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Chen inequalities for submanifolds of generalized space forms with a semi-symmetric metric connection

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Abstract

We investigate sharp inequalities for submanifolds in both generalized complex space forms and generalized Sasakian space forms with a semi-symmetric metric connection.

Keywords: Chen inequality, generalized complex space form, generalized Sasakian space form, semi-symmetric metric connection.

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1. Introduction

A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection on a Riemannian manifold in [10]. Later, H. A. Hayden [11] gave the definition of a semi-symmetric metric connection. In 1970, K. Yano [19] studied semi-symmetric metric connection and proved that a Rimannian manifold admits a semi-symmetric metric connection with vanishing curvature tensor if and only if the manifold is conformally flat. Then, in [12], [13] and [16] T. Imai and Z. Nakao considered some properties of a Riemannian manifold admitting a semi-symmetric metric connection and they studied submanifolds of a Riemannian manifold with a semi-symmetric metric connection.

On the other hand, B. Y. Chen introduced *Chen inequality* and he gave the definition of new types of curvature invariants (called extrinsic and intrinsic invariants) in [6]. Then,

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in [7], [8] and [9], he established sharp inequalities for different submanifolds in various ambient spaces.

In [3] and [4], K. Arslan, R. Ezentaş, I. Mihai, C. Murathan and C. Özgür studied Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds and (κ, μ)-contact space forms, respectively. Later, P. Alegre, A. Carriazo, Y. H. Kim and D. W. Yoon considered same inequalities for submanifolds of generalized space forms in [2].

Recently, in [14], A. Mihai and C. Özgür proved Chen inequalities for submanifolds of real space forms admitting a semi-symmetric metric connection. They also studied same problems for submanifolds of complex space forms and Sasakian space forms with a semi-symmetric metric connection in [15]. As a generalization of the results of [15], in tis study, we prove similar inequalities for submanifolds of generalized complex space forms and generalized Sasakian space forms with respect to a semi-symmetric metric connection.

2. Preliminaries

Let N be an $(n+p)$-dimensional Riemannian manifold with a Riemannian metric g. A linear connection $\widetilde{\nabla}$ on a Riemannian manifold N is called a *semi-symmetric connection* if the torsion tensor $\widetilde{T}$ of the connection $\widetilde{\nabla}$

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}] \tag{2.1}$$

satisfies

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = w(\widetilde{Y})\widetilde{X} - w(\widetilde{X})\widetilde{Y}, \tag{2.2}$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N, where w is a 1-form associated with the vector field U on N defined by

$$w(\widetilde{X}) = g(\widetilde{X}, U). \tag{2.3}$$

$\widetilde{\nabla}$ is called a *semi-symmetric metric connection* if

$$\widetilde{\nabla} g = 0.$$

If $\overset{\circ}{\widetilde{\nabla}}$ is the Levi-Civita connection of a Riemannian manifold N, the semi-symmetric metric connection $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \overset{\circ}{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + w(\widetilde{Y})\widetilde{X} - g(\widetilde{X},\widetilde{Y})U, \tag{2.4}$$

(see [19]).

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N. We will consider the induced semi-symmetric metric connection by ∇ and the induced Levi-Civita connection by $\overset{\circ}{\nabla}$ on the submanifold M.

Let $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ be curvature tensors of $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$ of a Riemannian manifold N, respectively. We also denote by R the curvature tensor of M with respect to ∇ and $\overset{\circ}{R}$ the curvature tensor of M with respect to $\overset{\circ}{\nabla}$. Then the Gauss formulas with a semi-symmetric metric connection ∇ and the Levi-Civita connection $\overset{\circ}{\nabla}$, respectively, are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$$

and

$$\overset{\circ}{\widetilde{\nabla}}_X Y = \overset{\circ}{\nabla}_X Y + \overset{\circ}{\sigma}(X,Y),$$

for any vector fields X, Y tangent to M, where $\overset{\circ}{\sigma}$ is the second fundamental form of M in N and σ is a $(0,2)$-tensor on M. Also, the mean curvature vector of M in N is denoted by $\overset{\circ}{H}$.

