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Mixed modulus of continuity in the Lebesgue spaces with Muckenhoupt weights and their properties

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Abstract: Main properties of the mixed modulus of continuity in the Lebesgue spaces with Muckenhoupt weights are investigated. We use the mixed modulus of continuity to obtain Potapov type direct and inverse estimates of angular trigonometric approximation of functions in these spaces. We prove an equivalence between the mixed modulus of continuity and K-functional and realization functional.

Key words: Direct theorem, inverse theorem, Muckenhoupt weights, modulus of continuity

1. Introduction and the main results

In this paper we consider the properties of the mixed modulus of continuity $\Omega(f,\delta,\xi)_{p,\omega}$ in the Lebesgue spaces $L^p_\omega(\mathbb{T}^2) := L^p(\mathbb{T}^2,\omega(x,y))$ with weights $\omega(x,y)$ belonging to the Muckenhoupt class $A_p(\mathbb{T}^2,\mathbb{J})$ where $\mathbb{J}$ is the set of rectangles in $\mathbb{T}^2 := \mathbb{T}\times\mathbb{T}$, $\mathbb{T} := [0,2\pi)$ with sides parallel to coordinate axes. In the particular case $\omega(x,y)\equiv 1$ on $\mathbb{T}^2$, classical mixed modulus of continuity was used to prove some results on trigonometric approximation by an angle for functions in the classical Lebesgue spaces $L^p(\mathbb{T}^d)$. For example, Potapov obtained a direct theorem [9, 12] and inverse estimate [13] on trigonometric approximation by an angle for functions in spaces $L^p(\mathbb{T}^d)$. Hardy–Littlewood, Marcinkiewicz, Littlewood–Paley, and embedding theorems were proved in [10, 11]. Transformed Fourier series and mixed modulus of continuity were investigated by Potapov et al. in [15] and [17]. Embeddings of the Besov–Nikolskii and Weyl–Nikolskii classes were studied in [14, 16]. $(L^p - L^q)$ Ulyanov type inequalities were proved in [18]. Mixed K-functionals were analyzed by Cottin in [2] and by Runovski in [20]. More information about mixed modulus of continuity and trigonometric approximation by an angle can be found in the survey [19]. In $L^p(\mathbb{T}^d)$ mixed modulus of continuity was defined by a difference operator based on the classical translation operator. Difference operators in $L^p(\mathbb{T}^d)$ can be defined various ways. For instance, partial difference, total difference, and mixed difference are successfully used in approximation problems in $L^p(\mathbb{T}^d)$ (see, e.g., the books of Timan [21] and Timan [22, Chapter 2]). We note that mixed differences are closely related to other differences [22, Chapter 2]. On the other hand, in the case of $L^p_\omega(\mathbb{T}^2)$ classical translation operator are not bounded, in general. Instead of classical translation

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operators we use Steklov type operators to define mixed modulus of continuity $\Omega(f,\delta,\xi)_{p,\omega}$ in $L^p_\omega(\mathbb{T}^2)$ (see (6)). Using some properties of $\Omega(f,\delta,\xi)_{p,\omega}$ we obtain an equivalence between $\Omega(f,\delta,\xi)_{p,\omega}$ and mixed K-functional $K(f,\delta,\xi,p,\omega,2,2)$ (see Definition 18):

Theorem 1 *If* $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$, $f\in L^p_\omega(\mathbb{T}^2)$, *then there exist constants depending only on Muckenhoupt's constant* $[\omega]_{A_p}$ *of* ω *and* p *so that the equivalence*

$$\Omega(f,\delta,\xi)_{p,\omega}\approx K(f,\delta,\xi,p,\omega,2,2) \text{ holds for } \delta,\xi\geq 0.$$

We use the properties of mixed modulus of continuity $\Omega(f,\delta,\xi)_{p,\omega}$ to obtain a Potapov type direct theorem on trigonometric approximation by an angle in $L^p_\omega(\mathbb{T}^2)$ with $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$:

Theorem 2 *If* $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$, $f\in L^p_\omega(\mathbb{T}^2)$, *then there exists a constant* $C_{[\omega]_{A_p},p}$ *depending only on Muckenhoupt's constant* $[\omega]_{A_p}$ *of* ω *and* p *so that*

$$Y_{m,n}(f)_{p,\omega}\leq C_{[\omega]_{A_p},p}\Omega\left(f,\frac{1}{m+1},\frac{1}{n+1}\right)_{p,\omega} \tag{1}$$

where $m,n\in\mathbb{N}$;

$$Y_{m,n}(f)_{p,\omega}=\inf\left\{\|f-T-U\|_{p,\omega}:T\in\mathcal{T}_{m,\circ},U\in\mathcal{T}_{\circ,n}\right\},$$

$\mathcal{T}_{m,\circ}$ *(respectively* $\mathcal{T}_{\circ,n}$ *) is the set of all trigonometric polynomials of degree at most* m *(at most* n*) with respect to variable* x *(variable* y*).*

If $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$, and $f\in L^p_\omega(\mathbb{T}^2)$, then we have $L^p_\omega(\mathbb{T}^2)\subset L^\lambda(\mathbb{T}^2)$, $\lambda>1$, and hence we can define the trigonometric Fourier series of $f\in L^p_\omega(\mathbb{T}^2)$. Partial sums of trigonometric Fourier series of f will be denoted by $S_{m,\circ}, S_{\circ,n}$, and $S_{m,n}$. Let $W^*_{m,n}f:=S_{m,\circ}(f)+S_{\circ,n}(f)-S_{m,n}(f)$. Then $\left\|f-W^*_{m,n}f\right\|_{p,\omega}\leq C_{[\omega]_{A_p},p}Y_{m,n}(f)_{p,\omega}$ and hence $Y_{m,n}(f)_{p,\omega}\searrow 0$ as $m,n\nearrow\infty$.

The following theorem gives the weak inverse of (1).

Theorem 3 *If* $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$, $f\in L^p_\omega(\mathbb{T}^2)$, *then there exist constants depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$\Omega\left(f,\frac{1}{m},\frac{1}{n}\right)_{p,\omega}\leq\frac{C_{[\omega]_{A_p},p}}{m^2n^2}\sum_{k=0}^{n}\sum_{l=0}^{n}(k+1)(l+1)Y_{k,l}(f)_{p,\omega}.$$

We obtain an equivalence between $\Omega(f,1/m,1/n)_{p,\omega}$ and mixed realization functional $R(f,m,n,p,\omega,2,2)$ (see (16)):

Theorem 4 *If* $1<p<\infty$, $\omega\in A_p(\mathbb{T}^2,\mathbb{J})$, $f\in L^p_\omega(\mathbb{T}^2)$, *then there exist constants depending only on* $[\omega]_{A_p}$ *and* p *so that the equivalence*

$$\Omega\left(f,m^{-1},n^{-1}\right)_{p,\omega}\approx R(f,m,n,p,\omega,2,2)$$

holds for $m,n\in\mathbb{N}$.

The rest of the work is organized as follows. Section 2 contains some preliminary definitions such as Muckenhoupt weights, weighted Lebesgue space, mixed modulus of continuity, and angular trigonometric approximation. Additionally, boundedness of partial sums and Cesàro and dela Valleè Poussin means of Fourier series of functions in $L_{\omega}^{p}\left(\mathbb{T}^{2}\right)$ are proved. In Section 3 we obtain Bernstein type inequalities and their several corollaries. In Section 4 Favard type Jackson inequalities are obtained. In Section 5, using some properties of $\Omega\left(f,\delta,\xi\right)_{p,\omega}$, we obtain an equivalence between $\Omega\left(f,\delta,\xi\right)_{p,\omega}$ and mixed K-functional $K(f,\delta,\xi,p,\omega,2,2)$. In Section 6 we obtain the Potapov type direct theorem and its improvement on trigonometric approximation by an angle in $L_{\omega}^{p}\left(\mathbb{T}^{2}\right)$. In Section 7 we obtain an equivalence between $\Omega\left(f,1/m,1/n\right)_{p,\omega}$ and mixed realization functional $R(f,m,n,p,\omega,2,2)$. In Section 8 we prove inverse estimates for functions in $L_{\omega}^{p}\left(\mathbb{T}^{2}\right)$.

Here and in what follows, $A\lesssim B$ will mean that there exists a positive constant $C_{u,v,\ldots}$, dependent only on the parameters $u,v,\ldots$ and it can be different in different places, such that the inequality $A\leq CB$ holds. If $A\lesssim B$ and $B\lesssim A$ we will write $A\approx B$.

2. Steklov type averages, difference operators, and modulus of continuity

Let $\mathbb{T}:=[0,2\pi)$, $\mathbb{T}^{2}:=\mathbb{T}\times\mathbb{T}$, $L^{1}\left(\mathbb{T}^{2}\right)$ be the collection of the Lebesgue integrable functions $f\left(x,y\right):\mathbb{T}^{2}\rightarrow\mathbb{R}:=(-\infty,\infty)$ such that $f\left(x,y\right)$ is 2π-periodic with respect to each variable x,y.

