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A presentation and some finiteness conditions for a new version of the Schützenberger product of monoids

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Abstract: In this paper we first define a *new version* of the Schützenberger product for any two monoids A and B, and then, by defining a generating and relator set, we present some finite and infinite consequences of the main result. In the final part of this paper, we give necessary and sufficient conditions for this new version to be periodic and locally finite.

Key words: Presentation, Schützenberger and wreath products, periodicity, local finiteness

1. Introduction and preliminaries

In [4, Theorem 2.2, Theorem 3.2], the generator and relator set for the wreath and Schützenberger products of arbitrary monoids A and B was defined. Further, in [6, see Theorems 7.1 and 7.2], the periodicity and local finiteness for semigroups under wreath products were studied. In fact these above results gave us the idea for this paper; since wreath and Schützenberger products have been studied a lot for many structures and some important properties have been obtained over them, we wonder what happens if we join both of these products under monoids. Thus, in this paper, we obtain a new monoid (see Section 2) by combining these two products and, also, define a presentation for this new monoid (see Theorem 2.2) by applying similar methods as processed previously [4]. Therefore, naturally, one can also wonder whether some algebraic properties (especially that have been investigated in [6]) still hold for this new structure. Hence, in the final section of this paper, we present necessary and sufficient conditions for this new monoid to be periodic and locally finite.

The aim of the rest of this section is just to give the standard definitions of (restricted) wreath and Schützenberger products for arbitrary monoids A and B.

It is well known that the cartesian product of B copies of the monoid A is denoted by $A^{\times B}$, while the corresponding direct product is denoted by $A^{\oplus B}$. One may think of $A^{\times B}$ as the set of all such functions from B to A, and $A^{\oplus B}$ as the set all such functions f having finite support, that is to say, having the property that $(x)f = 1_A$ for all but finitely many x in B. The unrestricted and restricted wreath products of the monoid A

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by the monoid B are the sets $A^{\times B} \times B$ and $A^{\oplus B} \times B$, respectively, with the multiplication defined by

$$(f,b)(g,b') = (f\ {}^{b}g, bb'),$$

where ${}^{b}g : B \to A$ is defined by

$$(x)^{b}g = (xb)g, \quad (x \in B). \tag{1}$$

It is also a well known fact that both these wreath products are monoids with the identity $(1, 1_B)$, where $x1 = 1_A$ for all $x \in B$. Now, for $a \in A$ and $b \in B$, let us define $\overline{a_b} : B \to A$ by

$$c\overline{a_b} = \begin{cases} a & ; \quad \text{if } c = b \\ 1_A & ; \quad \text{otherwise.} \end{cases} \tag{2}$$

For more details on the definition and applications of restricted (unrestricted) wreath products, we can refer, for instance, the reader to [2, 4, 5, 6, 7].

Now, for $P \subseteq A \times B$, $a \in A$, and $b \in B$, let us define the sets

$$Pb = \{(c, db) : (c,d) \in P\} \text{ and } aP = \{(ac, d) : (c,d) \in P\}.$$

The Schützenberger product of A and B, denoted by $A \diamond B$, is the set $A \times \wp(A \times B) \times B$ (where $\wp(.)$ defines the power set) with the multiplication

$$(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2).$$

Here $A \diamond B$ is a monoid [4] with the identity $(1_A, \emptyset, 1_B)$.

We recall that the benefit of definitions of wreath and Schützenberger products is to construct a new version of the Schützenberger product $A \diamond_v B$ (in Section 2) for any monoids A and B. The reason for us studying $A \diamond_v B$ is to obtain some new monoids. By computing new monoids, several questions arise, for example, if the monoids A and B have some nice properties (special type of rewriting systems, etc.), then does the monoid $A \diamond_v B$ also have this property? In the light of this idea, after giving some finite and infinite applications in Section 3, we will examine necessary and sufficient conditions for this new monoid to be periodic and locally finite (in Section 4) as we expressed in the first paragraph of this section.