The equation of Gauss for an n-dimensional submanifold M in an $(n+p)$-dimensional Riemannian manifold N is given by

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = \overset{\circ}{R}(X,Y,Z,W) + g(\overset{\circ}{\sigma}(X,Z), \overset{\circ}{\sigma}(Y,W)) - g(\overset{\circ}{\sigma}(Y,Z), \overset{\circ}{\sigma}(X,W)) \tag{2.5}$$

Then, $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ are related by

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) \;=\; & \overset{\circ}{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \\ & -\alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z), \end{aligned} \tag{2.6}$$

for any vector fields X, Y, Z, W on N [19], where $(0,2)$-tensor field α is given by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}} w\right) Y - w(X)w(Y) + \frac{1}{2} w(U) g(X,Y),$$

for $X, Y \in \chi(M)$, where the trace of α is denoted by

$$trace\alpha = \lambda.$$

Denote by $K(\pi)$ or $K(u,v)$ the sectional curvature of M associated with a 2-plane section $\pi \subset T_x M$ with respect to the induced semi-symmetric non-metric connection ∇, where $\{u, v\}$ is an orthonormal basis of π. The scalar curvature τ at $x \in M$ is denoted by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $\{e_1, ..., e_n\}$ is any orthonormal basis of $T_x M$ [8].

We will need the following Chen's lemma for later use:

2.1. Lemma. [6] Let $n \ge 2$ and $a_1, a_2, ..., a_n, b$ be real numbers such that

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right). \tag{2.7}$$

Then $2a_1 a_2 \ge b$, with equality holding if and only if

$$a_1 + a_2 = a_3 = ... = a_n.$$

Let M be an n-dimensional Riemannian manifold, L a k-plane section of T_xM, $x \in M$ and X a unit vector in L.

For an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$, the *Ricci curvature* (or k-Ricci curvature) of L at X is defined by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j. For any integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k of M is denoted by

$$\Theta_k(x) = \tfrac{1}{k-1} \inf_{L,X} Ric_L(X), \quad x \in M,$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.

3. Chen inequality for submanifolds of generalized complex space forms

We consider as an ambient space a generalized complex space form with a semi-symmetric metric connection.

A $2m$-dimensional almost Hermitian manifold (N, J, g) is said to be a *generalized complex space form* (see [17] and [18]) if there exist two functions F_1 and F_2 on N such that

$$\overset{\circ}{\widetilde{R}}(X,Y,Z,W) = F_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+ \tag{3.1}$$
$$+F_2[g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)],$$

for any vector fields X, Y, Z, W on N, where $\overset{\circ}{\widetilde{R}}$ is the curvature tensor of N with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. In such a case, we will write $N(F_1, F_2)$.

If $N(F_1, F_2)$ is a generalized complex space form with a semi-symmetric metric connection $\widetilde{\nabla}$, then by the use of (2.6) and (3.1), the curvature tensor $\widetilde{R}$ of $N(F_1, F_2)$ can be written as

$$\widetilde{R}(X,Y,Z,W) = F_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]+ \tag{3.2}$$
$$+F_2[g(X,JZ)g(JY,W) - g(Y,JZ)g(JX,W) + 2g(X,JY)g(JZ,W)]-$$
$$-\alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).$$

Let M be an n-dimensional, $n \geq 3$, submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$. We put

$$JX = PX + FX,$$

for any vector field X tangent to M, where PX and FX are tangential and normal components of JX, respectively.

We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j).$$

On the other hand, $\Theta^2(\pi)$ is denoted by $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(Je_1, e_2)$ in [2], where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in $[0, 1]$, independent of the choice of e_1 and e_2.

For submanifolds of generalized complex space forms with respect to the semi-symmetric metric connection we establish the following sharp inequality:

3.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\begin{aligned} \tau(x) - K(\pi) \quad &\leq \quad \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)F_1 - 2\lambda\right] - \\ &- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{F_2}{2} - trace(\alpha_{|\pi^\perp}), \end{aligned} \tag{3.3}$$

where π is a 2-plane section of T_xM, $x \in M$.