We suppose that $\mathbb{J}$ is the set of rectangles in $\mathbb{T}^{2}$ with the sides parallel to coordinate axes. A function $\omega:\mathbb{T}^{2}\rightarrow\mathbb{R}^{\geq}:=[0,\infty)$ is called a weight on $\mathbb{T}^{2}$ if $\omega\left(x,y\right)$ is measurable and positive almost everywhere on $\mathbb{T}^{2}$. We denote by $A_{p}\left(\mathbb{T}^{2},\mathbb{J}\right)$, $(1<p<\infty)$ the collection of locally integrable weights $\omega:\mathbb{T}^{2}\rightarrow\mathbb{R}^{\geq}$ such that $\omega\left(x,y\right)$ is 2π-periodic with respect to each variable x,y and

$$\left[\omega\right]_{A_{p}}:=\sup_{G\in\mathbb{J}}\left(\frac{1}{|G|}\int\int_{G}\omega\left(x,y\right)dxdy\right)\left(\frac{1}{|G|}\int\int_{G}\left[\omega\left(x,y\right)\right]^{-\frac{1}{p-1}}dxdy\right)^{p-1}<\infty. \tag{2}$$

Least constant $\left[\omega\right]_{A_{p}}$ in (2) will be called the Muckenhoupt constant of ω. Let $1<p<\infty$, $\omega\left(x,y\right)\in A_{p}\left(\mathbb{T}^{2},\mathbb{J}\right)$, and $L_{\omega}^{p}\left(\mathbb{T}^{2}\right)$ be the collection of the Lebesgue integrable functions $f\left(x,y\right):\mathbb{T}^{2}\rightarrow\mathbb{R}$ such that $f\left(x,y\right)$ is 2π-periodic with respect to each variable x,y and

$$\left\Vert f\right\Vert_{p,\omega}:=\left(\int\limits_{\mathbb{T}}\int\limits_{\mathbb{T}}\left|f\left(x,y\right)\right|^{p}\omega\left(x,y\right)dxdy\right)^{1/p}<\infty.$$

We define $\mathcal{T}_{m,\circ}$ (respectively $\mathcal{T}_{\circ,n}$) as the set of all trigonometric polynomials of degree at most m (at most n) with respect to variable x (variable y). $\mathcal{T}_{m,n}$ is defined as the set of all trigonometric polynomials of degree at most m with respect to variable x and of degree at most n with respect to variable y.

The best partial trigonometric approximation orders are defined as

$$\begin{aligned} Y_{m,\circ}(f)_{p,\omega} &= \inf\left\{\left\Vert f-T\right\Vert_{p,\omega}:T\in\mathcal{T}_{m,\circ}\right\}, \\ Y_{\circ,n}(f)_{p,\omega} &= \inf\left\{\left\Vert f-U\right\Vert_{p,\omega}:U\in\mathcal{T}_{\circ,n}\right\} \end{aligned}$$

where $1<p<\infty$, $\omega\in A_{p}\left(\mathbb{T}^{2},\mathbb{J}\right)$, and $f\in L_{\omega}^{p}\left(\mathbb{T}^{2}\right)$.

The best angular trigonometric approximation error is defined as

$$Y_{m,n}(f)_{p,\omega}=\inf\left\{\|f-T-U\|_{p,\omega}:T\in\mathcal{T}_{m,\circ},U\in\mathcal{T}_{\circ,n}\right\}$$

where $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, and $f\in L^p_\omega\left(\mathbb{T}^2\right)$.

We define Steklov type averages:

$$\begin{aligned}\sigma_{h,k}f\left(x,y\right) &= \frac{1}{hk}\int\limits_{x-h}^{x+h}\int\limits_{y-k}^{y+k}f\left(t,\tau\right)dtd\tau,\\ \sigma_{h,\circ}f\left(x,y\right) &= \frac{1}{h}\int\limits_{x-h}^{x+h}f\left(t,\tau\right)dt,\sigma_{\circ,k}f\left(x,y\right)=\frac{1}{k}\int\limits_{y-k}^{y+k}f\left(t,\tau\right)d\tau.\end{aligned}$$

Using Theorem 3.3 of [4] we obtain

$$\|\sigma_{h,k}f\|_{p,\omega}\lesssim\|f\|_{p,\omega} \tag{3}$$

for $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L^p_\omega\left(\mathbb{T}^2\right)$, where the constant C depends only on $[\omega]_{A_p}$ and p.

Lemma 5 *([6]) $A_p\left(\mathbb{T}^2,\mathbb{J}\right)$ over arbitrary rectangles implies $A_p\left(\mathbb{T}\right)$ in each variable uniformly with respect to the other variables for $1<p<\infty$. In other words, if $1<p<\infty,\ \omega\left(x,y\right)\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$ then, for any intervals $I,J\subset\mathbb{T}$*

$$\sup_I\left(\frac{1}{|I|}\int\limits_I\omega\left(x,y\right)dx\right)\left(\frac{1}{|I|}\int\limits_I\left[\omega\left(x,y\right)\right]^{-\frac{1}{p-1}}dx\right)^{p-1}<\infty,\ \textit{for a.e. } y$$

and

$$\sup_J\left(\frac{1}{|J|}\int\limits_J\omega\left(x,y\right)dy\right)\left(\frac{1}{|J|}\int\limits_J\left[\omega\left(x,y\right)\right]^{-\frac{1}{p-1}}dy\right)^{p-1}<\infty,\ \textit{for a.e. } x.$$

For weights in $A_p\left(\mathbb{R}^d,\mathbb{J}\right)$, $\left(1<p<\infty\right)$, Lemma 5 was given in [7, Lemma 2] and [3].

Now Corollary 4 of [8] and Lemma 5 give:

Lemma 6 *If $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L^p_\omega\left(\mathbb{T}^2\right)$, then*

$$\left\{\|\sigma_{h,\circ}f\|_{p,\omega},\|\sigma_{\circ,k}f\|_{p,\omega}\right\}\lesssim\|f\|_{p,\omega}, \tag{4}$$

where the constants depend only on $[\omega]_{A_p}$ and p.

For $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right),\ h,k>0$ we define the differences

$$\begin{aligned}
\bigtriangledown_{h,\circ} f(x,y) &= \left(\mathbb{I}-\sigma_{h,\circ}\right) f(x,y)=\frac{1}{h}\int\limits_{x-h}^{x+h}\left(f(x,y)-f(t,\tau)\right)dt,\\
\bigtriangledown_{\circ,k} f(x,y) &= \left(\mathbb{I}-\sigma_{\circ,k}\right) f(x,y)=\frac{1}{k}\int\limits_{y-k}^{y+k}\left(f(x,y)-f(t,\tau)\right)d\tau,\\
\bigtriangledown_{h,k} f(x,y) &= \bigtriangledown_{h,\circ}\left(\bigtriangledown_{\circ,k} f\right)(x,y)=\frac{1}{hk}\int\limits_{x-h}^{x+h}\int\limits_{y-k}^{y+k}\left(f(x,y)-f(t,\tau)\right)dtd\tau,
\end{aligned}$$

where $\mathbb{I}$ is the identity operator on $\mathbb{T}^2$. Using inequalities (4)–(3) we get

$$\left\{\left\|\bigtriangledown_{h,\circ} f\right\|_{p,\omega},\left\|\bigtriangledown_{\circ,k} f\right\|_{p,\omega},\left\|\bigtriangledown_{h,k} f\right\|_{p,\omega}\right\}\lesssim\|f\|_{p,\omega}, \tag{5}$$

for $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right)$ with constants depending only on $[\omega]_{A_p}$ and p.

The mixed modulus of continuity of $f\in L_\omega^p\left(\mathbb{T}^2\right),\ 1<p<\infty,\ \omega(x,y)\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, can be defined as

$$\Omega\left(f,\delta_1,\delta_2\right)_{p,\omega}=\sup_{\substack{0\leq h\leq\delta_1\\0\leq k\leq\delta_2}}\left\|\bigtriangledown_{h,k} f\right\|_{p,\omega}. \tag{6}$$

If $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right)$, then from (6) and (5)

$$\Omega\left(f,\delta_1,\delta_2\right)_{p,\omega}\lesssim\|f\|_{p,\omega}$$

with constant depending only on $[\omega]_{A_p}$ and p.

Note that from the definition of $\Omega\left(f,\cdot,\cdot\right)_{p,\omega}$, it has the following properties when $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right)$:

(1) $\Omega\left(f,0,0\right)_{p,\omega}=0$.

(2) $\Omega\left(f,\delta_1,\delta_2\right)_{p,\omega}$ is subadditive with respect to f.

(3) $\Omega\left(f,\delta_1,\delta_2\right)_{p,\omega}\leq\Omega\left(f,t_1,t_2\right)_{p,\omega}$ for $0\leq\delta_i\leq t_i; i=1,2$.

2.1. Some means of Fourier series

Let $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, and $f\in L_\omega^p\left(\mathbb{T}^2\right)$; then there is a $\lambda\in(1,\infty)$ such that $f\in L^\lambda\left(\mathbb{T}^2\right)$. Namely, we have $L_\omega^p\left(\mathbb{T}^2\right)\subset L^\lambda\left(\mathbb{T}^2\right)$ and this gives the possibility to define the corresponding Fourier series of f:

Lemma 7 *If $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$, and $f\in L_\omega^p\left(\mathbb{T}^2\right)$, then we have*

$$L_\omega^p\left(\mathbb{T}^2\right)\subset L^\lambda\left(\mathbb{T}^2\right) \tag{7}$$

for some $\lambda>1$.

Proof There exists a $r \in (1,p)$ so that $\omega \in A_r(\mathbb{T}^2, \mathbb{J})$. Then setting $\lambda := p/r > 1$ and using $f \in L^p_\omega(\mathbb{T}^2)$ we get $f^\lambda \omega^{1/r} \in L^r(\mathbb{T}^2)$ and $\omega^{-1/r} \in L^{\frac{r}{r-1}}(\mathbb{T}^2)$. From $\omega \in A_r(\mathbb{T}^2, \mathbb{J})$ we have $\omega(x,y)^{-\frac{1}{r-1}} \in A_{\frac{r}{r-1}}(\mathbb{T}^2, \mathbb{J})$ and using Hölder's inequality

$$\left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f(x,y)|^\lambda \omega(x,y)\, dxdy \right)^{\frac{1}{\lambda}} \leq \left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f(x,y)|^p \omega(x,y)\, dxdy \right)^{\frac{1}{p}} \times$$
$$\times \left(\int_{\mathbb{T}} \int_{\mathbb{T}} \omega(x,y)^{-\frac{1}{r-1}}\, dxdy \right)^{\frac{r-1}{p}} \lesssim \|f\|_{p,\omega}$$

where the constant depends only on $[\omega]_{A_p}$ and p. Hence, (7) is proved. □

Let $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$ and

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} A_{n_1,n_2}(x,y) \quad : \quad = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \mu_{n_1,n_2} [a_{n_1,n_2} \cos n_1 x \cos n_2 y + b_{n_1,n_2} \sin n_1 x \cos n_2 y +$$
$$+ c_{n_1,n_2} \cos n_1 x \sin n_2 y + d_{n_1,n_2} \sin n_1 x \sin n_2 y], \tag{8}$$

$$\mu_{n_1,n_2} = \begin{cases} 1/4 & , n_1 = n_2 = 0 \\ 1/2 & , n_1 = 0, n_2 > 0 \text{ or } n_2 = 0, n_1 > 0 \\ 1 & , n_1 > 0, n_2 > 0 \end{cases}$$

be the corresponding Fourier series for f.