2. A new version of the Schützenberger product

Let A, B be monoids and let $A^{\oplus B}$ be the set of all functions f from B into A having finite support. For $P \subseteq A^{\oplus B} \times B$ and $b \in B$, let us define a set

$$Pb = \{(f, db) : (f,d) \in P\}.$$

The new version of the Schützenberger product of A by B, denoted by $A \diamond_v B$, is the set $A^{\oplus B} \times \wp(A^{\oplus B} \times B) \times B$ with the multiplication

$$(f, P_1, b_1)(g, P_2, b_2) = (f\ {}^{b_1}g, P_1 b_2 \cup P_2, b_1 b_2).$$

We can easily show that $A \diamond_v B$ is a monoid with the identity $(\overline{1}, \emptyset, 1_B)$, where ${}^{b_1}g$ is defined as in (1).

In the following, by defining a generating set (in Lemma 2.1 below), we will give a presentation for this new product as the first main result of this paper (see Theorem 2.2).

Lemma 2.1 *Let us suppose that the monoids A and B are generated by the sets X and Y, respectively. Also let $b, d \in B$, and let*

$$\begin{array}{rcl} \overline{X_b} & = & \{(\overline{x_b}, \emptyset, 1_B) \ : \ x \in X\}, \\ \overline{Y} & = & \{(\overline{1}, \emptyset, y) \ : \ y \in Y\}, \\ \overline{P_d} & = & \{(\overline{1}, \{(\overline{x_d}, c)\}, 1_B) \ : \ x \in X, c \in B\}. \end{array}$$

Then the set $(\bigcup_{b\in B} \overline{X_b}) \cup \overline{Y} \cup (\bigcup_{d\in B} \overline{P_d})$ generates $A \diamondsuit_v B$.

Proof Let $\overline{x_b}$ be the function from B to A defined as in (2). For $x, x' \in X$, $b_1, b_2 \in B$, and $P_1, P_2 \subseteq A^{\oplus B} \times B$, we can easily show that the proof follows from

$$\left.\begin{array}{rcl} (\overline{x_{b_1}}, \emptyset, 1_B)(\overline{x'_{b_2}}, \emptyset, 1_B) & = & (\overline{x_{b_1}}\ {}^{1_B}\overline{x'_{b_2}}, \emptyset, 1_B) \\ & = & (\overline{x_{b_1} x'_{b_2}}, \emptyset, 1_B) \end{array}\right\}, \tag{3}$$

$$\left.\begin{array}{rcl} (\overline{1}, \emptyset, b_1)(\overline{1}, \emptyset, b_2) & = & (\overline{1}\ {}^{b_1}\overline{1}, \emptyset, b_1 b_2) \\ & = & (\overline{1}, \emptyset, b_1 b_2) \end{array}\right\}, \tag{4}$$

$$\left.\begin{array}{rcl} (\overline{1}, P_1, 1_B)(\overline{1}, P_2, 1_B) & = & (\overline{1}\ {}^{1_B}\overline{1}, P_1 \cup P_2, 1_B) \\ & = & (\overline{1}, P_1 \cup P_2, 1_B) \end{array}\right\}, \tag{5}$$

$$(\overline{x_{b_1}}, \emptyset, 1_B)(\overline{1}, \emptyset, b_2)(\overline{1}, P, 1_B) \ = \ (\overline{x_{b_1}}, P, b_2).$$

□

For each $d \in B$, let us denote the set P_d by $\{z_{x_d,c} : c \in B\}$. We then have the following theorem, which is the first main result of this paper.