Proof. Let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m}\}$ be an orthonormal basis of $T_x^\perp M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

Taking $X = W = e_i$ and $Y = Z = e_j$ such that $i \neq j$ and by the use of (3.2), we get

$$\widetilde{R}(e_i, e_j, e_j, e_i) = F_1 + 3F_2 g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{3.4}$$

From [16], the Gauss equation with respect to the semi-symmetric metric connection can be written as

$$\widetilde{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \tag{3.5}$$

Comparing the right hand sides of the equations (3.4) and (3.5), we obtain

$$\begin{aligned} &F_1 + 3F_2 g^2(Je_i, e_j) - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ = \quad &R(e_i, e_j, e_j, e_i) + g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - g(\sigma(e_i, e_i), \sigma(e_j, e_j)). \end{aligned}$$

Then, by summation over $1 \leq i, j \leq n$, the above equation turns into

$$\begin{aligned} &2\tau + \|\sigma\|^2 - n^2\|H\|^2 \\ = \quad &n(n-1)F_1 + 3F_2\sum_{i,j=1}^{n} g^2(Je_i, e_j) - 2(n-1)\lambda, \end{aligned} \tag{3.6}$$

where

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j))$$

and

$$H = \frac{1}{n} trace\sigma.$$

We set

$$\delta = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2\|P\|^2. \tag{3.7}$$

Then, the equation (3.6) can be written as follows

$$n^2 \|H\|^2 = (n-1)\left(\|\sigma\|^2 + \delta\right). \tag{3.8}$$

For a chosen orthonormal basis, the relation (3.8) takes the following form

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i=1}^{n} (\sigma_{ii}^{n+1})^2 + \sum_{i\neq j} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (\sigma_{ij}^{r})^2 + \delta\right].$$

So, by the use of Chen's Lemma, we have

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} = \sum_{1\leq i\neq j\leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (\sigma_{ij}^{r})^2 + \delta.$$

Let π be a 2-plane section of T_xM at a point x, where $\pi = sp\{e_1, e_2\}$. Then, the Gauss equation for $X = Z = e_1$ and $Y = W = e_2$ gives us

$$\begin{aligned}
K(\pi) &= F_1 + 3F_2 g^2(Je_1, e_2) - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \sum_{r=n+1}^{2m} [\sigma_{11}^r \sigma_{22}^r - (\sigma_{12}^r)^2] \geq \\
&\geq F_1 + 3F_2 g^2(Je_1, e_2) - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \\
&+ \frac{1}{2}\left(\sum_{1\leq i\neq j\leq n} (\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n} (\sigma_{ij}^{r})^2 + \delta\right) + \sum_{r=n+2}^{2m} \sigma_{11}^r\sigma_{22}^r - \sum_{r=n+1}^{2m} (\sigma_{12}^r)^2 \\
&= F_1 + 3F_2 g^2(Je_1, e_2) - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \\
&+ \frac{1}{2}\sum_{1\leq i\neq j\leq n} (\sigma_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{i,j>2}^{n} (\sigma_{ij}^{r})^2 + \\
&+ \frac{1}{2}\sum_{r=n+2}^{2m} (\sigma_{11}^r + \sigma_{22}^r)^2 + \sum_{j>2} [(\sigma_{1j}^{n+1})^2 + (\sigma_{2j}^{n+1})^2] + \frac{1}{2}\delta \geq \\
&\geq F_1 + 3F_2 g^2(Je_1, e_2) - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\delta
\end{aligned}$$

which implies

$$K(\pi) \geq F_1 + 3F_2 g^2(Je_1, e_2) - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\delta.$$

From (3.7), it is easy to see that

$$\begin{aligned}
K(\pi) \quad \geq \quad & \tau - \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)F_1 - 2\lambda\right] + \\
& + \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{F_2}{2} + trace(\alpha_{|\pi^\perp}),
\end{aligned}$$

where $trace(\alpha_{|\pi^\perp})$ is denoted by

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) = \lambda - trace(\alpha_{|\pi^\perp})$$

(see [15]). Hence, we finish the proof of the theorem. ∎

3.2. Proposition. The mean curvature H of M admitting semi-symmetric metric connection coincides with the mean curvature $\overset{\circ}{H}$ of M admitting Levi-Civita connection if and only if the vector field U is tangent to M.