For the Fourier series (8) of $f \in L^p_\omega(\mathbb{T}^2)$, $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$ we define the partial sums of (8) as

$$S_{m,\circ}(f)(x,y) = \sum_{n_1=0}^{m} \sum_{n_2=0}^{\infty} A_{n_1,n_2}(x,y,f), S_{\circ,n}(f)(x,y) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n} A_{n_1,n_2}(x,y,f),$$
$$S_{m,n}(f)(x,y) = S_{m,\circ}(S_{\circ,n}(f))(x,y) = \sum_{n_1=0}^{m} \sum_{n_2=0}^{n} A_{n_1,n_2}(x,y,f).$$

Then

$$S_{m,\circ}(f)(x,y) = \frac{1}{\pi} \int_{\mathbb{T}} f(x+t,y) D_m(t)\, dt, S_{\circ,n}(f)(x,y) = \frac{1}{\pi} \int_{\mathbb{T}} f(x,y+u) D_n(u)\, du,$$
$$S_{m,n}(f)(x,y) = \frac{1}{\pi^2} \int_{\mathbb{T}} \int_{\mathbb{T}} f(x+t,y+u) D_m(t) D_n(u)\, dudt$$

where

$$D_l(t) = (\sin(l+1/2)t))/(2\sin(t/2)) = \frac{1}{2} + \sum_{k=1}^{l} \cos kt$$

is the Dirichlet kernel.

Theorem 8 of [5] and Lemma 5 give:

Lemma 8 *If* $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right),$ *then*

$$\left\{\left\|S_{m,\circ}(f)\right\|_{p,\omega},\left\|S_{\circ,n}(f)\right\|_{p,\omega},\left\|S_{m,n}(f)\right\|_{p,\omega}\right\}\lesssim\|f\|_{p,\omega},$$

with constants depending only on $[\omega]_{A_p}$ *and* p*.*

For the Fourier series (8) of $f\in L_\omega^p\left(\mathbb{T}^2\right),\ 1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$ we define the partial Cesaro means and dela Valleè Poussin means of f as

$$\begin{aligned} C_{m,\circ}(f)(x,y) &= \frac{1}{m+1}\sum_{k=0}^{m}S_{k,\circ}(f),C_{\circ,n}(f)(x,y)=\frac{1}{n+1}\sum_{l=0}^{n}S_{\circ,l}(f),\\ C_{m,n}(f)(x,y) &= C_{m,\circ}(C_{\circ,n}(f))(x,y)=\frac{1}{(n+1)(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n}S_{k,l}(f)\end{aligned}$$

and

$$V_{m,\circ}(f)(x,y) = \frac{1}{m+1}\sum_{k=m}^{2m-1}S_{k,\circ}(f),V_{\circ,n}(f)(x,y)=\frac{1}{n+1}\sum_{l=n}^{2n-1}S_{\circ,l}(f), \tag{9}$$

$$V_{m,n}(f)(x,y) = V_{m,\circ}(V_{\circ,n}(f))(x,y)=\frac{1}{(n+1)(m+1)}\sum_{k=m}^{2m-1}\sum_{l=n}^{2n-1}S_{k,l}(f). \tag{10}$$

In this case

$$\begin{aligned} C_{m,\circ}(f)(x,y) &= \frac{1}{\pi}\int\limits_{\mathbb{T}}f(x+u,y)K_m(u)\,du,C_{\circ,n}(f)(x,y)=\frac{1}{\pi}\int\limits_{\mathbb{T}}f(x,y+v)K_n(v)\,dv,\\ C_{m,n}(f)(x,y) &= \frac{1}{\pi^2}\int\limits_{\mathbb{T}}\int\limits_{\mathbb{T}}f(x+u,y+v)D_m(u)D_n(v)\,dudv\end{aligned}$$

where

$$K_l(t)=\frac{1}{2(l+1)}\left(\frac{\sin(2l+1)t/2}{\sin t/2}\right)^2$$

is the Fejer kernel.

As a corollary of Lemma 8 and (9)-(10) we have:

Corollary 9 *If* $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right),$ *then*

$$\left\{\left\|V_{m,\circ}(f)\right\|_{p,\omega},\left\|V_{\circ,n}(f)\right\|_{p,\omega},\left\|V_{m,n}(f)\right\|_{p,\omega}\right\}\lesssim\|f\|_{p,\omega},$$

with constants depending only on $[\omega]_{A_p}$ *and* p*.*

Lemma 10 *Let* $W_{m,n}f(x,y) = (V_{m,\circ}(f) + V_{\circ,n}(f) - V_{m,n}(f))(x,y)$. *Then*

$$\|f - W_{m,n}f\|_{p,\omega} \lesssim Y_{m,n}(f)_{p,\omega}$$

holds for $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$, *where the constant depends only on* $[\omega]_{A_p}$ *and* p.

Proof We take polynomials $T_1 \in \mathcal{T}_{m,\circ}, T_2 \in \mathcal{T}_{\circ,n}, T_3 \in \mathcal{T}_{m,n}$ and set

$$\varphi(x,y) = f(x,y) - T_1(x,y) - T_2(x,y) + T_3(x,y).$$

Then

$$f - W_{m,n}f = \varphi - V_{m,\circ}(\varphi) - V_{\circ,n}(\varphi) + V_{m,n}(\varphi).$$

Hence, using Corollary 9,

$$\begin{aligned} \|f - W_{m,n}f\|_{p,\omega} &= \|\varphi - V_{m,\circ}(\varphi) - V_{\circ,n}(\varphi) + V_{m,n}(\varphi)\|_{p,\omega} \\ &\lesssim \|\varphi\|_{p,\omega} = \|f - T_1 - T_2 + T_3\|_{p,\omega}, \end{aligned}$$

with a constant independent of T_1, T_2, T_3. Since T_1, T_2, T_3 are arbitrarily chosen

$$\|f - W_{m,n}f\|_{p,\omega} \lesssim Y_{m,n}(f)_{p,\omega}$$

holds. □

Using Theorems 10 and 11 of [8] and Lemma 5, we obtain:

Lemma 11 *If* $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$, *then*

$$\left\{\|C_{m,\circ}(f)\|_{p,\omega}, \|C_{\circ,n}(f)\|_{p,\omega}, \|C_{m,n}(f)\|_{p,\omega}\right\} \lesssim \|f\|_{p,\omega},$$

with constants depending only on $[\omega]_{A_p}$ *and* p.

Lemma 11 is a consequence of a pointwise estimate through a strong maximal function and its boundedness in weighted Lebesgue spaces.

3. Bernstein type inequalities

Let $T_1 \in \mathcal{T}_{m,\circ}, T_2 \in \mathcal{T}_{\circ,n}, T_3 \in \mathcal{T}_{m,n}$. Then

$$\begin{aligned} T_1(x,y) &= \frac{1}{\pi}\int_{\mathbb{T}} T_1(t,y) D_m(t-x)\,dt, T_2(x,y) = \frac{1}{\pi}\int_{\mathbb{T}} T_2(x,s) D_n(s-y)\,ds, \\ T_3(x,y) &= \frac{1}{\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}} T_3(t,s) D_m(t-x) D_n(s-y)\,dtds \end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial}{\partial x}T_1(x,y) &= \frac{-1}{\pi}\int\limits_{\mathbb{T}} T_1(t,y)\frac{\partial}{\partial x}(D_m(t-x))\,dt,\\
\frac{\partial}{\partial y}T_2(x,y) &= \frac{-1}{\pi}\int\limits_{\mathbb{T}} T_2(x,s)\frac{\partial}{\partial y}(D_n(s-y))\,ds,\\
\frac{\partial^2}{\partial x\partial y}T_3(x,y) &= \frac{1}{\pi^2}\int\limits_{\mathbb{T}}\int\limits_{\mathbb{T}} T_3(t,s)\frac{\partial}{\partial x}(D_m(t-x))\frac{\partial}{\partial y}(D_n(s-y))\,dtds.
\end{aligned}$$

The following Bernstein type inequalities hold.

Lemma 12 *([4]) Let* $1<p<\infty,\ \omega\in A_p(\mathbb{T}^2,\mathbb{J}),\ T_1\in\mathcal{T}_{m,\circ},T_2\in\mathcal{T}_{\circ,n},T_3\in\mathcal{T}_{m,n}$. *Then*

$$\left\|\frac{\partial}{\partial x}T_1\right\|_{p,\omega}\lesssim m\left\|T_1\right\|_{p,\omega},\ \left\|\frac{\partial}{\partial y}T_2\right\|_{p,\omega}\lesssim n\left\|T_2\right\|_{p,\omega}, \tag{11}$$

and

$$\left\|\frac{\partial^2}{\partial x\partial y}T_3\right\|_{p,\omega}\lesssim mn\left\|T_3\right\|_{p,\omega} \tag{12}$$

with constants depending only on $[\omega]_{A_p}$ *and* p.

