Araştırma Makalesi **BAUN Fen Bil. Enst. Dergisi, 26(1), 188-195, (2024)**

DOI:10.25092/baunfbed. 1356259 **J. BAUN Inst. Sci. Technol., 26(1), 188-195, (2024)**

Fejér means in variable exponent Lebesgue spaces on the real axis

Ebru ALTIPARMAK[1,*], Ali GUVEN[2]

[1] *Erzurum Technical University Faculty of Science, Department of Mathematics, Erzurum.*
[2]*Balıkesir University, Faculty of Arts and Science, Department of Mathematics, Balikesir.*

Geliş Tarihi (Received Date): 07.09.2023
Kabul Tarihi (Accepted Date): 25.10.2023

Abstract

Variable exponent Lebesgue spaces are generalizations of classical Lebesgue spaces and have importance in many branches of Mathematical Analysis. Especially, direct and converse theorems and their improvements are studied by many mathematicians in these spaces. In this article, direct and converse predictions for the rate of convergence of Fejér means of functions belonging to the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$ *are established by using an appropriate* K*-functional. In this way, the result of Z. Ditzian on Fejér means in classical Lebesgue spaces* $L^p(\mathbb{R})(1 < p < \infty)$ *is generalized.*

Keywords: *Fejér means, fourier transform, variable exponent Lebesgue space.*

Reel eksende değişken üslü Lebesgue uzaylarda Fejér ortalamalar

Öz

Değişken üslü Lebesgue uzayları klasik Lebesgue uzaylarının genellemeleridir ve Matematiksel Analizin birçok dalında öneme sahiptir. Özellikle direkt ve ters teoremler ve bunların geliştirilmesi bu uzaylarda birçok matematikçi tarafından incelenmektedir. Bu makalede, değişken üslü Lebesgue uzayı $L^{p(\cdot)}(\mathbb{R})$*'ye ait fonksiyonların Fejér ortalamalarının yakınsaklık hızına ilişkin doğrudan ve ters tahminler, uygun bir K-fonksiyonu kullanılarak oluşturulmuştur. Bu şekilde, Z. Ditzian'ın klasik Lebesgue uzaylarında* $L^p(\mathbb{R})(1 < p < \infty)$ *Fejér ortalamalarına ilişkin sonucu genelleştirilmiştir.*

Anahtar kelimeler: *Fejér ortalamaları, fourier dönüşümü, değişken üslü Lebesgue uzayı.*

[*]Ebru ALTIPARMAK, ebru.altiparmak@erzurum.edu.tr, https://orcid.org/0000-0001-6722-0807
Ali GÜVEN, guvenali@gmail.com, https://orcid.org/0000-0001-8878-250X

1. Introduction

The classical Lebesgue spaces L^p $(1 \leq p \leq \infty)$ are very important spaces for studying approximation theory problems. There are several direct and converse theorems of approximation theory on L^p spaces defined on the real axis $\mathbb{R}$ or on its intervals. These theorems can be found in the monographs [1] and [2]. In recent years, approximation problems that replace a fixed exponent p with a variable exponent $p(\cdot)$ in variable exponent Lebesgue spaces, which are a generalization of classical Lebesgue spaces, have also been studied. Most of these studies are concerned with variable exponent Lebesgue spaces of 2π-periodic functions and summability methods of trigonometric Fourier series (see, for example, [3]-[11]). In this work, we established an approximation theorem for Fejér means of non-periodic functions belonging to variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$, and generalized Theorem 1 of [12].

Variable exponent Lebesgue spaces have been used in many areas of Mathematical Analysis in recent years. In this part, we give a brief knowledge of variable exponent Lebesgue spaces. More details about these spaces can be found in [13, Chapters 2-5].

The set of all Lebesgue measurable functions $\mathcal{P}(\mathbb{R})$ are denoted by

$p(\cdot): \mathbb{R} \to [1, \infty]$,

which are referred to as exponent functions. For $p(\cdot) \in \mathcal{P}(\mathbb{R})$ we set

$$p_{-} := \operatorname*{essinf}_{x\in\mathbb{R}} p(x), p_{+} := \operatorname*{esssup}_{x\in\mathbb{R}} p(x).$$

A function $p(\cdot) \in \mathcal{P}(\mathbb{R})$ is referred to as locally log-Hölder continuous if there is a constant c_0 such that

$$|p(x) - p(y)| \leq \frac{c_0}{-\log(|x-y|)}, x, y \in \mathbb{R}, |x-y| < \frac{1}{2}, \tag{1.1}$$

and log-Hölder continuous at infinity if there are constants c_∞ and p_∞ such that

$$|p(x) - p_\infty| \leq \frac{c_\infty}{\log(e+|x|)}, x \in \mathbb{R}. \tag{1.2}$$

The class of exponent functions which satisfies both (1.1) and (1.2) will be denoted by $LH(\mathbb{R})$.