Theorem 2.2 *Let us suppose that the monoids A and B are defined by presentations $[X; R]$ and $[Y; S]$, respectively. Let $b \in B$ (not necessarily distinct from d). Also let $X_b = \{x_b : x \in X\}$ be the corresponding copy of X, and let R_b be the corresponding copy of R. Then the new version of the Schützenberger product of A by B is defined by generators*

$$Z = (\bigcup_{b\in B} X_b) \cup Y \cup (\bigcup_{d\in B} P_d)$$

and relations

$$R_b \, (b \in B), \quad S, \tag{6}$$

$$x_b x'_e = x'_e x_b \quad (x, x' \in X, \ b, e \in B, \ b \neq e), \tag{7}$$

$$y x_b = (\prod_{m \in by^{-1}} x_m) y \quad (x \in X, \ y \in Y, \ b \in B), \tag{8}$$

$$\left.\begin{array}{ll} z^2_{x_d,c} = z_{x_d,c}, & z_{x_d,c} z_{x'_e,w} = z_{x'_e,w} z_{x_d,c} \\ & (x, x' \in X, \ c, d, e, w \in B) \end{array}\right\}, \tag{9}$$

$$\left.\begin{array}{ll} z_{x_d,c}y = yz_{x_d,cy}, & x_e^{'}z_{x_d,c} = z_{x_d,c}x_e^{'} \\ & (x, x^{'} \in X,\ y \in Y,\ c, d, e \in B) \end{array}\right\}. \tag{10}$$

Before giving the proof, for a set T, let us denote the set of all words in T by T^* and, for $b, y \in B$, let us denote the set $\{m \in B : b = my\}$ by by^{-1}.

Proof Let

$$\theta : Z^* \longrightarrow A\diamond_v B$$

be the homomorphism defined by

$$\begin{array}{rcl} (x_b)\theta & = & (\overline{x_b}, \emptyset, 1_B), \quad x \in X,\ b \in B, \\ (y)\theta & = & (\overline{1}, \emptyset, y), \quad y \in Y, \\ (z_{x_d,c})\theta & = & (\overline{1}, \{(\overline{x_d}, c)\}, 1_B), \quad c, d \in B. \end{array}$$

Then, by Lemma 2.1, we say that θ is onto. Now let us check whether $A\diamond_v B$ satisfies relations (6)–(10). In fact relations (6), (7), and (9) follow from (3), (4), and (5). For relation (10), we have

$$\begin{array}{rcl} (\overline{1}, \{(\overline{x_d}, c)\}, 1_B)(\overline{1}, \emptyset, y) & = & (\overline{1}, \{(\overline{x_d}, cy)\}, y) = \\ & = & (\overline{1}, \emptyset, y)(\overline{1}, \{(\overline{x_d}, cy)\}, 1_B)\,, \end{array}$$

$$\begin{array}{rcl} (\overline{x_e'}, \emptyset, 1_B)(\overline{1}, \{(\overline{x_d}, c)\}, 1_B) & = & (\overline{x_e'}, \{(\overline{x_d}, c)\}, 1_B) = \\ & = & (\overline{1}, \{(\overline{x_d}, c)\}, 1_B)(\overline{x_e'}, \emptyset, 1_B)\,. \end{array}$$

Now, as similarly in the proof of [4, Theorem 2.2], let us show that the relation (8) holds. To do that we use the equality

$$(\overline{1}, \emptyset, y)\,(\overline{x_b}, \emptyset, 1_B) = ({}^{y}\overline{x_b}, \emptyset, y).$$

For each $d \in B$,

$$\begin{array}{rcl} (d)^{y}\overline{x_b} = (dy)\overline{x_b} & = & \left\{\begin{array}{ll} x, & b = dy \\ 1_A, & \text{otherwise} \end{array}\right. = \left\{\begin{array}{ll} x, & d \in by^{-1} \\ 1_A, & \text{otherwise} \end{array}\right. \\ & = & \displaystyle\prod_{m \in by^{-1}} d\overline{x_m} = d(\prod_{m \in by^{-1}} \overline{x_m}). \end{array}$$

Therefore, we have ${}^{y}\overline{x_b} = \prod\limits_{m \in by^{-1}} \overline{x_m}$. Hence

$$(\overline{1}, \emptyset, y)(\overline{x_b}, \emptyset, 1_B) = (\prod_{m \in by^{-1}} (\overline{x_m}, \emptyset, 1_B))(\overline{1}, \emptyset, y),$$

for all $x, x^{'} \in X$, $y \in Y$, and $b \in B$. Therefore, θ induces an epimorphism $\overline{\theta}$ from the monoid M defined by (6)–(10) onto $A\diamond_v B$.