Proof (12) was obtained in [4, Theorem 4.2]. Using Lemma 11 the inequalities in (11) can be obtained in the same way. □

Corollary 13 *Let* $1<p<\infty,\ \omega\in A_p(\mathbb{T}^2,\mathbb{J}),\ T_1\in\mathcal{T}_{m,\circ},T_2\in\mathcal{T}_{\circ,n},T_3\in\mathcal{T}_{m,n},\ k,l\in\mathbb{N}$. *Then*

$$\left\|\frac{\partial^k}{\partial x^k}T_1\right\|_{p,\omega}\lesssim m^k\left\|T_1\right\|_{p,\omega},\ \left\|\frac{\partial^l}{\partial x^l}T_2\right\|_{p,\omega}\lesssim n^l\left\|T_2\right\|_{p,\omega},$$

and as a result

$$\left\|\frac{\partial^{k+l}}{\partial x^k\partial x^l}T_3\right\|_{p,\omega}\lesssim m^k n^l\left\|T_3\right\|_{p,\omega}$$

with constants depending only on $[\omega]_{A_p}$ *and* p.

Lemma 14 *For* $1<p<\infty,\ \omega\in A_p(\mathbb{T}^2,\mathbb{J}),\ f\in L^p_\omega(\mathbb{T}^2)$ *there exists a constant depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$\left\|\frac{\partial^{k+l}}{\partial x^k\partial x^l}\varphi_{i,j}(f)\right\|_{p,\omega}\lesssim 2^{ik}2^{jl}Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}$$

where

$$V_{2^i,2^j}(f)-V_{2^i,\lfloor 2^{j-1}\rfloor}(f)-V_{\lfloor 2^{i-1}\rfloor,2^j}(f)+V_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)=:\varphi_{i,j}(f)\in\mathcal{T}_{2^{i+1}-1,2^{j+1}-1}$$

and $\lfloor x\rfloor:=\max\{z\in\mathbb{Z}:z\leq x\}$.

Proof Since

$$\begin{aligned}\varphi_{i,j}(f) &= V_{2^i,2^j}(f) - V_{2^i,\lfloor 2^{j-1}\rfloor}(f) - V_{\lfloor 2^{i-1}\rfloor,2^j}(f) + V_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)\\ &= W_{2^i,2^j}(f) - W_{2^i,\lfloor 2^{j-1}\rfloor}(f) - W_{\lfloor 2^{i-1}\rfloor,2^j}(f) + W_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)\\ &= W_{2^i,2^j}(f) - f + f - W_{2^i,\lfloor 2^{j-1}\rfloor}(f) + f - W_{\lfloor 2^{i-1}\rfloor,2^j}(f) - f + W_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)\end{aligned}$$

we have by Lemma 10

$$\begin{aligned}\|\varphi_{i,j}(f)\|_{p,\omega} &\leq \left\|W_{2^i,2^j}(f) - f\right\|_{p,\omega} + \left\|f - W_{2^i,\lfloor 2^{j-1}\rfloor}(f)\right\|_{p,\omega} +\\ &\quad + \left\|f - W_{\lfloor 2^{i-1}\rfloor,2^j}(f)\right\|_{p,\omega} + \left\|f + W_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)\right\|_{p,\omega}\\ &\lesssim Y_{2^i,2^j}(f)_{p,\omega} + Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega} + Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega} + Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}\\ &\lesssim Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}.\end{aligned}$$

Now using Corollary 13 and the last inequality,

$$\left\|\frac{\partial^{k+l}}{\partial x^k \partial x^l}\varphi_{i,j}(f)\right\|_{p,\omega} \lesssim 2^{ik}2^{jl}\|\varphi_{i,j}(f)\|_{p,\omega} \lesssim 2^{ik+jl} Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}$$

holds. □

Lemma 15 *For $1 < p < \infty$, $\omega \in A_p(\mathbb{T}^2, \mathbb{J})$, $f \in L^p_\omega(\mathbb{T}^2)$ there exists a constant depending only on $[\omega]_{A_p}$ and p so that*

$$\left\|\frac{\partial^k}{\partial x^k}\psi_{i,j}(f)\right\|_{p,\omega} \lesssim 2^{ik} Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega}$$

where $V_{2^i,\circ}\left(f - V_{\circ,2^j}(f)\right) - V_{\lfloor 2^{i-1}\rfloor,\circ}\left(f - V_{\circ,2^j}(f)\right) =: \psi_{i,j}(f) \in \mathcal{T}_{2^{i+1}-1,\circ}$.

Proof Since

$$\psi_{i,j}(f) = V_{2^i,\circ}\left(f - V_{\circ,2^j}(f)\right) - V_{\lfloor 2^{i-1}\rfloor,\circ}\left(f - V_{\circ,2^j}(f)\right) = W_{2^i,2^j}(f) - W_{\lfloor 2^{i-1}\rfloor,2^j}(f)$$

we have by Lemma 10

$$\begin{aligned}\|\psi_{i,j}(f)\|_{p,\omega} &\leq \left\|W_{2^i,2^j}(f) - f\right\|_{p,\omega} + \left\|f - W_{\lfloor 2^{i-1}\rfloor,2^j}(f)\right\|_{p,\omega}\\ &\lesssim Y_{2^i,2^j}(f)_{p,\omega} + Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega} \lesssim Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega}.\end{aligned}$$

Now using Corollary 13 and the last inequality,

$$\left\|\frac{\partial^k}{\partial x^k}\psi_{i,j}(f)\right\|_{p,\omega} \lesssim 2^{ik}\|\psi_{i,j}(f)\|_{p,\omega} \lesssim 2^{ik} Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega}$$

holds. □

Lemma 16 *For* $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L_\omega^p\left(\mathbb{T}^2\right)$ *there exists a constant depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$\left\|\frac{\partial^l}{\partial y^l}h_{i,j}(f)\right\|_{p,\omega}\lesssim 2^{jl}Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}$$

where

$$V_{\circ,2^j}\left(f-V_{2^i,\circ}(f)\right)-V_{\circ,\lfloor 2^{j-1}\rfloor}\left(f-V_{2^i,\circ}(f)\right)=:h_{i,j}(f)\in\mathcal{T}_{\circ,2^{j+1}-1}.$$

Proof Since

$$h_{i,j}(f)=V_{\circ,2^j}\left(f-V_{2^i,\circ}(f)\right)-V_{\circ,\lfloor 2^{j-1}\rfloor}\left(f-V_{2^i,\circ}(f)\right)=W_{2^i,2^j}(f)-W_{2^i,\lfloor 2^{j-1}\rfloor}(f)$$

we have by Lemma 10

$$\begin{aligned}\|h_{i,j}(f)\|_{p,\omega}&\leq\left\|W_{2^i,2^j}(f)-f\right\|_{p,\omega}+\left\|f-W_{2^i,\lfloor 2^{j-1}\rfloor}(f)\right\|_{p,\omega}\\&\lesssim Y_{2^i,2^j}(f)_{p,\omega}+Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}\lesssim Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}.\end{aligned}$$

Now using Corollary 13 and the last inequality,

$$\left\|\frac{\partial^l}{\partial x^l}h_{i,j}(f)\right\|_{p,\omega}\lesssim 2^{jl}\|h_{i,j}(f)\|_{p,\omega}\lesssim 2^{jl}Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}$$

holds. □

4. Favard type Jackson inequalities

Let $W_{p,\omega}^{r,s}$ (respectively $W_{p,\omega}^{r,\circ};\ W_{p,\omega}^{\circ,s}$) denote the collection of functions $f\in L^1\left(\mathbb{T}^2\right)$ such that $f^{(r,s)}:=\frac{\partial^{r+s}f}{\partial x^r\partial y^s}\in L_\omega^p\left(\mathbb{T}^2\right)$ (respectively $f^{(r,\circ)}:=\frac{\partial^r f}{\partial x^r}\in L_\omega^p\left(\mathbb{T}^2\right);\quad f^{(\circ,s)}:=\frac{\partial^s f}{\partial y^s}\in L_\omega^p\left(\mathbb{T}^2\right)$).

The following Favard type Jackson inequalities hold.

Lemma 17 *For* $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right)$ *there exist constants depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$Y_{m,n}(g_1)_{p,\omega}\lesssim\frac{1}{(m+1)^2}\left\|g_1^{(2,\circ)}\right\|_{p,\omega},\quad g_1\in W_{p,\omega}^{2,\circ},\tag{13}$$

$$Y_{m,n}(g_2)_{p,\omega}\lesssim\frac{1}{(n+1)^2}\left\|g_2^{(\circ,2)}\right\|_{p,\omega},\quad g_2\in W_{p,\omega}^{\circ,2},$$

and

$$Y_{m,n}(g)_{p,\omega}\lesssim\frac{1}{(m+1)^2(n+1)^2}\left\|g^{(2,2)}\right\|_{p,\omega}\tag{14}$$

hold for $g\in W_{p,\omega}^{2,2}$.