For an exponent function $p(\cdot) \in \mathcal{P}(\mathbb{R})$ and a Lebesgue measurable function f, the modular associated with $p(\cdot)$ is defined in the following

$$\rho_{p(\cdot)}(f) := \int\limits_{\mathbb{R}\setminus\mathbb{R}_\infty^{p(\cdot)}} |f(x)|^{p(x)} dx + \|f\|_{L^\infty\left(\mathbb{R}_\infty^{p(\cdot)}\right)},$$

where $\mathbb{R}_\infty^{p(\cdot)} := \{x \in \mathbb{R}: p(x) = \infty\}$. The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R})$ is the set of Lebesgue measurable functions f such that $\rho_{p(\cdot)}(f/\alpha) < \infty$ for some $\alpha > 0$, which is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R})} := \inf\{\alpha > 0: \rho_{p(\cdot)}(f/\alpha) \leq 1\}.$$

(see [13]). In case $p(x) = p$, $1 \leq p < \infty$, $L^{p(\cdot)}(\mathbb{R})$ coincides with the classical Lebesgue space $L^p(\mathbb{R})$.

The analog of Minkowski's inequality holds for variable exponent Lebesgue spaces (see [13, Corollary 2.38]).

If $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is Lebesgue measurable and $f(\cdot,y)\in L^{p(\cdot)}(\mathbb{R})$ for almost all $y\in\mathbb{R}$, then

$$\left\|\int_{\mathbb{R}} f(\cdot,y)dy\right\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim \int_{\mathbb{R}} \|f(\cdot,y)\|_{L^{p(\cdot)}(\mathbb{R})}dy. \quad (1.3)$$

Note that, throughout the paper, the expression $X\lesssim Y$ means that there is a constant $K>0$ such that $X\le K.Y$ holds.

A non-negative, measurable function w is called a weight if $0<w(x)<\infty$ almost everywhere. For $1\le p<\infty$, the weighted Lebesgue space $L_w^p(\mathbb{R})$ with the weight w is defined by the space of all Lebesgue measurable functions f such that

$$\|f\|_{L_w^p(\mathbb{R})}:=\left(\int_{\mathbb{R}}|f(x)|^p w(x)dx\right)^{\frac{1}{p}}<\infty.$$

For $1<p<\infty$, a weight function w is in the Muckenhoupt class $A_p(\mathbb{R})$ if

$$\sup_I\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}dx\right)^{p-1}<\infty,$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$, and $p'=p/p-1$ ($|I|$ is the length of the interval I). The Muckenhoupt class $A_1(\mathbb{R})$ is defined as the class of weights w such that

$$\operatorname*{esssup}_{x\in\mathbb{R}}\frac{M(w)(x)}{w(x)}<\infty,$$

where M is the Hardy-Littlewood maximal operator, that is,

$$M(f): f\to Mf(x):=\sup_{I\ni x}\frac{1}{|I|}\int_{\mathrm{I}} f(y)dy.$$

The following extrapolation theorem can be deduced from Theorem 3.16 and Corollary 5.32 of [13].

Theorem 1: Let $p(\cdot)\in LH(\mathbb{R})$ and $1<p_-\le p_+<\infty$ and $p_0\ge 1$. Suppose that for a family of functions $\mathcal{F}$, the inequality

$$\int_{\mathbb{R}} F(x)^{p_0}w(x)dx\lesssim\int_{\mathbb{R}} G(x)^{p_0}w(x)dx, (F,G)\in\mathcal{F}$$

where (F,G) are pairs of non-negative, measurable functions, holds $\forall w\in A_{p_0}(\mathbb{R})$. Then, for $(F,G)\in\mathcal{F}$,

$$\|F\|_{L^{p(\cdot)}(\mathbb{R})}\lesssim\|G\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Note that the conditions $p(\cdot)\in LH(\mathbb{R})$ and $1<p_-\le p_+<\infty$ are sufficient for the boundedness of the Hardy-Littlewood maximal operator on the space $L^{p(\cdot)}(\mathbb{R})$ ([13, p.89]).

