

GEOMETRIC PROPERTIES OF FIXED POINTS AND SIMULATION FUNCTIONS

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ABSTRACT. Geometric properties of the fixed point set $Fix(f)$ of a self-mapping f on a metric or a generalized metric space is an attractive issue. The set $Fix(f)$ can contain a geometric figure (a circle, an ellipse, etc.) or it can be a geometric figure. In this paper, we consider the set of simulation functions for geometric applications in the fixed point theory both on metric and some generalized metric spaces (S -metric spaces and b -metric spaces). The main motivation of this paper is to investigate the geometric properties of non unique fixed points of self-mappings via simulation functions.

1. Introduction

Recently, the set of simulation functions defined in [7] has been used for the solutions of many recent problems such as fixed-circle problem (resp. fixed-disc problem) and Rhoades' open problem on discontinuity (see [11, 18, 21]). On the other hand, simulation functions have been studied by various aspects in metric fixed-point theory (see for example [2, 6, 7, 8, 13, 22, 23]). For example, in [13], Olgun et al. gave a new class of Picard operators on complete metric spaces via simulation functions. Simulation functions have been used to study the best proximity points in metric spaces. For example, Kostić et al. presented several best proximity point results involving simulation functions (for more details see [6, 8] and the references therein).

In [22], the set of simulation functions has been enlarged by A. F. Roldán-López-de-Hierro et al. Every simulation function in the original Khojasteh et al.'s sense is also a simulation function in A. F. Roldán-López-de-Hierro et al.'s sense but the converse is not true (see [22] for more details). In this paper, we focus on the set of simulation functions in both sense and using their properties, we consider some recent problems in fixed-point theory.

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Recall that the function $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is said to be a simulation function in the Khojasteh et al.'s sense, if the following hold:

- (ζ_1) $\zeta(0, 0) = 0$,
- (ζ_2) $\zeta(t, s) < s - t$ for all $s, t > 0$,
- (ζ_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0,$$

then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

The set of all the simulation functions is denoted by \mathcal{Z} (see [2] and [7] for more details).

In [22], Roldán-López-de-Hierro et al. modified this definition of simulation functions and so enlarged the family of all simulation functions. To do this, only the condition (ζ_3) was replaced by the following condition (ζ_3)* as follows:

- (ζ_3)* If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$$

and

$$t_n < s_n \text{ for all } n \in \mathbb{N},$$

then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

Every simulation function in the Khojasteh et al.'s sense is also a simulation function in the Roldán-López-de-Hierro et al.'s sense, the converse is not true (for example see Example 3.3 in [22]). In applications of the simulation functions to the study of discontinuity problem and the geometric study of the non unique fixed points of self-mappings, the condition (ζ_3) is not used. So, both definitions of the simulation functions and examples can be used to study such kind applications. Some examples of simulation functions $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ are

- 1) $\zeta(t, s) = \lambda s - t$, where $\lambda \in [0, 1)$,
- 2) $\zeta(t, s) = s - \varphi(s) - t$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$,
- 3) $\zeta(t, s) = s\phi(s) - t$, where $\phi : [0, \infty) \rightarrow [0, 1)$ is a mapping such that $\limsup_{t \rightarrow r^+} \phi(t) < 1$ for all $r > 0$,
- 4) $\zeta(t, s) = \eta(s) - t$, where $\eta : [0, \infty) \rightarrow [0, \infty)$ be an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$,

5) $\zeta(t, s) = s - \int_0^t \psi(t) dt$, where $\psi : [0, \infty) \rightarrow [0, 1)$ is a function such that $\int_0^t \psi(t) dt$ exists and $\int_0^\varepsilon \psi(u) du > \varepsilon$ for each $\varepsilon > 0$.

The main motivation of this paper is the study of the geometric properties of the fixed point set of a self-mapping in the non unique fixed point case. There are some examples of self-mappings where the fixed point set of the self-mapping contains a geometric figure such as a circle, a disc or an ellipse. For example, let us consider the metric space (\mathbb{C}, d) with the metric defined for the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as follows:

$$d(z_1, z_2) = \sqrt{\frac{(x_1 - x_2)^2}{9} + 4(y_1 - y_2)^2},$$

where d is the metric induced by the norm function $\|z\| = \|x + iy\| = \sqrt{\frac{x^2}{9} + 4y^2}$. Consider the circle $C_{0,1}$ and define the self-mapping g on \mathbb{C} by

$$gz = \begin{cases} z & ; x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0 \\ \frac{36z}{-35(z^2 + \bar{z}^2) + 74z\bar{z}} & ; x < 0, y > 0 \text{ or } x > 0, y < 0 \end{cases},$$

for each $z = x + iy \in \mathbb{C}$, where $\bar{z} = x - iy$ is the complex conjugate of z . Then, it is easy to verify that the fixed point set of g contains the circle $C_{0,1}$, that is, $C_{0,1}$ is a fixed circle of g . We use the notion of “inversion in an ellipse” to construct this self-mapping (see Proposition 1 given in [25]).