Proof For (13) we have

$$
\begin{aligned}
& \left\| g_1 - S_{m,\circ}(g_1) - S_{\circ,n}(g_1) + S_{m,n}(g_1) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} A_{i,j}(x,y,g_1) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^2} i^2 A_{i,j}(x,y,g_1) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \cos \pi \frac{1}{i^2} A_{i,j}\left(x+(\pi/2), y, g_1^{(2,\circ)}\right) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^2} A_{i,j}\left(x, y, g_1^{(2,\circ)}\right) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{1}{i^2} \left(S_{i,j}\left(g_1^{(2,\circ)}\right) - S_{i,j-1}\left(g_1^{(2,\circ)}\right) - S_{i-1,j}\left(g_1^{(2,\circ)}\right) + S_{i-1,j-1}\left(g_1^{(2,\circ)}\right) \right) \right\|_{p,\omega} \\
= & \left\| \sum_{i=m+1}^{\infty} \left[\frac{1}{(i+1)^2} - \frac{1}{i^2} \right] S_{i,m}\left(g_1^{(2,\circ)}\right) + \frac{1}{(m+1)^2} S_{m,n}\left(g_1^{(2,\circ)}\right) \right\|_{p,\omega} \\
\leq & \sum_{i=m+1}^{\infty} \left| \frac{1}{(i+1)^2} - \frac{1}{i^2} \right| \left\| S_{i,n}\left(g_1^{(2,\circ)}\right) \right\|_{p,\omega} + \frac{1}{(m+1)^2} \left\| S_{m,n}\left(g_1^{(2,\circ)}\right) \right\|_{p,\omega} \\
\leq & \left\| g_1^{(2,\circ)} \right\|_{p,\omega} \left(\sum_{i=m+1}^{\infty} \left(\frac{1}{i^2} - \frac{1}{(i+1)^2} \right) + \frac{1}{(m+1)^2} \right) \\
\leq & \frac{C}{(m+1)^2} \left\| g_1^{(2,\circ)} \right\|_{p,\omega}.
\end{aligned}
$$

Since

$$
\begin{aligned}
Y_{m,n}(g_1)_{p,\omega} &= Y_{m,n}\left(g_1 - S_{m,\circ}(g_1) - S_{\circ,n}(g_1) + S_{m,n}(g_1)\right)_{p,\omega} \\
&\leq \left\| g_1 - S_{m,\circ}(g_1) - S_{\circ,n}(g_1) + S_{m,n}(g_1) \right\|_{p,\omega}
\end{aligned}
$$

inequality (13) follows.

Similarly, we have

$$
\left\| g_2 - S_{\circ,n}(g_2) - S_{m,\circ}(g_2) + S_{m,n}(g_2) \right\|_{p,\omega} \lesssim \frac{1}{(n+1)^2} \left\| g_2^{(\circ,2)} \right\|_{p,\omega}
$$

and hence

$$\begin{aligned} Y_{m,n}(g_2)_{p,\omega} &= Y_{m,n}(g_2 - S_{\circ,n}(g_2) - S_{m,\circ}(g_2) + S_{m,n}(g_2))_{p,\omega} \\ &\leq \|g_1 - S_{\circ,n}(g_1) - S_{m,\circ}(g_1) + S_{m,n}(g_1)\|_{p,\omega} \\ &\lesssim \frac{1}{(n+1)^2}\left\|g_2^{(\circ,2)}\right\|_{p,\omega}. \end{aligned}$$

For (14)

$$\begin{aligned} &\|g - S_{m,\circ}(g) - S_{\circ,n}(g) + S_{m,n}(g)\|_{p,\omega} \\ = &\left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} A_{i,j}(x,y,g)\right\|_{p,\omega} = \left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \frac{1}{i^2j^2} i^2j^2 A_{i,j}(x,y,g)\right\|_{p,\omega} \\ = &\left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \frac{1}{i^2j^2} A_{i,j}\left(x,y,g^{(2,2)}\right)\right\|_{p,\omega} = \left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \frac{1}{i^2j^2} A_{i,j}(x,y,\Upsilon)\right\|_{p,\omega} \\ = &\left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \frac{1}{i^2j^2} A_{i,j}(x,y,\Upsilon)\right\|_{p,\omega} \\ = &\left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty} \frac{1}{i^2j^2}\left(S_{i,j}(\Upsilon) - S_{i,j-1}(\Upsilon) - S_{i-1,j}(\Upsilon) + S_{i-1,j-1}(\Upsilon)\right)\right\|_{p,\omega} \\ = &\left\|\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\left[\frac{1}{(i+1)^2} - \frac{1}{i^2}\right]\left[\frac{1}{(j+1)^2} - \frac{1}{j^2}\right] S_{i,j}(\Upsilon) + \right. \\ &+\frac{1}{(m+1)^2}\sum_{j=n+1}^{\infty}\left[\frac{1}{(j+1)^2} - \frac{1}{j^2}\right] S_{m,j}(\Upsilon) + \frac{1}{(n+1)^2}\sum_{i=n+1}^{\infty}\left[\frac{1}{(i+1)^2} - \frac{1}{i^2}\right] S_{i,m}(\Upsilon) + \\ &\left. +\frac{1}{(m+1)^2}\frac{1}{(n+1)^2} S_{m,n}(\Upsilon)\right\|_{p,\omega} \\ \leq &\sum_{i=m+1}^{\infty}\sum_{j=n+1}^{\infty}\left(\frac{1}{i^2} - \frac{1}{(i+1)^2}\right)\left(\frac{1}{j^2} - \frac{1}{(j+1)^2}\right)\|S_{i,j}(\Upsilon)\|_{p,\omega} + \\ &+\frac{1}{(m+1)^2}\sum_{j=n+1}^{\infty}\left[\frac{1}{j^2} - \frac{1}{(j+1)^2}\right]\|S_{m,j}(\Upsilon)\|_{p,\omega} + \\ &+\frac{1}{(n+1)^2}\sum_{i=m+1}^{\infty}\left[\frac{1}{i^2} - \frac{1}{(i+1)^2}\right]\|S_{i,n}(\Upsilon)\|_{p,\omega} + \frac{1}{(m+1)^2}\frac{1}{(n+1)^2}\|S_{m,n}(\Upsilon)\|_{p,\omega} \\ \lesssim &\frac{1}{(m+1)^2(n+1)^2}\|\Upsilon\|_{p,\omega} = \frac{1}{(m+1)^2(n+1)^2}\left\|g^{(2,2)}\right\|_{p,\omega}. \end{aligned}$$

Hence,

$$\begin{aligned} Y_{m,n}(g)_{p,\omega} &= Y_{m,n}\left(g - S_{m,\circ}(g) - S_{\circ,n}(g) + S_{m,n}(g)\right)_{p,\omega} \\ &\leq \left\| g - S_{m,\circ}(g) - S_{\circ,n}(g) + S_{m,n}(g) \right\|_{p,\omega} \lesssim \frac{1}{(m+1)^2 (n+1)^2} \left\| g^{(2,2)} \right\|_{p,\omega}. \end{aligned}$$

The proof is completed. □

5. Mixed K-functional

Definition 18 *The mixed K-functional is defined as*

$$\begin{aligned} K(f,\delta,\xi,p,\omega,r,s) \quad : \quad &= \inf_{g_1,g_2,g} \left\{ \|f-g_1-g_2-g\|_{p,\omega} + \delta^r \left\| \frac{\partial^r g_1}{\partial x^r} \right\|_{p,\omega} + \right. \\ &\left. + \xi^s \left\| \frac{\partial^s g_2}{\partial y^s} \right\|_{p,\omega} + \delta^r \xi^s \left\| \frac{\partial^{r+s} g}{\partial x^r \partial y^s} \right\|_{p,\omega} \right\} \end{aligned}$$

where the infimum is taken from all g_1, g_2, g so that $g_1 \in W^{r,\circ}_{p,\omega}$, $g_2 \in W^{\circ,s}_{p,\omega}, g \in W^{r,s}_{p,\omega}$ where $1 < p < \infty$, $\omega \in A_p\left(\mathbb{T}^2, \mathbb{J}\right)$, $f \in L^p_\omega\left(\mathbb{T}^2\right)$.

Here we give the proof of Theorem 1, but first we need the following theorem.

Theorem 19 *If $1 < p < \infty$, $\omega \in A_p\left(\mathbb{T}^2, \mathbb{J}\right)$, then there exist constants depending only on $[\omega]_{A_p}$ and p so that*

$$\begin{aligned} \Omega\left(g_1,\delta,\cdot\right)_{p,\omega} &\lesssim \delta^2 \left\| \frac{\partial^2 g_1}{\partial x^2} \right\|_{p,\omega}, \quad g_1 \in W^{2,\circ}_{p,\omega}, \\ \Omega\left(g_2,\cdot,\xi\right)_{p,\omega} &\lesssim \xi^2 \left\| \frac{\partial^2 g_2}{\partial y^2} \right\|_{p,\omega}, \quad g_2 \in W^{\circ,2}_{p,\omega}, \\ \Omega\left(g,\delta,\xi\right)_{p,\omega} &\lesssim \delta^2 \xi^2 \left\| \frac{\partial^4 g}{\partial x^2 \partial y^2} \right\|_{p,\omega}, \quad g \in W^{2,2}_{p,\omega} \end{aligned} \tag{15}$$

hold for $\delta, \xi > 0$.

Proof Since

$$\begin{aligned}
\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k}g_1\right)\right\|_{p,\omega} &= \left\|\left(\mathbb{I}-\sigma_{h,\circ}\right)\left(\mathbb{I}-\sigma_{\circ,k}\right)g_1\right\|_{p,\omega} \\
&= \left\|\left(\mathbb{I}-\sigma_{h,\circ}\right)F\right\|_{p,\omega} \\
&= \left\|\frac{1}{2h}\int_{-h}^{h}\left(F\left(x,y\right)-F\left(x+t,y\right)\right)dt\right\|_{p,\omega} \\
&= \left\|\frac{-1}{2h}\int_{0}^{h}\int_{0}^{t}\int_{-u}^{u}\frac{d^2}{dx^2}F\left(x+s,y\right)dsdudt\right\|_{p,\omega} \\
&\leq \frac{1}{2h}\int_{0}^{h}\int_{0}^{t}2u\left\|\frac{1}{2u}\int_{-u}^{u}\frac{d^2}{dx^2}F\left(x+s,y\right)ds\right\|_{p,\omega}dudt \\
&= \frac{1}{2h}\int_{0}^{h}\int_{0}^{t}2u\left\|\sigma_{u,\circ}\left(\frac{d^2}{dx^2}F\right)\right\|_{p,\omega}dudt \\
&\lesssim h^2\left\|\frac{d^2}{dx^2}F\right\|_{p,\omega}=h^2\left\|\frac{d^2}{dx^2}\left[\left(\mathbb{I}-\sigma_{\circ,k}\right)g_1\right]\right\|_{p,\omega} \\
&= h^2\left\|\left(\mathbb{I}-\sigma_{\circ,k}\right)\left(\frac{d^2}{dx^2}g_1\right)\right\|_{p,\omega} \\
&\lesssim h^2\left\|\frac{d^2}{dx^2}g_1\right\|_{p,\omega}=h^2\left\|g_1^{(2,\circ)}\right\|_{p,\omega}
\end{aligned}$$

we have $\Omega\left(g_1,\delta,\cdot\right)_{p,\omega}\lesssim\delta^2\left\|g_1^{(2,\circ)}\right\|_{p,\omega},\quad g_1\in W_{p,\omega}^{2,\circ}$. Similarly,

$$\Omega\left(g_2,\cdot,\xi\right)_{p,\omega}\lesssim\xi^2\left\|g_2^{(\circ,2)}\right\|_{p,\omega},\quad g_2\in W_{p,\omega}^{\circ,2},\text{ and}$$

$$\Omega\left(g,\delta,\xi\right)_{p,\omega}\lesssim\delta^2\xi^2\left\|g^{(2,2)}\right\|_{p,\omega},\quad g\in W_{p,\omega}^{2,2}\text{ hold.}$$