2. Approximation by Fejér means in variable exponent Lebesgue spaces

Fejér means of the function $f \in L^{p(\cdot)}(\mathbb{R})$ are defined as

$$F_\lambda(f)(x) := (f * G_\lambda)(x) = \int_{\mathbb{R}} f(x-t)G_\lambda(t)dt,$$

where

$$G_\lambda(t) := \frac{1}{2\pi\lambda}\left(\frac{\sin\frac{\lambda t}{2}}{\frac{t}{2}}\right)^2, \lambda > 0$$

is the Fejér kernel.

The Hilbert transform of $f \in L^{p(\cdot)}(\mathbb{R})$ is given by

$$H(f)(x) := P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-t)}{t}dt,$$

where P. V. means Cauchy Principal Value. By [14, Theorem 9] we have

$$\|H(f)\|_{L^p_w(\mathbb{R})} \lesssim \|f\|_{L^p_w(\mathbb{R})}, f \in L^p_w(\mathbb{R}),$$

where $1 < p < \infty$ and $w \in A_p(\mathbb{R})$, hence Theorem 1 yields $H(f) \in L^{p(\cdot)}(\mathbb{R})$ for $f \in L^{p(\cdot)}(\mathbb{R})$, provided that $p(\cdot) \in LH(\mathbb{R})$ and $1 < p_- \leq p_+ < \infty$. We consider the subspace $\mathcal{H}^{p(\cdot)}(\mathbb{R})$ of $L^{p(\cdot)}(\mathbb{R})$ defined by

$$\mathcal{H}^{p(\cdot)}(\mathbb{R}) := \left\{g \in L^{p(\cdot)}(\mathbb{R}): \frac{d}{dx}H(g) \in L^{p(\cdot)}(\mathbb{R})\right\}.$$

This article is devoted to estimating the rate of approximation of the Fejér means in the $L^{p(\cdot)}(\mathbb{R})$. The main result of the article is the following Theorem 2.

Theorem 2: Let $p(\cdot) \in LH(\mathbb{R})$ and $1 < p_- \leq p_+ < \infty$. *For* $f \in L^{p(\cdot)}(\mathbb{R})$ we have

$$K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})} \lesssim \|F_\lambda(f) - f\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})},$$

where

$$K(f,t)_{L^{p(\cdot)}(\mathbb{R})} := \inf\left\{\|f-g\|_{L^{p(\cdot)}(\mathbb{R})} + t\left\|\frac{d}{dx}H(g)\right\|_{L^{p(\cdot)}(\mathbb{R})} : g \in \mathcal{H}^{p(\cdot)}(\mathbb{R})\right\}.$$

To prove the above theorem, we require the following Lemma.

Lemma 1: Let $p(\cdot) \in LH(\mathbb{R})$ and $1 < p_- \leq p_+ < \infty$. Then

$$\|F_\lambda(f) - f\|_{L^{p(\cdot)}(\mathbb{R})} \to 0, \lambda \to \infty \tag{2.1}$$

for $f \in L^{p(\cdot)}(\mathbb{R})$.

Proof: Fix $\epsilon > 0$. Since $L^\infty_c(\mathbb{R})$, the space of bounded functions of compact support, is dense in $L^{p(\cdot)}(\mathbb{R})$ ([13, Theorem 2.72]), there exists a function $g \in L^\infty_c(\mathbb{R})$ (not identically

zero) such that $\|f-g\|_{L^{p(\cdot)}(\mathbb{R})}<\epsilon$.
By [15, Theorem 2.1],

$$\|F_\lambda(h)\|_{L^p_w(\mathbb{R})} \lesssim \|h\|_{L^p_w(\mathbb{R})}, h \in L^p_w(\mathbb{R})$$

for $1<p<\infty$ and $w\in A_p(\mathbb{R})$. Thus, by Theorem 1, we get

$$\|F_\lambda(h)\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim \|h\|_{L^{p(\cdot)}(\mathbb{R})}, h \in L^{p(\cdot)}(\mathbb{R}). \quad (2.2)$$