Let $w \in Z^*$ be any nonempty word. By using relations (7), (8), and (10), there exist words $w(b)$ in X^* $(b \in B)$, $w^{'} \in Y^*$, and $w^{''} \in P_d^*$ such that

$$w = (\prod_{b \in B} (w(b))_b) w^{'} w^{''}$$

in M. Moreover, we can use relations (9) to prove that there exists a set $P(w) \subseteq A^{\oplus B} \times B$ such that

$$w^{''} = \prod_{(\overline{x_d}, c) \in P(w)} z_{x_d, c} \ .$$

Therefore, for any word $w \in Z^*$, we have

$$\begin{aligned}
(w)\theta &= (\prod_{b \in B} (w(b))_b w^{'} w^{''})\theta \\
&= (\prod_{b \in B} \overline{(w(b))}_b, \emptyset, 1_B)(\overline{1}, \emptyset, w^{'})(\overline{1}, P(w), 1_B) \\
&= (\prod_{b \in B} \overline{(w(b))}_b, P(w), w^{'}).
\end{aligned}$$

Now, for each $w \in X^*$ and each c in B, we have

$$c\overline{w_b} = \begin{cases} w & ; \quad \text{if } c = b \\ 1 & ; \quad \text{otherwise} \end{cases},$$

where 1 denotes the empty word. Hence

$$c(\prod_{b \in B} \overline{(w(b))}_b) = \prod_{b \in B} c\overline{(w(b))}_b = w(c),$$

for all $c \in B$. Therefore, if $(w_1)\theta = (w_2)\theta$ for some $w_1, w_2 \in Z^*$ then, by the equality of these components, we deduce that $w_1(c) = w_2(c)$ in A (for every $c \in B$), $w_1^{'} = w_2^{'}$ in A and $P(w_1) = P(w_2)$. Relations (6) imply that $w_1(c) = w_2(c)$ and $w_1^{'} = w_2^{'}$ hold in M, so that $w_1 = w_2$ holds as well. Thus $\overline{\theta}$ is injective. □

Corollary 2.3 *Let A be a monoid and let B be a group. If $[X; R]$ and $[Y; S]$ are monoid presentations for A and B, respectively, then, for $x, x', x'', x''' \in X$, $y \in Y$, $b, c, d, e, w \in B$, the presentation $\mathcal{P}'_{A \diamond_v B}$ has a generating set $\{R, S\}$ and has a relation set that contains the relators*

$$\begin{aligned}
x(b^{-1}x'b) = (b^{-1}x'b)x, \quad z^2_{d^{-1}x''d,c} &= z_{d^{-1}x''d,c}, \\
z_{d^{-1}x''d,c} z_{e^{-1}x'''e,w} &= z_{e^{-1}x'''e,w} z_{d^{-1}\ x''d,c}, \\
z_{d^{-1}x''d,c}\, y &= y z_{d^{-1}x''d,cy}, \\
(e^{-1}x^{'''}e) z_{d^{-1}x''d,c} &= z_{d^{-1}x''d,c}(e^{-1}x^{'''}e)
\end{aligned}$$

which defines the monoid $A \diamond_v B$.