As a consequence of Proposition 3.2 we can give the following result:

3.3. Theorem. If the vector field U is tangent to M, then the equality case of (3.3) holds at a point $x \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, ...e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, ..., e_{2m}\}$ of $T_x^{\perp}M$ such that the shape operators of M in $N(F_1, F_2)$ at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu$$

and

$$A_{e_r} = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \cdots & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n + 2 \leq i \leq 2m,$$

where we denote by $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n + 2 \leq r \leq 2m$.

Proof. Equality case holds at a point $x \in M$ if and only if the equality holds in each of the previous inequalities and hence the Lemma yields equality.

$$\sigma_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$\sigma_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n + 1, ..., 2m,$$

$$\sigma_{11}^r + \sigma_{22}^r = 0, \quad \forall r = n + 2, ..., 2m,$$

$$\sigma_{1j}^{n+1} = \sigma_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}.$$

If we choose $\{e_1, e_2\}$ such that $\sigma_{12}^{n+1} = 0$ and denote by $a = \sigma_{11}^r$, $b = \sigma_{22}^r$, $\mu = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}$, then the shape operators take the desired forms. ∎

4. Ricci curvature for submanifolds of generalized complex space forms

In this section we establish relationship between the Ricci curvature of a submanifold M in a generalized complex space form $N(F_1, F_2)$ with a semi-symmetric metric connection, and the squared mean curvature $\|H\|^2$.

Now, let begin with the following theorem:

4.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)} \|P\|^2 . \tag{4.1}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m}\}$ be an orthonormal basis of $T_x^{\perp}M$ at $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

Then, the equation (3.7) can be written as follows

$$n^2 \|H\|^2 = 2\tau + \|\sigma\|^2 + 2(n-1)\lambda - n(n-1)F_1 - 3F_2 \|P\|^2 . \tag{4.2}$$

For a choosen orthonormal basis, let $e_1, e_2, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then, the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and

$$A_{e_r} = (\sigma_{ij}^r), \quad i,j = 1, ..., n; \quad r = n+2, ..., 2m, \quad traceA_{e_r} = 0.$$

By the use of (4.2), we obtain

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \\ &\quad +2(n-1)\lambda - n(n-1)F_1 - 3F_2 \|P\|^2 . \end{aligned} \tag{4.3}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{i<j} a_i a_j,$$

we get

$$n^2 \|H\|^2 = \left(\sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2,$$

which means

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 . \tag{4.4}$$

Thus, in view of (4.4) in (4.3) we get (4.1), which completes the proof of the theorem. ∎

In view of Theorem 4.1, we can give the following theorem:

4.2. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $2m$-dimensional generalized complex space form $N(F_1, F_2)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M. Then, for any integer k, $2 \leq k \leq n$ and for any point $x \in M$, we have:

$$\|H\|^2 (x) \geq \Theta_k(\pi) + \frac{2}{n}\lambda - F_1 - \frac{3F_2}{n(n-1)} \|P\|^2 . \tag{4.5}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of T_xM at $x \in M$. The k-plane section spanned by $e_{i_1}, ..., e_{i_k}$ is denoted by $L_{i_1 \ldots i_k}$. Then, by the definitions, we can write

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1 \ldots i_k\}} Ric_{L_{i_1 \ldots i_k}}(e_i) \tag{4.6}$$

and

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \le i_1 \le \ldots \le i_k \le n} \tau(L_{i_1 \ldots i_k}). \tag{4.7}$$

By making use of (4.6) and (4.7) in (4.1), we obtain

$$\tau(x) \ge \frac{n(n-1)}{2} \Theta_k(\pi),$$

which gives us (4.5). ∎

5. Chen inequality for submanifolds of generalized Sasakian space forms

Let N be a $(2m+1)$-dimensional *almost contact metric manifold* [5] with an almost contact metric structure (φ, ξ, η, g) consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g on N satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

for all vector fields X, Y on N. Such a manifold is said to be a *contact metric manifold* if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental* 2*-form* of N [5].

On the other hand, the almost contact metric structure of N is said to be *normal* if

$$[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi,$$

for any vector fields X, Y on N, where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a *Sasakian manifold* [5].