There are several papers for the cases a fixed circle and a fixed disc (see [1, 10, 12, 17, 18, 19, 20, 24, 28] and the references therein). The fixed ellipse case is also considered in the recent studies [3] and [5]. In [3], the cases a fixed Apollonius circle, fixed hyperbola and fixed Cassini curve are considered extensively on metric and some generalized metric spaces. Therefore, the study of geometric properties of the fixed point set of a self-mapping seems to be an interesting problem in case where the fixed point is non unique. In this paper, we study on the geometric properties of the fixed point set of a self-mapping via simulation functions on metric (resp. S -metric and b -metric) spaces. The relationships among a metric, an S -metric and a b -metric are well known, so we refer the reader to [14], [26] and [27] for more details.

2. Simulation functions and the geometry of fixed points

In this section, we study on geometric properties of the fixed point set $Fix(f) = \{x \in X : fx = x\}$ for a self-mapping f of a metric (resp. S -metric, b -metric) space. The set of simulation functions has been used in [11], [18] and [21] to obtain new

results on the fixed-circle (resp. fixed-disc) problem. In [21], together with some properties of simulation functions, the numbers $M(x, y)$ and ρ defined by

$$M(x, y) = \max \left\{ \begin{array}{l} ad(x, fx) + (1-a)d(y, fy), \\ (1-a)d(x, fx) + ad(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \end{array} \right\}, \quad 0 \leq a < 1, \quad (2.1)$$

$$\rho = \inf \{d(x, fx) : x \neq fx, x \in X\}, \quad (2.2)$$

and an auxiliary function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\varphi(t) < t$ for each $t > 0$, were used to get new fixed-circle (resp. fixed-disc) results. Using these numbers and an auxiliary function, we present new results on the geometric study of the fixed point set of a self-mapping.

2.1. Geometric study of fixed points on metric spaces. Let (X, d) be a metric space and $f : X \rightarrow X$ be a self-mapping. First, we recall that the circle $C_{x_0, \rho} = \{x \in X : d(x, x_0) = \rho\}$ (resp. the disc $D_{x_0, \rho} = \{x \in X : d(x, x_0) \leq \rho\}$) is a fixed circle (resp. a fixed disc) of f if $fx = x$ for all $x \in C_{x_0, \rho}$ (resp. for all $x \in D_{x_0, \rho}$) (see [17], [18]). More generally, a geometric figure \mathcal{F} (a circle, an ellipse, a hyperbola, a Cassini curve etc.) contained in the fixed point set $Fix(f)$ is called a *fixed figure* (a fixed circle, a fixed ellipse, a fixed hyperbola, a fixed Cassini curve, etc.) of the self-mapping f .

Let $E_r(x_1, x_2)$ be the ellipse defined as

$$E_r(x_1, x_2) = \{x \in X : d(x, x_1) + d(x, x_2) = r\}.$$

Clearly, we have

$$r = 0 \Rightarrow x_1 = x_2 \text{ and } E_r(x_1, x_2) = C_{x_1, r} = \{x_1\}.$$

Now, we use the set of simulation functions and the number ρ to obtain some results for the case where the fixed point set $Fix(f)$ contains an ellipse or an ellipse with its interior.

Theorem 2.1. *Let (X, d) be a metric space, $f : X \rightarrow X$ be a self-mapping, $\zeta \in \mathcal{Z}$ be a simulation function and the number ρ be defined as in (2.2). If there exist some points $x_1, x_2 \in X$ such that*

(a) *For all $x \in E_\rho(x_1, x_2)$, there exists $\delta(\rho) > 0$ satisfying*

$$\frac{\rho}{2} \leq M(x, x_1) + M(x, x_2) < \frac{\rho}{2} + \delta(\rho) \implies d(fx, x_1) + d(fx, x_2) \leq \rho,$$

(b) *For all $x \in X$,*

$$d(fx, x) > 0 \implies \zeta(d(fx, x), M(x, x_1)) \geq 0 \text{ and } \zeta(d(fx, x), M(x, x_2)) \geq 0,$$

(c) For all $x \in X$,

$$d(fx, x) > 0 \implies \zeta \left(d(fx, x), \frac{d(x, x_1) + d(fx, x_1) + d(x, x_2) + d(fx, x_2)}{2} \right) \geq 0,$$

then $fx_1 = x_1$, $fx_2 = x_2$ and $\text{Fix}(f)$ contains the ellipse $E_\rho(x_1, x_2)$.