Proof It easy to see that

$$x_b = b^{-1}x_{1_B}b$$

holds in $A\Diamond_v B$. By using this above relation, if we eliminate x_b $(x \in X,\ b \in B-\{1_B\})$ from relations (6)–(10) in Theorem 2.2 then we obtain the relators $\mathcal{P}'_{A\Diamond_v B}$, as required. □

Between here and the next section, we will make an important remark, which is generally mentioning that the presentation, given in Theorem 2.2, can also be derived from two wreath products that are submonoids of the our new Schützenberger products. To do that let us use the same notation as above. Therefore, for monoids A and B, let $A^{\oplus B}$ be the set of all functions f having finite support, and also, for $P \subseteq A^{\oplus B} \times B$ and $b \in B$, let $Pb = \{(f, db) : (f, d) \in P\}$. Here, since the first element f of each pair (f, db) does not really play any role, there is no real reason to take $P \subseteq A^{\oplus B} \times B$; we could have $P \subseteq E \times B$ for any set E and define an action of B in the same way. Thus the new Schützenberger product $A\Diamond_v B$ becomes the set $A^{\oplus B} \times \wp(E \times B) \times B$ with the same multiplication given at the beginning of this section. In particular, we can think of P in the form $\bigcup\{(E_x, x) : x \in B\}$ where each E_x is a subset of E (for each $x \in B$ we are just collecting together all the pairs (e, x) that have x in the second component). We then have that

$$Pb = \bigcup\{(E_x, xb) : x \in B\} = \bigcup\{(E_{xb^{-1}}, x) : x \in B\}.$$

We can think of $\bigcup\{(E_x, x) : x \in B\}$ as being an element of $B^{\oplus\wp(E)}$ or $B^{\times\wp(E)}$ as appropriate.

Now let i be the element of $A^{\oplus B}$ such that $xi = 1_A$ for all $x \in B$ (by (1)). Note that ${}^{b}i = i$ for any $b \in B$. Additionally, let U be the submonoid of $A \diamond_v B$ consisting of all elements of the form $(f, \emptyset, b)$ and V be the submonoid of $A \diamond_v B$ consisiting of all elements of the form (i, P, b). In U we have

$$(f, \emptyset, b_1)(g, \emptyset, b_2) = (f\ {}^{b_1}g, \emptyset, b_1b_2).$$

This is just the the restricted wreath product of A by B.

Let us now consider V; here we have

$$(i, P_1, b_1)(i, P_2, b_2) = (i, P_1b_2 \cup P_2, b_1b_2).$$

Now

$$P_1b_2 \cup P_2 = \{(f, db_2) : (f, d) \in P_1\} \cup \{(f, d) : (f, d) \in P_2\}.$$

If we consider V^{op}, then we get

$$(i, P_2, b_2)(i, P_1, b_1) = (i, P_1b_2 \cup P_2, b_1b_2) = (i, P_2 \cup P_1b_2, b_1b_2).$$

By the above comments, this is isomorphic to the wreath product of $\wp(A^{\oplus B})$ by B^{op} (or, more generally, $\wp(E)$ by B^{op} as pointed out above).

Note also that

$$(f, \emptyset, b_1)(i, P, b_2) = (f, P, b_1b_2),$$

in $A \diamond_v B$, so that $A \diamond_v B = UV$ in an entirely natural way.

Now there are standard ways of deriving presentations for wreath products U and V (as in the paper [4]). Therefore, since $A \diamond_v B = UV$, our presentation defined in Theorem 2.2 can also be derived from these.

3. Some applications

The aim of this section is to define a presentation for $A\Diamond_v B$ while A and B are some special (finite and infinite) monoids.

<u>*Finite case*:</u>

Here, we will work on finite cyclic monoids, the fundamental facts of which can be found in [3, "Monogenic Semigroups"]. (One can look at the paper [1] for some examples, applications, and algebraic structures on cyclic monoids). Thus let us suppose that A and B are finite cyclic monoids with presentations

$$\mathcal{P}_A = [x\ ;\ x^k = x^l\ (k > l)] \text{ and } \mathcal{P}_B = [y\ ;\ y^s = y^t\ (s > t)],$$

respectively. Thus, as an application of Theorem 2.2, we obtain the following result.