□

Proof of Theorem 1 We prove the upper estimate. We define

$$U_{h,\circ}f\left(x,y\right):=\frac{1}{h^3}\int_{0}^{h}\int_{0}^{t_1}\int_{-u_1}^{u_1}f\left(x+s_1,y\right)ds_1du_1dt_1, U_{\circ,k}f\left(x,y\right):=\frac{1}{k^3}\int_{0}^{k}\int_{0}^{t_2}\int_{-u_2}^{u_2}f\left(x,y+s_2\right)ds_2du_2dt_2,$$

$$U_{h,k}f\left(x,y\right):=U_{h,\circ}\left(U_{\circ,k}f\right)\left(x,y\right)=\frac{1}{h^3k^3}\int_{0}^{h}\int_{0}^{k}\int_{0}^{t_1}\int_{0}^{t_2}\int_{-u_1}^{u_1}\int_{-u_2}^{u_2}f\left(x+s_1,y+s_2\right)ds_1ds_2du_1du_2dt_1dt_2,$$

and

$$\begin{aligned}
g_1\left(x,y\right) &:= U_{h,\circ}\left(\mathbb{I}-U_{\circ,k}\right)f\left(x,y\right), g_2\left(x,y\right):=U_{\circ,k}\left(\mathbb{I}-U_{h,\circ}\right)f\left(x,y\right), \\
g\left(x,y\right) &:= U_{h,\circ}\left(U_{\circ,k}f\right)\left(x,y\right)=U_{h,k}f\left(x,y\right).
\end{aligned}$$

Then

$$\begin{aligned}
\|f-g_1-g_2-g\|_{p,\omega} &= \|f-U_{h,\circ}f-U_{\circ,k}f+U_{h,k}f\|_{p,\omega} \\
&= \|(\mathbb{I}-U_{h,\circ})(\mathbb{I}-U_{\circ,k})f\|_{p,\omega}=\|(\mathbb{I}-U_{h,\circ})\,\mathrm{F}\|_{p,\omega} \\
&= \left\|\frac{1}{h^3}\int_0^h\int_0^{t_1}\int_{-u_1}^{u_1}(\mathrm{F}(x,y)\text{-}\mathrm{F}(x+s_1,y))\,ds_1du_1dt_1\right\|_{p,\omega} \\
&\lesssim \frac{1}{h^3}\int_0^h\int_0^{t_1}u_1\left\|\frac{1}{2u_1}\int_{-u_1}^{u_1}(\mathrm{F}(x,y)\text{-}\mathrm{F}(x+s_1,y))\,ds_1\right\|_{p,\omega}du_1dt_1 \\
&= \frac{1}{h^3}\int_0^h\int_0^{t_1}u_1\|(\mathbb{I}-\sigma_{u_1,\circ})\,\mathrm{F}\|_{p,\omega}\,du_1dt_1 \\
&\lesssim \sup_{0\le u\le h}\|(\mathbb{I}-\sigma_{u,\circ})\,\mathrm{F}\|_{p,\omega}\frac{1}{h^3}\int_0^h\int_0^{t_1}u_1du_1dt_1 \\
&\lesssim \sup_{0\le u\le h}\|(\mathbb{I}-\sigma_{u,\circ})\,\mathrm{F}\|_{p,\omega}=C\sup_{0\le u\le h}\|(\mathbb{I}-\sigma_{u,\circ})(\mathbb{I}-U_{\circ,k})f\|_{p,\omega} \\
&\lesssim \sup_{0\le u\le h}\|(\mathbb{I}-U_{\circ,k})(\mathbb{I}-\sigma_{u,\circ})f\|_{p,\omega}=\sup_{0\le u\le h}\|(\mathbb{I}-U_{\circ,k})\mathfrak{F}\|_{p,\omega}.
\end{aligned}$$

By the same procedure

$$\|(\mathbb{I}-U_{\circ,k})\mathfrak{F}\|_{p,\omega}\lesssim\sup_{0\le v\le k}\|(\mathbb{I}-\sigma_{\circ,v})\mathfrak{F}\|_{p,\omega}=\sup_{0\le v\le k}\|(\mathbb{I}-\sigma_{\circ,v})(\mathbb{I}-\sigma_{u,\circ})f\|_{p,\omega}$$

and hence

$$\|f-g_1-g_2-g\|_{p,\omega}\lesssim\sup_{\substack{0\le u\le h\\0\le v\le k}}\|(\mathbb{I}-\sigma_{\circ,v})(\mathbb{I}-\sigma_{u,\circ})f\|_{p,\omega}.$$

We have

$$\begin{aligned}
\left\|\frac{\partial^2 g_1}{\partial x^2}\right\|_{p,\omega} &= \left\|\frac{\partial^2}{\partial x^2}U_{h,\circ}(\mathbb{I}-U_{\circ,k})f\right\|_{p,\omega} \\
&= \left\|\frac{\partial^2}{\partial x^2}\frac{1}{h^3}\int_0^h\int_0^{t_1}\int_{-u_1}^{u_1}[(\mathbb{I}-U_{\circ,k})f](x+s_1,y)\,ds_1du_1dt_1\right\|_{p,\omega} \\
&= \frac{2}{h^2}\|(\mathbb{I}-U_{h,\circ})(\mathbb{I}-U_{\circ,k})f\|_{p,\omega}
\end{aligned}$$

and

$$\begin{aligned}
h^2\left\|\frac{\partial^2 g_1}{\partial x^2}\right\|_{p,\omega} &= 2\|(\mathbb{I}-U_{h,\circ})(\mathbb{I}-U_{\circ,k})f\|_{p,\omega} \\
&\le 2\sup_{\substack{0\le u\le h\\0\le v\le k}}\|(\mathbb{I}-\sigma_{\circ,v})(\mathbb{I}-\sigma_{u,\circ})f\|_{p,\omega}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\left\|\frac{\partial^2 g_2}{\partial y^2}\right\|_{p,\omega} &= \left\|\frac{\partial^2}{\partial y^2}U_{\circ,k}\left(\mathbb{I}-U_{h,\circ}\right)f\right\|_{p,\omega}\\ &= \left\|\frac{\partial^2}{\partial y^2}\frac{1}{k^3}\int\limits_0^h\int\limits_0^{t_2}\int\limits_{-u_2}^{u_2}\left[\left(\mathbb{I}-U_{h,\circ}\right)f\right](x,y+s_2)\,ds_2du_2dt_2\right\|_{p,\omega}\\ &= \frac{2}{h^2}\left\|\left(\mathbb{I}-U_{\circ,k}\right)\left(\mathbb{I}-U_{h,\circ}\right)f\right\|_{p,\omega}\end{aligned}$$

and

$$\begin{aligned}k^2\left\|\frac{\partial^2 g_1}{\partial x^2}\right\|_{p,\omega} &= 2\left\|\left(\mathbb{I}-U_{\circ,k}\right)\left(\mathbb{I}-U_{h,\circ}\right)f\right\|_{p,\omega}\\ &\leq 2\sup_{\substack{0\leq u\leq h\\0\leq v\leq k}}\left\|\left(\mathbb{I}-\sigma_{\circ,v}\right)\left(\mathbb{I}-\sigma_{u,\circ}\right)f\right\|_{p,\omega}.\end{aligned}$$

Also,

$$\begin{aligned}k^2h^2\left\|\frac{\partial^4 g}{\partial x^2\partial y^2}\right\|_{p,\omega} &= k^2h^2\left\|\frac{\partial^4}{\partial y^2\partial x^2}U_{h,\circ}\left(U_{\circ,k}f\right)\right\|_{p,\omega}\\ &= k^2\left\|\frac{\partial^2}{\partial y^2}h^2\frac{\partial^2}{\partial x^2}U_{h,\circ}\left(U_{\circ,k}f\right)\right\|_{p,\omega}\\ &= k^2\left\|\frac{\partial^2}{\partial y^2}\left(\mathbb{I}-\sigma_{h,\circ}\right)\left(U_{\circ,k}f\right)\right\|_{p,\omega}\\ &= k^2\left\|\frac{\partial^2}{\partial y^2}U_{\circ,k}\left(\left(\mathbb{I}-\sigma_{h,\circ}\right)f\right)\right\|_{p,\omega}\\ &= \left\|k^2\frac{\partial^2}{\partial y^2}U_{\circ,k}\left(\left(\mathbb{I}-\sigma_{h,\circ}\right)f\right)\right\|_{p,\omega}\\ &= \left\|\left(\mathbb{I}-\sigma_{\circ,k}\right)\left(\mathbb{I}-\sigma_{h,\circ}\right)f\right\|_{p,\omega}\\ &\leq \sup_{\substack{0\leq u\leq h\\0\leq v\leq k}}\left\|\left(\mathbb{I}-\sigma_{\circ,v}\right)\left(\mathbb{I}-\sigma_{u,\circ}\right)f\right\|_{p,\omega}.\end{aligned}$$