Given an almost contact metric manifold N with an almost contact metric structure (φ, ξ, η, g), N is called a *generalized Sasakian space form* [1] if there exist three functions f_1, f_2 and f_3 on N such that

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(X, Y, Z, W) = {} & f_1\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}+ \\ & +f_2\{g(X, \varphi Z)g(\varphi Y, W) - g(Y, \varphi Z)g(\varphi X, W) + 2g(X, \varphi Y)g(\varphi Z, W)\}+ \\ & +f_3\{\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z)\}, \end{aligned} \tag{5.1}$$

for any vector fields X, Y, Z, W on N, where $\overset{\circ}{\widetilde{R}}$ denotes the curvature tensor of N with respect to the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. In such a case, we will write $N(f_1, f_2, f_3)$. If $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, then N is a Sasakian space form.

If $N(f_1, f_2, f_3)$ is a $(2m+1)$-dimensional generalized Sasakian space form with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then, from (2.6) and (5.1) the curvature tensor $\widetilde{R}$ of $N(f_1, f_2, f_3)$ can be written as follows

$$
\begin{aligned}
\widetilde{R}(X,Y,Z,W) &= f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}+ \\
&+f_2\{g(X,\varphi Z)g(\varphi Y,W) - g(Y,\varphi Z)g(\varphi X,W) + 2g(X,\varphi Y)g(\varphi Z,W)\}+ \\
&+f_3\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z)\}- \\
&-\alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z).
\end{aligned} \tag{5.2}
$$

Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form. We put

$$\varphi X = PX + FX,$$

for any vector field X tangent to M, where PX and FX are tangential and normal components of φX, respectively.

We also set

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\varphi e_i, e_j).$$

Decompose

$$\xi = \xi^\top + \xi^\perp,$$

where $\xi^\top$ and $\xi^\perp$ denote the tangential and normal components of ξ.

From [2], recall $\Theta^2(\pi) = g^2(Pe_1, e_2) = g^2(\varphi e_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π, is a real number in $[0, 1]$, independent of the choice of e_1 and e_2.

Now, let begin with the following theorem which gives us a sharp inequality for submanifolds of generalized Sasakian space forms with respect to the semi-symmetric metric connection:

5.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$
\begin{aligned}
\tau(x) - K(\pi) \quad &\leq \quad (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\
&- \left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)\left\|\xi^\top\right\|^2]f_3 - \\
&-trace(\alpha_{|\pi^\perp}),
\end{aligned} \tag{5.3}
$$

where π is a 2-plane section of T_xM, $x \in M$.

Proof. Let $\{e_1, e_2, ...e_n\}$ be an orthonormal basis of T_xM and $\{e_{n+1}, ..., e_{2m+1}\}$ be an orthonormal basis of $T_x^\perp M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H.

For $X=W=e_i$ and $Y=Z=e_j$ such that $i\neq j$, the equation (5.2) can be written as

$$\widetilde{R}(e_i,e_j,e_j,e_i)=f_1+3f_2g^2(\varphi e_i,e_j)-f_3[\eta(e_i)^2+\eta(e_j)^2]-\alpha(e_1,e_1)-\alpha(e_2,e_2). \tag{5.4}$$

Comparing the right hand sides of the equations (3.5) and (5.4) we can write

$$\begin{aligned}&f_1+3f_2g^2(\varphi e_i,e_j)-f_3[\eta(e_i)^2+\eta(e_j)^2]-\alpha(e_1,e_1)-\alpha(e_2,e_2)\\ =\;&R(e_i,e_j,e_j,e_i)+g(\sigma(e_i,e_j),\sigma(e_i,e_j))-g(\sigma(e_i,e_i),\sigma(e_j,e_j)).\end{aligned}$$

Then, by summation over $1\leq i,j\leq n$, the above relation reduces to

$$2\tau+\|\sigma\|^2-n^2\|H\|^2=n(n-1)f_1+3f_2\|P\|^2-2(n-1)f_3\left\|\xi^\top\right\|^2-2(n-1)\lambda. \tag{5.5}$$