Corollary 3.1 *The product $A\Diamond_v B$ has a presentation $\mathcal{P}''_{A\Diamond_v B}$ as the form*

$$\begin{aligned}
[x^{(i)},\ z_{x^{(j)},y^m},\ y\ ;\ y^s = y^t,\ x^{(i)}x^{(j)} = x^{(j)}x^{(i)} \quad (i<j),\\
{x^{(i)}}^k = {x^{(i)}}^l,\ yx^{(t)} = x^{(s-1)}y,\\
yx^{(i)} = x^{(i-1)}y \quad (0 < i \leq s-1)\\
z^2_{x^{(j)},y^m} = z_{x^{(j)},y^m},\\
z_{x^{(j)},y^m} z_{x^{(i)},y^n} = z_{x^{(i)},y^n} z_{x^{(j)},y^m},\\
x^{(i)} z_{x^{(j)},y^m} = z_{x^{(j)},y^m} x^{(i)},\ z_{x^{(j)},y^m} y = y z_{x^{(j)},y^{m+1}}],
\end{aligned}$$

where $0 \leq i,j,m,n \leq s-1$.

Proof Now let us consider the relator

$$yx_b = (\prod_{m\in by^{-1}} x_m)y$$

given in Theorem 2.2. In this relator, for each representative element y^i in the monoid B, let us label x_{y^i} by $x^{(i)}$ where $0 < i \leq s-1$. Then we obtain the relator $yx^{(i)} = x^{(i-1)}y$. Moreover, for the monoid B, since we have $y^s = y^t$ in $\mathcal{P}_B$ as a relator, we can write this relator as $y^t = y^{s-1}y$, which implies that $yx^{(t)} = x^{(s-1)}y$, for $b = y^t$ and $m = y^{s-1}$, by keeping the same idea as in the previous sentence. The remaining relators in $\mathcal{P}''_{A\Diamond_v B}$ can be seen easily from Theorem 2.2. Hence the result. □

<u>*Infinite case*:</u>

Let us consider $A\Diamond_v B$ while A is the free abelian monoid rank 2 and B is the finite cyclic monoid. Then, again as an application of Theorem 2.2, we have the following result.

Corollary 3.2 *Let $\mathcal{P}_A^{Ab} = [x_1, x_2\ ;\ x_1x_2 = x_2x_1]$ and $\mathcal{P}_B = [y\ ;\ y^s = y^t\ (s > t)]$ be monoid presentations for the monoids A and B. Then the product $A\Diamond_v B$ has a presentation $\mathcal{P}'''_{A\Diamond_v B}$ such that the generators are*

$$x_1^{(i)},\ x_2^{(k)},\ z_{x_1^{(j)},y^m},\ z_{x_2^{(l)},y^n},\ y,$$

where $0 \leq i,j,k,l,m,n \leq s-1$*, and the relators are*

$$x_p^{(i)}x_q^{(k)} = x_q^{(k)}x_p^{(i)} \quad (i<k,\ p,q \in \{1,2\}),$$
$$y^s = y^t, \quad yx_1^{(i)} = x_1^{(i-1)}y \quad (1 \leq i \leq s-1),$$
$$yx_2^{(k)} = x_2^{(k-1)}y \quad (1 \leq k \leq s-1), \quad yx_1^{(t)} = x_1^{(s-1)}y,$$
$$yx_2^{(t)} = x_2^{(s-1)}y, \quad z^2_{x_1^{(j)},y^m} = z_{x_1^{(j)},y^m},$$
$$z^2_{x_2^{(l)},y^n} = z_{x_2^{(l)},y^n}, \quad z_{x_1^{(j)},y^m}z_{x_1^{(i)},y^n} = z_{x_1^{(i)},y^n}z_{x_1^{(j)},y^m},$$
$$z_{x_2^{(k)},y^m}z_{x_2^{(l)},y^n} = z_{x_2^{(l)},y^n}z_{x_2^{(k)},y^m},$$
$$z_{x_1^{(j)},y^m}z_{x_2^{(l)},y^n} = z_{x_2^{(l)},y^n}z_{x_1^{(j)},y^m},$$