Then

$$K(f,\delta,\xi,p,\omega,2,2)\lesssim\Omega\left(f,\delta,\xi\right)_{p,\omega}.$$

For the lower inequality

$$\begin{aligned}\Omega\left(f,\delta,\xi\right)_{p,\omega} &\leq \Omega\left(f-g_1-g_2-g,\delta,\xi\right)_{p,\omega}+\Omega\left(g_1,\delta,\xi\right)_{p,\omega}\\ &\quad+\Omega\left(g_2,\delta,\xi\right)_{p,\omega}+\Omega\left(g,\delta,\xi\right)_{p,\omega}\\ &\lesssim \left\|f-g_1-g_2-g\right\|_{p,\omega}+\delta^2\left\|\frac{\partial^2 g_1}{\partial x^2}\right\|_{p,\omega}+\xi^2\left\|\frac{\partial^2 g_2}{\partial y^2}\right\|_{p,\omega}+\delta^2\xi^2\left\|\frac{\partial^4 g}{\partial x^2\partial y^2}\right\|_{p,\omega}.\end{aligned}$$

From the last inequality we take the infimum and the lower estimate

$$\Omega\left(f,\delta,\xi\right)_{p,\omega}\lesssim K(f,\delta,\xi,p,\omega,2,2)$$

holds. □

Corollary 20 *If* $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L^p_\omega\left(\mathbb{T}^2\right),$ *then there exist constants depending only on* $[\omega]_{A_p}$ *and* p *so that*

$$\Omega\left(f,\lambda\delta,\eta\xi\right)_{p,\omega}\lesssim(1+\lambda)^2(1+\eta)^2\,\Omega\left(f,\delta,\xi\right)_{p,\omega}\quad \text{for } \delta,\xi>0$$

and

$$\frac{\Omega\left(f,\delta_1,\delta_2\right)_{p,\omega}}{\delta_1^2\delta_2^2}\lesssim\frac{\Omega\left(f,t_1,t_2\right)_{p,\omega}}{t_1^2t_2^2}\quad \text{for } 0<t_i\leq\delta_i\ ;\ i=1,2.$$

6. Potapov type direct inequality

Here we give the proof of Potapov type direct Theorem 2.

Proof of Theorem 2 For any $g_1,\ g_2,\ g$ with $g_1\in W^{r,\circ}_{p,\omega},\ g_2\in W^{\circ,s}_{p,\omega}, g\in W^{r,s}_{p,\omega}$,

$$\begin{aligned}Y_{m,n}\left(f,\delta,\xi\right)_{p,\omega} &\leq Y_{m,n}\left(f-g_1-g_2-g,\delta,\xi\right)_{p,\omega}+Y_{m,n}\left(g_1,\delta,\xi\right)_{p,\omega}\\ &\quad+Y_{m,n}\left(g_2,\delta,\xi\right)_{p,\omega}+Y_{m,n}\left(g,\delta,\xi\right)_{p,\omega}\\ &\lesssim \left\|f-g_1-g_2-g\right\|_{p,\omega}+\frac{1}{(m+1)^2}\left\|\frac{\partial^2 g_1}{\partial x^2}\right\|_{p,\omega}+\\ &\quad+\frac{1}{(n+1)^2}\left\|\frac{\partial^2 g_2}{\partial y^2}\right\|_{p,\omega}+\frac{1}{(m+1)^2}\frac{1}{(n+1)^2}\left\|\frac{\partial^4 g}{\partial x^2\partial y^2}\right\|_{p,\omega}.\end{aligned}$$

Taking the infimum on $g_1,\ g_2,\ g$ we have that the inequality

$$Y_{m,n}\left(f\right)_{p,\omega}\lesssim K(f,\frac{1}{m+1},\frac{1}{n+1},p,\omega,2,2)\lesssim\Omega\left(f,\frac{1}{m+1},\frac{1}{n+1}\right)_{p,\omega}$$

holds. □

7. Realization functional

We define the mixed realization functional as

$$R(f,m,n,p,\omega,2,2):=\left\|f-S_{m,\circ}\left(f\right)-S_{\circ,n}\left(f\right)+S_{m,n}\left(f\right)\right\|_{p,\omega}+m^{-2}\left\|\frac{\partial^2 S_{m,\circ}\left(f-S_{\circ,n}\left(f\right)\right)}{\partial x^2}\right\|_{p,\omega}+$$

$$+n^{-2}\left\|\frac{\partial^2 S_{\circ,n}\left(f-S_{m,\circ}\left(f\right)\right)}{\partial y^2}\right\|_{p,\omega}+m^{-2}n^{-2}\left\|\frac{\partial^4 S_{m,n}\left(f\right)}{\partial x^2\partial y^2}\right\|_{p,\omega},\qquad(16)$$

where $1<p<\infty,\ \omega\in A_p\left(\mathbb{T}^2,\mathbb{J}\right),\ f\in L^p_\omega\left(\mathbb{T}^2\right).$

We give the proof of Theorem 4.

Proof of Theorem 3 $\forall h,k>0$ and $m,n\in\mathbb{N}$, and we get

$$\begin{aligned}\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k} f\right)\right\|_{p,\omega} &= \left\|\nabla_{h,\circ}\left(\nabla_{\circ,k}\left[f-S_{m,\circ}(f)-S_{\circ,n}(f)+S_{m,n}(f)\right]\right)\right\|_{p,\omega}+\\ &+\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k} S_{m,\circ}\left(f-S_{\circ,n}(f)\right)\right)\right\|_{p,\omega}+\\ &+\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k}\left(S_{\circ,n}\left(f-S_{m,\circ}(f)\right)\right)\right)\right\|_{p,\omega}+\\ &+\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k} S_{m,n} f\right)\right\|_{p,\omega}:=I_1+I_2+I_3+I_4.\end{aligned}$$

Let $f(x,y)-S_{m,\circ}(f)(x,y)-S_{\circ,n}(f)(x,y)+S_{m,n}(f)(x,y)=:\varphi(x,y)$. From Lemma 5 for a.e. y

$$\left(\int_{\mathbb{T}}\left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right)\varphi(x,y)\right|^p\omega(x,y)\,dx\right)^{1/p}\lesssim\left(\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dx\right)^{1/p}.$$

Then

$$\int_{\mathbb{T}}\left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right)\varphi(x,y)\right|^p\omega(x,y)\,dx\lesssim\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dx$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right)\varphi(x,y)\right|^p\omega(x,y)\,dxdy\lesssim\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dxdy.$$

Hence

$$I_1=\left\|\nabla_{h,\circ}\left(\nabla_{\circ,k}\varphi(x,y)\right)\right\|_{p,\omega}\lesssim\left\|\nabla_{\circ,k}\varphi(x,y)\right\|_{p,\omega}.$$

For a.e. x

$$\left(\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dy\right)^{1/p}\lesssim\left(\int_{\mathbb{T}}|\varphi(x,y)|^p\omega(x,y)\,dy\right)^{1/p}.$$

Then

$$\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dy\lesssim\int_{\mathbb{T}}|\varphi(x,y)|^p\omega(x,y)\,dy$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\nabla_{\circ,k}\varphi(x,y)\right|^p\omega(x,y)\,dydx\lesssim\int_{\mathbb{T}}\int_{\mathbb{T}}|\varphi(x,y)|^p\omega(x,y)\,dydx.$$

Hence

$$\begin{aligned}\left\|\nabla_{\circ,k}\varphi(x,y)\right\|_{p,\omega} &\lesssim \|\varphi(x,y)\|_{p,\omega}\\ &= \left\|f-S_{m,\circ}(f)-S_{\circ,n}(f)+S_{m,n}(f)\right\|_{p,\omega}.\end{aligned}$$

Let $f(x,y) - S_{\circ,n}(f)(x,y) =: \psi(x,y)$. From Lemma 5 for a.e. x

$$\left(\int_{\mathbb{T}} \left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right) S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dy\right)^{1/p} \lesssim \left(\int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dy\right)^{1/p}.$$

Then

$$\int_{\mathbb{T}} \left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right) S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dy \lesssim \int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dy$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left|\nabla_{h,\circ}\left(\nabla_{\circ,k}\right) S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dydx \lesssim \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dxdy.$$

Hence

$$I_2 = \left\|\nabla_{h,\circ}\left(\nabla_{\circ,k} S_{m,\circ}(\psi)\right)\right\|_{p,\omega} \lesssim \left\|\nabla_{\circ,k} S_{m,\circ}(\psi)\right\|_{p,\omega}.$$

Using Nikolskii–Stechkin type one-dimensional inequality (see [1, Lemma 1, p. 70])

$$\left(\int_{\mathbb{T}} \left|\nabla_{h} T_m(x)\right|^p \omega(x)\,dx\right)^{1/p} \lesssim \frac{1}{m^2}\left(\int_{\mathbb{T}} \left|\frac{d^2}{dx^2} T_m(x)\right|^p \omega(x)\,dx\right)^{1/p}, 0 < h < \frac{1}{m},$$

we have for a.e. y and $0 < h < \frac{1}{m}$

$$\left(\int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dx\right)^{1/p} \lesssim \frac{1}{m^2}\left(\int_{\mathbb{T}} \left|S_{m,\circ}^{(2,\circ)}(\psi)(x,y)\right|^p \omega(x,y)\,dx\right)^{1/p}.$$

Then

$$\int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dx \lesssim \frac{1}{m^{2p}} \int_{\mathbb{T}} \left|S_{m,\circ}^{(2,\circ)}(\psi)(x,y)\right|^p \omega(x,y)\,dx$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left|\nabla_{h,\circ} S_{m,\circ}(\psi)(x,y)\right|^p \omega(x,y)\,dxdy \lesssim \frac{1}{m^{2p}} \int_{\mathbb{T}}\int_{\mathbb{T}} \left|S_{m,\circ}^{(2,\circ)}(\psi)(x,y)\right|^p \omega(x,y)\,dxdy.$$

Hence

$$\left\|\nabla_{h,\circ} S_{m,\circ}(\psi)\right\|_{p,\omega} \lesssim \frac{1}{m^2}\left\|S_{m,\circ}^{(2,\circ)}(\psi)\right\|_{p,\omega}.$$

From this we get

$$I_2 \lesssim \frac{1}{m^2}\left\|S_{m,\circ}^{(2,\circ)}\left(f - S_{\circ,n}(f)\right)\right\|_{p,\omega}.$$

By a similar method we can find

$$I_3 \lesssim \frac{1}{n^2}\left\|S_{\circ,n}^{(\circ,2)}\left(f - S_{m,\circ}(f)\right)\right\|_{p,\omega}, 0 < k < \frac{1}{n}$$

and

$$I_3 \lesssim \frac{1}{m^2}\frac{1}{n^2}\left\|S_{m,n}^{(2,2)}(f)\right\|_{p,\omega}, 0<k<\frac{1}{n}, 0<h<\frac{1}{m}.$$

To sum up the obtained estimates we have $\Omega\left(f,m^{-1},n^{-1}\right)_{p,\omega} \lesssim R(f,m,n,p,\omega,2,2)$.