Proof. We have

$$M(x_1, x_1) = d(fx_1, x_1) \text{ and } M(x_2, x_2) = d(fx_2, x_2).$$

First, we show that $fx_1 = x_1$ and $fx_2 = x_2$. If $fx_1 \neq x_1$ and $fx_2 \neq x_2$ then $d(fx_1, x_1) > 0$ and $d(fx_2, x_2) > 0$. Using the condition (ζ_2) , we find

$$\begin{aligned} \zeta(d(fx_1, x_1), M(x_1, x_1)) &= \zeta(d(fx_1, x_1), d(fx_1, x_1)) \\ &< d(fx_1, x_1) - d(fx_1, x_1) = 0 \end{aligned}$$

and

$$\begin{aligned} \zeta(d(fx_2, x_2), M(x_2, x_2)) &= \zeta(d(fx_2, x_2), d(fx_2, x_2)) \\ &< d(fx_2, x_2) - d(fx_2, x_2) = 0, \end{aligned}$$

which are contradictions with the condition (b). Hence it should be $fx_1 = x_1$ and $fx_2 = x_2$.

If $\rho = 0$, then we have $E_\rho(x_1, x_2) = C_{x_1, \rho} = \{x_1\}$ and $x_1 = x_2$. Hence, the proof is completed.

Now we assume $\rho \neq 0$. Let $x \in E_\rho(x_1, x_2)$ be any point such that $fx \neq x$. Then $d(x, fx) > 0$ and we have

$$M(x, x_1) = \max \left\{ ad(x, fx), (1-a)d(x, fx), \frac{d(x, x_1) + d(x_1, fx)}{2} \right\}$$

and

$$M(x, x_2) = \max \left\{ ad(x, fx), (1-a)d(x, fx), \frac{d(x, x_2) + d(x_2, fx)}{2} \right\}.$$

Using the condition (a), we get

$$\frac{\rho}{2} \leq M(x, x_1) + M(x, x_2) < \frac{\rho}{2} + \delta(\rho) \implies d(fx, x_1) + d(fx, x_2) \leq \rho. \quad (2.3)$$

Now, using the inequality (2.3) and the conditions (c), (ζ_2) , we obtain

$$\begin{aligned}
0 &\leq \zeta \left(d(fx, x), \frac{d(x, x_1) + d(fx, x_1) + d(fx, x_2) + d(x, x_2)}{2} \right) \\
&< \frac{d(x, x_1) + d(fx, x_1) + d(fx, x_2) + d(x, x_2)}{2} - d(fx, x) \\
&= \frac{d(x, x_1) + d(x, x_2)}{2} + \frac{d(fx, x_1) + d(fx, x_2)}{2} - d(fx, x) \\
&\leq \frac{\rho}{2} + \frac{\rho}{2} - d(fx, x) = \rho - d(fx, x)
\end{aligned}$$

and hence

$$d(fx, x) < \rho.$$

This is a contradiction by the definition of the number ρ . Because of this contradiction, it should be $fx = x$. Consequently, we have $E_\rho(x_1, x_2) \subset \text{Fix}(f)$. \square

Remark 2.1. If $x_1 = x_2$ then we have $E_\rho(x_1, x_2) = C_{x_1, \frac{\rho}{2}}$ and Theorem 2.1 is reduced to a fixed-circle theorem as follows:

Theorem 2.2. Let (X, d) be a metric space, $f : X \rightarrow X$ be a self-mapping, $\zeta \in \mathcal{Z}$ be a simulation function and the number ρ be defined as in (2.2). If there exists some point $x_0 \in X$ such that

(a) For all $x \in C_{x_0, \rho}$, there exists $\delta(\rho) > 0$ satisfying

$$\frac{\rho}{4} \leq M(x, x_0) < \frac{\rho}{4} + \delta(\rho) \implies d(fx, x_0) \leq \rho,$$

(b) For all $x \in X$,

$$d(fx, x) > 0 \implies \zeta(d(fx, x), M(x, x_0)) \geq 0,$$

then $fx_0 = x_0$ and the set $\text{Fix}(f)$ contains the circle $C_{x_0, \rho}$.