$$x_1^{(i)}z_{x_1^{(j)},y^m} = z_{x_1^{(j)},y^m}x_1^{(i)}, \qquad x_1^{(i)}z_{x_2^{(l)},y^n} = z_{x_2^{(l)},y^n}x_1^{(i)},$$
$$x_2^{(k)}z_{x_1^{(j)},y^m} = z_{x_1^{(j)},y^m}x_2^{(k)}, \qquad x_2^{(k)}z_{x_2^{(l)},y^n} = z_{x_2^{(l)},y^n}x_2^{(k)},$$
$$z_{x_1^{(j)},y^m}y = yz_{x_1^{(j)},y^{m+1}}, \qquad z_{x_2^{(l)},y^n}y = yz_{x_2^{(l)},y^{n+1}},$$

where $0 \leq i,j,m,n \leq s-1$.

In fact, the above corollary can be proved similarly as in Corollary 3.1 by considering A as a free abelian monoid rank 2 in the proof of Theorem 2.2. It is clear that Corollary 3.2 can be generalized for the free abelian monoid A rank $n>2$.

Another application of Theorem 2.2 can be given as follows.

Let A be the free monoid with a presentation $\mathcal{P}_A = [x;\ \ ]$ and let B be the monoid $Z_s \times Z_m$ with a presentation

$$\mathcal{P}_B = [y_1,y_2\ ;\ y_1y_2 = y_2y_1,\ y_1^s = y_1^t,\ y_2^m = y_2^n\ \ (s>t, m>n)].$$

For a representative element $y_1^iy_2^j$ in the monoid B, let us label $x_{y_1^iy_2^j}$ by $x^{(i,j)}$ where $0 \leq i \leq s-1$, $0 \leq j \leq m-1$. Then, for each element in B, we have the generating set

$$\{x^{(i,j)}, y_1, y_2, z_{x^{(r,q)},y_1^ky_2^l}\}, \tag{11}$$

for the monoid $A\Diamond_v B$, where $0 \leq i,k,r \leq s-1$ and $0 \leq j,q,l \leq m-1$. Therefore, by suitable changes in presentation given in Theorem 2.2, we obtain the following result.

Corollary 3.3 *Let* A *and* B *be as above. Then, for* $0 \leq i,h,k,r \leq s-1$ *and* $0 \leq j,q,l,w \leq m-1$*, the product* $A\Diamond_v B$ *has a presentation with the generating set* (11) *and the relator set*

$$\{y_1^s = y_1^t,\ y_2^m = y_2^n, \qquad x^{(i,j)}x^{(l,k)} = x^{(l,k)}x^{(i,j)} \quad ((i,j)<(l,k)),$$
$$y_1x^{(i,j)} = x^{(i-1,j)}y_1 \qquad (1 \leq i \leq s-1,\ 0 \leq j \leq m-1),$$
$$y_2x^{(i,j)} = x^{(i,j-1)}y_1 \qquad (0 \leq i \leq s-1,\ 1 \leq j \leq m-1),$$

$$\begin{aligned}
y_1 x^{(t,j)} = x^{(s-1,j)} y_1,\ y_2 x^{(i,n)} &= x^{(i,m-1)} y_2, \\
z^2_{x^{(r,q)},y_1^k y_2^l} &= z_{x^{(r,q)},y_1^k y_2^l}, \\
z_{x^{(r,q)},y_1^k y_2^l} z_{x^{(i,j)},y_1^h y_2^w} &= z_{x^{(i,j)},y_1^h y_2^w} z_{x^{(r,q)},y_1^k y_2^l}, \\
x^{(i,j)} z_{x^{(r,q)},y_1^k y_2^l} &= z_{x^{(r,q)},y_1^k y_2^l} x^{(i,j)}, \\
z_{x^{(r,q)},y_1^k y_2^l} y_1 &= y_1 z_{x^{(r,q)},y_1^{k+1} y_2^l}, \\
z_{x^{(r,q)},y_1^k y_2^l} y_2 &= y_2 z_{x^{(r,q)},y_1^k y_2^{l+1}} \}.
\end{aligned}$$

4. Periodicity and local finiteness

In this last section, our aim is to prove that the new Schützenberger product agrees well with periodicity and local finiteness. Recall that a monoid M is called *periodic* if every element $m \in M$ has finite order and called *locally finite* if every finitely generated submonoid of M is finite.