For the inverse of the last inequality we use Theorem 2 to obtain

$$\begin{aligned} A_1 &= \left\|f-S_{m,\circ}(f)-S_{\circ,n}(f)+S_{m,n}(f)\right\|_{p,\omega} \\ &\lesssim Y_{m,n}(f)_{p,\omega} \lesssim \Omega\left(f,m^{-1},n^{-1}\right)_{p,\omega}. \end{aligned}$$

Set $A_2=\left\|S_{m,\circ}^{(2,\circ)}\left(f-S_{\circ,n}(f)\right)\right\|_{p,\omega}$ and $\gamma(x,y)=f(x,y)-S_{\circ,n}(f)(x,y)$. We know that the one-dimensional inequality (see [1, Lemma 2, p. 72])

$$\left(\int_{\mathbb{T}}\left|\frac{d^2}{dx^2}T_m(x)\right|^p\omega(x)\,dx\right)^{1/p} \lesssim m^2\left(\int_{\mathbb{T}}\left|\nabla_{\frac{1}{m}}T_m(x)\right|^p\omega(x)\,dx\right)^{1/p}.$$

For a.e. y

$$\left(\int_{\mathbb{T}}\left|\frac{d^2}{dy^2}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dx\right)^{1/p} \lesssim m^2\left(\int_{\mathbb{T}}\left|\nabla_{\frac{1}{m},\circ}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dx\right)^{1/p}.$$

Then

$$\int_{\mathbb{T}}\left|\frac{d^2}{dy^2}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dx \lesssim m^{2p}\int_{\mathbb{T}}\left|\nabla_{\frac{1}{m},\circ}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dx$$

and

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\frac{d^2}{dy^2}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dxdy \lesssim m^{2p}\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\nabla_{\frac{1}{m},\circ}S_{m,\circ}(\gamma)\right|^p\omega(x,y)\,dxdy.$$

Hence

$$\begin{aligned}
A_2 &\lesssim m^2 \left\| \nabla_{\frac{1}{m},\circ} S_{m,\circ}(\gamma) \right\|_{p,\omega} = m^2 \left\| S_{m,\circ}\left(\nabla_{\frac{1}{m},\circ} \gamma \right) \right\|_{p,\omega} \\
&\lesssim m^2 \left\| \nabla_{\frac{1}{m},\circ} \gamma \right\|_{p,\omega} = m^2 \left\| \nabla_{\frac{1}{m},\circ} \left(f - S_{\circ,n}(f) \right) \right\|_{p,\omega} \\
&= m^2 \left\| \nabla_{\frac{1}{m},\circ} f - \nabla_{\frac{1}{m},\circ} S_{\circ,n}(f) \right\|_{p,\omega} \\
&= m^2 \left\| \nabla_{\frac{1}{m},\circ} f - S_{0,\circ}\left(\nabla_{\frac{1}{m},\circ} f \right) - S_{\circ,n}\left(\nabla_{\frac{1}{m},\circ} f \right) + S_{0,n}\left(\nabla_{\frac{1}{m},\circ} f \right) \right\|_{p,\omega} \\
&\lesssim m^2 Y_{0,n}\left(\nabla_{\frac{1}{m},\circ} f \right)_{p,\omega} \lesssim m^2 \Omega\left(f, 1, \frac{1}{n+1} \right)_{p,\omega} \\
&= m^2 \sup_{\substack{0\le h\le 1 \\ 0\le k\le 1/n}} \left\| \nabla_{h,\circ} \nabla_{\circ,k} \left(\nabla_{\frac{1}{m},\circ} f \right) \right\|_{p,\omega} \\
&\lesssim m^2 \sup_{0\le k\le 1/n} \left\| \nabla_{\circ,k} \left(\nabla_{\frac{1}{m},\circ} f \right) \right\|_{p,\omega} \\
&\lesssim m^2 \sup_{\substack{0\le h\le 1/m \\ 0\le k\le 1/n}} \left\| \nabla_{h,\circ} \left(\nabla_{\circ,k} f \right) \right\|_{p,\omega} = m^2 \Omega\left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}.
\end{aligned}$$

Similarly

$$\begin{aligned}
A_3 &= \left\| S_{\circ,n}^{(\circ,2)}\left(f - S_{m,\circ}(f) \right) \right\|_{p,\omega} \lesssim n^2 \Omega\left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}, \\
A_4 &= \left\| S_{m,n}^{(2,2)}(f) \right\|_{p,\omega} \lesssim m^2 n^2 \Omega\left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}.
\end{aligned}$$

Then $R(f,m,n,p,\omega,2,2) \lesssim \Omega\left(f, m^{-1}, n^{-1}\right)_{p,\omega}$ and $\Omega\left(f, m^{-1}, n^{-1}\right)_{p,\omega} \approx R(f,m,n,p,\omega,2,2)$. □

8. Inverse estimate

Proof of Theorem 4 We have

$$\Omega\left(f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \le \Omega\left(f - W_{2^\mu,2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} + \Omega\left(W_{2^\mu,2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega}$$

and

$$\Omega\left(f - W_{2^\mu,2^\nu} f, \frac{1}{m}, \frac{1}{n} \right)_{p,\omega} \lesssim \left\| f - W_{2^\mu,2^\nu} \right\|_{p,\omega} \lesssim Y_{2^\mu,2^\nu}(f)_{p,\omega}.$$

Since

$$W_{2^\mu,2^\nu}f - W_{0,0}f \leq \sum_{i=0}^{\mu}\left(W_{2^i,2^\nu}f - W_{\lfloor 2^{i-1}\rfloor,2^\nu}f\right) + \sum_{j=0}^{\nu}\left(W_{2^\mu,2^j}f - W_{2^\mu,\lfloor 2^{j-1}\rfloor}f\right)$$

$$-\sum_{i=0}^{\mu}\sum_{j=0}^{\nu} W_{2^i,2^j}f - W_{2^i,\lfloor 2^{j-1}\rfloor}f - W_{\lfloor 2^{i-1}\rfloor,2^j}f + W_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}f,$$

$$\begin{aligned}
\Omega\left(W_{2^\mu,2^\nu}f,\frac{1}{m},\frac{1}{n}\right)_{p,\omega} &= \Omega\left(W_{2^\mu,2^\nu}f - W_{0,0}f,\frac{1}{m},\frac{1}{n}\right)_{p,\omega} \\
&\leq \sum_{i=0}^{\mu}\Omega\left(\psi_{i,\nu}(f),\frac{1}{m},\frac{1}{n}\right)_{p,\omega} + \sum_{j=0}^{\nu}\Omega\left(h_{\mu,j}(f),\frac{1}{m},\frac{1}{n}\right)_{p,\omega} \\
&\quad + \sum_{i=0}^{\mu}\sum_{j=0}^{\nu}\Omega\left(\varphi_{i,j}(f),\frac{1}{m},\frac{1}{n}\right)_{p,\omega} \\
&\lesssim \frac{1}{m^2}\sum_{i=0}^{\mu}\left\|(\psi_{i,\nu}(f))^{(2,\circ)}\right\|_{p,\omega} + \frac{1}{n^2}\sum_{j=0}^{\nu}\left\|(h_{\mu,j}(f))^{(\circ,2)}\right\|_{p,\omega} \\
&\quad + \frac{1}{m^2}\frac{1}{n^2}\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}\left\|(\varphi_{i,j}(f))^{(2,2)}\right\|_{p,\omega}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \frac{1}{m^2}\sum_{i=0}^{\mu}2^{2i}Y_{\lfloor 2^{i-1}\rfloor,2^j}(f)_{p,\omega} + \frac{1}{n^2}\sum_{j=0}^{\nu}2^{2j}Y_{2^i,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega} \\
&\quad + \frac{1}{m^2}\frac{1}{n^2}\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}2^{2i+2j}Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}.
\end{aligned}$$

Taking $2^\mu \leq m < 2^{\mu+1}, 2^\nu \leq n < 2^{\nu+1}$

$$\Omega\left(f,\frac{1}{m},\frac{1}{n}\right)_{p,\omega} \lesssim \frac{1}{m^2}\frac{1}{n^2}\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}2^{2i+2j}Y_{\lfloor 2^{i-1}\rfloor,\lfloor 2^{j-1}\rfloor}(f)_{p,\omega}$$

and the inequality

$$\Omega\left(f,\frac{1}{m},\frac{1}{n}\right)_{p,\omega} \lesssim \frac{1}{m^2n^2}\sum_{i=0}^{m}\sum_{j=0}^{n}(i+1)(j+1)Y_{i,j}(f)_{p,\omega}$$

holds. □

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