Proof. The proof follows by Theorem 2.1 and Remark 2.1. \square

Example 2.1. Let $X = \{-3, -1, 1, 3, 12, 18\}$ with the metric $d(x, y) = |x - y|$. Define the self-mapping $f : X \rightarrow X$ by

$$fx = \begin{cases} x + 6 & , \quad x = 12 \\ x & , \quad x \in \{-3, -1, 1, 3, 18\} \end{cases}.$$

Then the self-mapping f satisfies the conditions of Theorem 2.1 for the points $x_1 = -1$ and $x_2 = 1$ and the simulation function $\zeta(t, s) = \frac{1}{2}s - t$. Indeed, we have

$$\rho = \min \{d(x, fx) : x \in X, x \neq fx\} = 6$$

and

$$E_6(-1, 1) = \{-3, 3\}.$$

For all $x \in E_6(-1, 1)$, there exists $\delta(\rho) = 4 > 0$ satisfying

$$3 \leq M(x, -1) + M(3, 1) < 3 + 4 \implies d(fx, -1) + d(fx, 1) = 6 \leq \rho,$$

hence the condition (a) is satisfied.

For $x = 12$, we have $d(x, fx) \neq 0$, $M(12, -1) = 16$, $M(12, 1) = 14$ and so, we obtain

$$\begin{aligned} \zeta(d(fx, x), M(x, x_1)) &= \zeta(6, 16) = \frac{16}{2} - 6 \\ &= 2 > 0 \end{aligned}$$

and

$$\begin{aligned} \zeta(d(fx, x), M(x, x_2)) &= \zeta(6, 14) = \frac{14}{2} - 6 \\ &= 1 > 0. \end{aligned}$$

This shows that the condition (b) is also satisfied by f .

Since we have $d(x, fx) > 0$ for $x = 12$, we find

$$\begin{aligned} \zeta\left(d(fx, x), \frac{d(x, x_1) + d(fx, x_1) + d(fx, x_2) + d(x, x_2)}{2}\right) &= \zeta(6, 30) \\ &= \frac{30}{2} - 6 = 9 > 0, \end{aligned}$$

hence the condition (c) is satisfied.

Clearly, we have $\text{Fix}(f) = \{-3, -1, 1, 3, 18\}$ and the ellipse $E_6(-1, 1) = \{-3, 3\}$ is contained in the set $\text{Fix}(f)$. That is, the ellipse $E_6(-1, 1)$ is a fixed ellipse of the self-mapping f .

On the other hand, it is easy to check that the self-mapping f satisfies the conditions of Theorem 2.2 for the point $x_0 = 3$ and the simulation function $\zeta(t, s) = \frac{2}{3}s - t$. Clearly, the set $\text{Fix}(f)$ contains the circle $C_{3,6} = \{-3\}$.

Definition 2.1. Let $\zeta \in \mathcal{Z}$ be any simulation function. The self-mapping f is said to be a \mathcal{Z}_E -contraction with respect to ζ if there exist $x_1, x_2 \in X$ such that the following condition holds for all $x \in X$:

$$d(fx, x) > 0 \implies \zeta(d(fx, x), d(fx, x_1) + d(fx, x_2)) \geq 0.$$

If f is a \mathcal{Z}_E -contraction with respect to ζ , then we have

$$d(fx, x) < d(fx, x_1) + d(fx, x_2), \quad (2.4)$$

for all $x \in X$ with $fx \neq x_1$ or $fx \neq x_2$. Indeed, if $fx = x$ then the inequality (2.4) is satisfied trivially. If $fx \neq x$ then $d(fx, x) > 0$ and by the definition of a \mathcal{Z}_E -contraction and the condition (ζ_2) , we obtain

$$0 \leq \zeta(d(fx, x), d(fx, x_1) + d(fx, x_2)) < d(fx, x_1) + d(fx, x_2) - d(fx, x)$$

and so Equation (2.4) is satisfied.

Now we give the following theorem.

Theorem 2.3. *Let f be a \mathcal{Z}_E -contraction with respect to ζ with $x_1, x_2 \in X$ and consider the set*

$$\overline{E}_\rho(x_1, x_2) = \{x \in X : d(x, x_1) + d(x, x_2) \leq \rho\}.$$

If the condition $0 < d(fx, x_1) + d(fx, x_2) \leq \rho$ holds for all $x \in \overline{E}_\rho(x_1, x_2) - \{x_1, x_2\}$ then $Fix(f)$ contains the set $\overline{E}_\rho(x_1, x_2)$.