Now we can give the following theorems as another main result of this paper.

Theorem 4.1 *The product $A\Diamond_v B$ of two monoids A and B is periodic if and only if both A and B are periodic.*

Proof $(\Rightarrow)$ If $A\Diamond_v B$ is periodic then, being a homomorphic image of it, B is also periodic. Furthermore, for a given $a \in A$, we can choose some $f_a \in A^{\oplus B}$ such that $1_B \mapsto a$ and $b \mapsto 1_A$. Now let us consider the set $\{(f_a, \emptyset, 1_B) : a \in A\}$. One can easily see that the monoid defined by this set is isomorphic to A. Thus A becomes a submonoid of $A\Diamond_v B$, which means A is periodic as well.

$(\Leftarrow)$ Let (f, P, b) be an arbitrary element of $A\Diamond_v B$. We should note that for any monoid element y and positive integer p, y has finite order if and only if y^p has finite order. Thus, since B is periodic, we may assume that the element $b = d$ is idempotent. Moreover, since A is periodic, by [6, Proposition 2.1], f has a finite image $X \subseteq A$, for $f \in A^{\oplus B}$. Since X is a finite set of periodic elements, we may find some positive integers $m < n$ such that $x^m = x^n$, for all $x \in X$. Therefore, for all $b \in B$, we have $(bd)f \in X$ and so

$$\begin{aligned}
(b)(f(\ ^d f)^m) = (b)f\ ((bd)f)^m &= (b)f\ ((bd)f)^n \\
&= (b)(f(\ ^d f)^n).
\end{aligned}$$

It follows that $(\ ^d f)^m = (\ ^d f)^n$. Therefore $(f, P, d)^{m+1} = (f, P, d)^{n+1}$, which proves that $A\Diamond_v B$ is periodic. □

Theorem 4.2 *The product $A\Diamond_v B$ of two monoids A and B is locally finite if and only if both A and B are locally finite.*

Proof $(\Rightarrow)$ If $A\Diamond_v B$ is locally finite, then so is B (being a homomorphic image). Now, as we did in the necessity part of the proof of Theorem 4.1, A is a submonoid of $A\Diamond_v B$. Thus we can easily conclude that A is locally finite.

$(\Leftarrow)$ Let $D \subseteq A\Diamond_v B$ be a finite set, and let

$$(f, P, b) = (f_1, P_1, b_1)(f_2, P_2, b_2) \cdots (f_r, P_r, b_r) \in \langle D \rangle.$$

We must show that there are only finitely many choices for f, P, and b with the assumption that A and B are locally finite. Let $C_1 = \langle\{y \ : \ (\exists g)(\exists P)\ (g,P,y) \in D\}\rangle, C_2 = \{^{b}g \ : \ b \in C_1, (\exists c)(\exists P)(g,P,c) \in D\}$ and $Pd = \{(g,bd); b,d \in C_1, g \in C_2\}$. By assumption and [6, Proposition 2.1], we say that C_1, C_2 and Pd are finite. Therefore, we have only finitely many choices for $b = b_1b_2\cdots b_r$, $f = f_1\, {}^{b_1}f_2 \cdots {}^{b_1b_2\cdots b_{r-1}}f_r$ and $P = P_1d_2d_3\cdots d_r \cup P_2d_3d_4\cdots d_r \cup \cdots \cup P_r$. Hence the result. $\square$

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