$$e) \quad r(u) \quad = \quad \cosh u.$$

Figure 3. The flat and minimal canal surfaces of a straight line

4. Conclusion

In this paper, a method of canal surfaces is investigated. For demonstrating the performance of the proposed method, parameters of canal surfaces models were constructed from the spine curve γ. Infect, canal surfaces are solid models that can fairly simple parametrization of representing a large variety of standard geometric models. They have wide applications in CAGD, such as construction of blending surfaces. This makes them much more convenient for representing shape reconstruction, transition surfaces and pipe surfaces. By the use of main results of differential geometry we calculated the Gaussian and mean curvatures of canal surfaces in $\mathbb{E}^3$. Moreover, this frame work can be used for the modelling of 3-D shapes in $\mathbb{E}^3$. For future work it will be necessary to improve the system to allow for the canal surface in $\mathbb{E}^4$.

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