If we put

$$\delta=2\tau-\frac{n^2(n-2)}{n-1}\|H\|^2+2(n-1)\lambda-n(n-1)f_1-3f_2\|P\|^2+2(n-1)f_3\left\|\xi^\top\right\|^2, \tag{5.6}$$

the equation (5.5) turns into

$$n^2\|H\|^2=(n-1)\left(\|\sigma\|^2+\delta\right). \tag{5.7}$$

For a chosen orthonormal basis, the relation (5.7) takes the following form

$$\left(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\right)^2=(n-1)\left[\sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2+\sum_{i\neq j}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2+\delta\right].$$

So, by the use of Chen's Lemma, we have

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1}=\sum_{1\leq i\neq j\leq n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2+\delta.$$

Let π be a 2-plane section of T_xM at a point x, where $\pi=sp\{e_1,e_2\}$. We need to denote $\xi_\pi=pr_\pi\xi$ for the later use as follows

$$\|\xi_\pi\|^2=\eta(e_1)^2+\eta(e_2)^2.$$

Then, from the Gauss equation for $X=Z=e_1$ and $Y=W=e_2$ we get

$$\begin{aligned}K(\pi)&=f_1+3f_2g^2(Pe_1,e_2)-f_3\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\sum_{r=n+1}^{2m+1}[\sigma_{11}^r\sigma_{22}^r-(\sigma_{12}^r)^2]\geq\\
&\geq f_1+3f_2g^2(Pe_1,e_2)-f_3\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\\
&+\frac{1}{2}\left(\sum_{1\leq i\neq j\leq n}(\sigma_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2+\delta\right)+\sum_{r=n+2}^{2m+1}\sigma_{11}^r\sigma_{22}^r-\sum_{r=n+1}^{2m+1}(\sigma_{12}^r)^2\\
&=f_1+3f_2g^2(Pe_1,e_2)-f_3\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\\
&+\frac{1}{2}\sum_{1\leq i\neq j\leq n}(\sigma_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}^{n}(\sigma_{ij}^r)^2+\\
&+\frac{1}{2}\sum_{r=n+2}^{2m+1}(\sigma_{11}^r+\sigma_{22}^r)^2+\sum_{j>2}[(\sigma_{1j}^{n+1})^2+(\sigma_{2j}^{n+1})^2]+\frac{1}{2}\delta\geq\\
&\geq f_1+3f_2g^2(Pe_1,e_2)-f_3\|\xi_\pi\|^2-\alpha(e_1,e_1)-\alpha(e_2,e_2)+\frac{1}{2}\delta,\end{aligned}$$

which implies

$$K(\pi) \geq f_1 + 3f_2 g^2(Pe_1, e_2) - f_3 \|\xi_\pi\|^2 - \alpha(e_1, e_1) - \alpha(e_2, e_2) + \frac{1}{2}\delta.$$

From (5.6), it easy to see that

$$\begin{aligned} K(\pi) \quad \geq \quad & \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ & -\left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} - [\|\xi_\pi\|^2 - (n-1)\left\|\xi^\top\right\|^2]f_3 + \\ & + trace(\alpha_{|\pi^\perp}), \end{aligned}$$

which gives us (5.3). Hence, we complete the proof of the theorem. ∎

5.2. Corollary. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$.

If the structure vector field ξ is tangent to M, we have

$$\begin{aligned} \tau(x) - K(\pi) \quad \leq \quad & (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ & -\left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)]f_3 - \\ & - trace(\alpha_{|\pi^\perp}). \end{aligned} \tag{5.8}$$

If the structure vector field ξ is normal to M, we have

$$\begin{aligned} \tau(x) - K(\pi) \quad \leq \quad & (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ & -\left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} - trace(\alpha_{|\pi^\perp}). \end{aligned} \tag{5.9}$$

As a consequence of Proposition 3.2, for both submanifolds of generalized Sasakian space forms, we can give the following corollary:

5.3. Corollary. Under the same assumptions as in the Theorem 5.1, if the vector field U is tangent to M, then we have:

$$\begin{aligned} \tau(x) - K(\pi) \quad \leq \quad & (n-2)\left[\frac{n^2}{2(n-1)}\left\|\overset{\circ}{H}\right\|^2 + (n+1)\frac{f_1}{2} - \lambda\right] - \\ & -\left[6\Theta^2(\pi) - 3\|P\|^2\right]\frac{f_2}{2} + [\|\xi_\pi\|^2 - (n-1)]f_3 - \\ & - trace(\alpha_{|\pi^\perp}). \end{aligned}$$