Then by (2.2),

$$\|F_\lambda(f)-f\|_{L^{p(\cdot)}(\mathbb{R})} \le \|F_\lambda(f-g)\|_{L^{p(\cdot)}(\mathbb{R})} + \|F_\lambda(g)-g\|_{L^{p(\cdot)}(\mathbb{R})} + \|f-g\|_{L^{p(\cdot)}(\mathbb{R})}$$
$$\lesssim 2\|f-g\|_{L^{p(\cdot)}(\mathbb{R})} + \|F_\lambda(g)-g\|_{L^{p(\cdot)}(\mathbb{R})}$$
$$< 2\epsilon + \|F_\lambda(g)-g\|_{L^{p(\cdot)}(\mathbb{R})}.$$

Since $\epsilon>0$ is arbitrary, showing

$$\|F_\lambda(g)-g\|_{L^{p(\cdot)}(\mathbb{R})} \to 0, \lambda\to\infty$$

will be sufficient to complete the proof. By [13, Theorem 2.58] the last statement is equivalent to modular convergence, i.e.

$$\int_{\mathbb{R}} |F_\lambda(g)(x)-g(x)|^{p(x)}dx \to 0, \lambda\to\infty.$$

It is clear that $g_0\in L_c^\infty(\mathbb{R})$ and $\|g_0\|_\infty \le 1/2$ for the function

$$g_0(x) := \frac{1}{2\|g\|_\infty} g(x).$$

Since $\|G_\lambda\|_1 = 1$, (see for detail [12]) we get

$$|F_\lambda(g_0)(x)| = |(G_\lambda * g_0)(x)| = \left|\int_{\mathbb{R}} g_0(x-t)G_\lambda(t)dt\right|$$
$$\le \|G_\lambda\|_1\|g_0\|_\infty \le 1/2,$$

which implies $\|F_\lambda(g_0)\|_\infty \le 1/2$, and hence

$$\|F_\lambda(g_0)-g_0\|_\infty = \|G_\lambda * g_0 - g_0\|_\infty \le 1.$$

Therefore,

$$\int_{\mathbb{R}} |F_\lambda(g)(x)-g(x)|^{p(x)}dx = \int_{\mathbb{R}} |(G_\lambda * g)(x)-g(x)|^{p(x)}dx$$
$$= \int_{\mathbb{R}} |(G_\lambda * 2\|g\|_\infty g_0)(x) - 2\|g\|_\infty g_0(x)|^{p(x)}dx$$
$$= \int_{\mathbb{R}} (2\|g\|_\infty)^{p(x)} |(G_\lambda * g_0)(x)-g_0(x)|^{p(x)}dx$$
$$\le (2\|g\|_\infty+1)^{p+} \int_{\mathbb{R}} |(G_\lambda * g_0)(x)-g_0(x)|^{p(x)}dx$$

$$\leq (2\|g\|_\infty + 1)^{p_+} \int_{\mathbb{R}} |(G_\lambda * g_0)(x) - g_0(x)|^{p_-} dx.$$

Since $g_0 \in L^{p_-}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |(G_\lambda * g_0)(x) - g_0(x)|^{p_-} dx \to 0, \lambda \to \infty$$

(see, for example [16, Theorem 2.6.4]) and (2.1) follows.

Proof of Theorem 2: The inequality (2.2) implies $F_\lambda = G_\lambda * f \in L^{p(\cdot)}(\mathbb{R})$ for $f \in L^{p(\cdot)}(\mathbb{R})$. Thus, the identity

$$G_\lambda * f - G_\lambda * (G_\lambda * f) = \frac{1}{\lambda}\frac{d}{dx} H(G_\lambda * f), \tag{2.3}$$

which is proved in [12], yields $\frac{d}{dx} H(G_\lambda * f) \in L^{p(\cdot)}(\mathbb{R})$, hence $G_\lambda * f \in \mathcal{H}^{p(\cdot)}(\mathbb{R})$. By considering this fact, (2.3) and (2.2), we obtain

$$K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})} \leq \|f - G_\lambda * f\|_{L^{p(\cdot)}(\mathbb{R})} + \frac{1}{\lambda}\left\|\frac{d}{dx} H(G_\lambda * f)\right\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$= \|f - G_\lambda * f\|_{L^{p(\cdot)}(\mathbb{R})} + \|G_\lambda * f - G_\lambda * (G_\lambda * f)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$\lesssim \|f - G_\lambda * f\|_{L^{p(\cdot)}(\mathbb{R})},$$

which yield the converse inequality.