5.4. Theorem. The equality case of (5.3) holds at a point $x \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, ...e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1}, ..., e_{2m+1}\}$ of $T_x^\perp M$ such that the shape operators of M in $N(f_1, f_2, f_3)$ at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu$$

and

$$A_{e_r} = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 & \cdots & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad n+2 \leq i \leq 2m+1,$$

where we denote by $\sigma_{ij}^r = g(\sigma(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+2 \leq r \leq 2m+1$.

Proof. Equality case holds at a point $x \in M$ if and only if the equality holds in each of the previous inequalities and hence the Lemma yields equality.

$$\sigma_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$\sigma_{ij}^r = 0, \quad \forall i \neq j, i, j > 2, r = n+1, ..., 2m+1,$$

$$\sigma_{11}^r + \sigma_{22}^r = 0, \quad \forall r = n+2, ..., 2m+1,$$

$$\sigma_{1j}^{n+1} = \sigma_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}.$$

If we choose $\{e_1, e_2\}$ such that $\sigma_{12}^{n+1} = 0$ and denote by $a = \sigma_{11}^r$, $b = \sigma_{22}^r$, $\mu = \sigma_{33}^{n+1} = ... = \sigma_{nn}^{n+1}$, then the shape operators take the mentioned forms. ∎

6. Ricci curvature for submanifolds of generalized Sasakian space forms

In this section we establish relationship between the Ricci curvature of a submanifold M of a generalized Sasakian space form $N(f_1, f_2, f_3)$ with a semi-symmetric metric connection and the squared mean curvature $\|H\|^2$.

Now, let begin with the following theorem:

6.1. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$. Then we have:

$$\begin{aligned} \|H\|^2 \quad \geq \quad & \frac{2\tau}{n(n-1)} + \frac{2}{n}\lambda - f_1 - \frac{3f_2}{n(n-1)} \|P\|^2 + \\ & + \frac{2}{n} f_3 \left\| \xi^\top \right\|^2 . \end{aligned} \tag{6.1}$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of $T_x M$ and $\{e_{n+1}, ..., e_{2m+1}\}$ be an orthonormal basis of $T_x^\perp M$, $x \in M$, where e_{n+1} is parallel to the mean curvature vector H . Then, the equation (5.5) can be written as follows

$$\begin{aligned} n^2 \|H\|^2 \quad = \quad & 2\tau + \|\sigma\|^2 + 2(n-1)\lambda - n(n-1) f_1 \\ & -3 f_2 \|P\|^2 + 2(n-1) f_3 . \end{aligned} \tag{6.2}$$

For a choosen orthonormal basis, let $e_1, e_2, ..., e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then, the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

and

$$A_{e_r} = (\sigma_{ij}^r), \quad i, j = 1, ..., n; \quad r = n+2, ..., 2m+1, \quad traceA_{e_r} = 0.$$

By the use of (6.2), we obtain

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 + \\ &\quad +2(n-1)\lambda - n(n-1)f_1 - 3f_2 \|P\|^2 + 2(n-1)f_3. \end{aligned} \tag{6.3}$$

On the other hand, we know that

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2. \tag{6.4}$$

Hence, by the use of (6.4) in (6.3), we obtain (6.1). ∎

In view of Theorem 6.1, we can give the following theorem:

6.2. Theorem. Let M, $n \geq 3$, be an n-dimensional submanifold of a $(2m+1)$-dimensional generalized Sasakian space form $N(f_1, f_2, f_3)$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ such that the vector field U is tangent to M. Then, for any integer k, $2 \leq k \leq n$ and for any point $x \in M$, we have:

$$\|H\|^2(x) \geq \Theta_k(\pi) + \frac{2}{n}\lambda - f_1 - \frac{3f_2}{n(n-1)} \|P\|^2 + \frac{2}{n} f_3 \left\| \xi^\top \right\|^2. \tag{6.5}$$

Proof. Similar to the proof of the Theorem 4.2, we easily get (6.5). ∎

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