Let $\epsilon > 0$ be arbitrary. By definition of $K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})}$ there exist a function $g_\epsilon \in \mathcal{H}^{p(\cdot)}(\mathbb{R})$ such that

$$\|f - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})} + \frac{1}{\lambda}\left\|\frac{d}{dx} H(g_\epsilon)\right\|_{L^{p(\cdot)}(\mathbb{R})} < (1+\epsilon) K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})}. \tag{2.4}$$

By considering (2.2),

$$\|f - F_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R})} \leq \|f - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})} + \|g_\epsilon - G_\lambda * g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$+ \|G_\lambda * (g_\epsilon - f)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$\lesssim \|f - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})} + \|g_\epsilon - G_\lambda * g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}.$$

We have

$$\|g_\epsilon - G_\lambda * g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})} \leq \|G_\lambda * g_\epsilon - G_\lambda * (G_\lambda * g_\epsilon)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$+ \|G_\lambda * (G_\lambda * g_\epsilon) - G_r * (G_r * g_\epsilon)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$+ \|G_r * (G_r * g_\epsilon) - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$=: I_1 + I_2(r) + I_3(r)$$

for $r > 0$. Using (2.2) again,

$$I_3(r) = \|G_r * (G_r * g_\epsilon) - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$\leq \|G_r * (G_r * g_\epsilon - g_\epsilon)\|_{L^{p(\cdot)}(\mathbb{R})} + \|G_r * g_\epsilon - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$\lesssim \|G_r * g_\epsilon - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})}.$$

By Lemma 1, $\|G_r * g_\epsilon - g_\epsilon\|_{L^{p(\cdot)}(\mathbb{R})} \to 0$, and hence $I_3(r) \to 0$ as $r \to \infty$. Thus we have

$$I_3(r) < \epsilon \frac{1}{\lambda} \left\| \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})}, r \geq r_\epsilon.$$

Since

$$\frac{d}{dx} H(G_\lambda * g_\epsilon)(x) = \left(G_\lambda * \frac{d}{dx} H(g_\epsilon) \right)(x),$$

by (2.3) and (2.2) we get

$$I_1 = \|G_\lambda * g_\epsilon - G_\lambda * (G_\lambda * g_\epsilon)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$= \frac{1}{\lambda} \left\| G_\lambda * \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim \frac{1}{\lambda} \left\| \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})}.$$

It is known that (see [12])

$$\frac{d}{dt} \big(G_t * (G_t * g_\epsilon) \big) = \frac{2}{t^2} \left(G_t * \frac{d}{dx} H(g_\epsilon) \right).$$

Considering this equality, (1.3) and (2.2),

$$I_2(r) = \|G_\lambda * (G_\lambda * g_\epsilon) - G_r * (G_r * g_\epsilon)\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$= \left\| \int_\lambda^r \frac{d}{dt} \big(G_t * (G_t * g_\epsilon) \big) dt \right\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$= 2 \left\| \int_\lambda^r \frac{1}{t^2} \left(G_t * \frac{d}{dx} H(g_\epsilon) \right) dt \right\|_{L^{p(\cdot)}(\mathbb{R})}$$

$$\lesssim \int_\lambda^r \left\| G_t * \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})} \frac{dt}{t^2}$$

$$\lesssim \left\| \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})} \int_\lambda^r \frac{dt}{t^2}$$

$$\leq \frac{1}{\lambda} \left\| \frac{d}{dx} H(g_\epsilon) \right\|_{L^{p(\cdot)}(\mathbb{R})}.$$

By combining these estimates for $I_1, I_2(r)$ and $I_3(r)$, and considering (2.4) we get $\|f - F_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R})} \lesssim (1+\epsilon)^2 K\left(f, \frac{1}{\lambda}\right)_{L^{p(\cdot)}(\mathbb{R})}$.

Since $\epsilon > 0$ is arbitrary, this gives the direct estimate.
By taking $p(x) \equiv p$, i. e. in the $L^p(\mathbb{R})$ $(1 < p < \infty)$, this theorem was obtained by Z. Ditzian in [12].

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