Proof. If $\rho = 0$, then we have $\overline{E}_\rho(x_1, x_2) = D_{x_1, \rho} = \{x_1\}$ and this theorem coincides with Theorem 2.2 in [18]. In this case, we have $fx_1 = x_1$. Assume that $\rho \neq 0$. If $x_1 = x_2$ then $\overline{E}_\rho(x_1, x_2) = D_{x_1, \frac{\rho}{2}}$ and again this case is reduced to Theorem 2.2 in [18].

Assume that $x_1 \neq x_2$ and let $x \in \overline{E}_\rho(x_1, x_2)$ be such that $fx \neq x$. By the definition of ρ , we have $0 < \rho \leq d(x, fx)$ and using the condition (ζ_2) we find

$$\begin{aligned} \zeta(d(fx, x), d(fx, x_1) + d(fx, x_2)) &< d(fx, x_1) + d(fx, x_2) - d(fx, x) \\ &\leq \rho - d(fx, x) \leq \rho - \rho = 0, \end{aligned}$$

a contradiction with the \mathcal{Z}_E -contractive property of f . This contradiction leads $fx = x$, so the set $Fix(f)$ contains the set $\overline{E}_\rho(x_1, x_2)$. \square

Example 2.2. *Let us consider the self-mapping f defined in Example 2.1. f is an \mathcal{Z}_E -contraction with respect to $\zeta(t, s) = \frac{1}{2}s - t$ with the points $x_1 = -1$ and $x_2 = 1$. Indeed, we get*

$$\begin{aligned} \zeta(d(fx, x), d(fx, x_1) + d(fx, x_2)) &= \zeta(6, 19 + 17) = \zeta(6, 36) \\ &= \frac{36}{2} - 6 = 12 \geq 0, \end{aligned}$$

for $x = 12$ with $d(fx, x) > 0$. Also we have,

$$0 < d(fx, -1) + d(fx, 1) \leq 6,$$

for all $x \in \overline{E}_6(-1, 1) - \{-1, 1\}$. Therefore, f satisfies the conditions of Theorem 2.3. Notice that the set $Fix(f)$ contains the set $\overline{E}_6(-1, 1) = \{-3, -1, 1, 3\}$.

Let $r \in [0, \infty)$. Now we give a fixed-circle theorem using the auxiliary function $\varphi_r : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ defined by

$$\varphi_r(u) = \begin{cases} u - r & ; u > 0 \\ 0 & ; u = 0 \end{cases}, \quad (2.5)$$

for all $u \in \mathbb{R}^+ \cup \{0\}$ (see [15]).

Theorem 2.4. *Let (X, d) be a metric space, $\zeta \in \mathcal{Z}$ be a simulation function and $C_{x_0, r}$ be any circle on X . If there exists a self-mapping $f : X \rightarrow X$ satisfying*

- i) $d(x_0, fx) = r$ for each $x \in C_{x_0, r}$,*
- ii) $\zeta(r, d(fx, fy)) \geq 0$ for each $x, y \in C_{x_0, r}$ with $x \neq y$,*
- iii) $\zeta(d(fx, fy), d(x, y) - \varphi_r(d(x, fx))) \geq 0$ for each $x, y \in C_{x_0, r}$,*
- iv) f is one to one on the circle $C_{x_0, r}$,*

then the circle $C_{x_0, r}$ is a fixed circle of f .

Proof. Let $x \in C_{x_0, r}$ be an arbitrary point. By the condition (i), we have $d(x_0, fx) = r$, that is, $fx \in C_{x_0, r}$. Now we show that $fx = x$ for all $x \in C_{x_0, r}$. Conversely, assume that $x \neq fx$ for any $x \in C_{x_0, r}$. Then we have $d(x, fx) > 0$ and using the conditions (ii), (iv) and (ζ_2) , we get

$$0 \leq \zeta(r, d(fx, f^2x)) < d(fx, f^2x) - r$$

and so

$$r < d(fx, f^2x). \quad (2.6)$$

Using the definition of the function φ_r and the conditions (iii), (iv) and (ζ_2) , we obtain

$$\begin{aligned} 0 &\leq \zeta(d(fx, f^2x), d(x, fx) - \varphi_r(d(x, fx))) = \zeta(d(fx, f^2x), d(x, fx) - (d(x, fx) - r)) \\ &= \zeta(d(fx, f^2x), r) < r - d(fx, f^2x) \end{aligned}$$

and hence

$$d(fx, f^2x) < r,$$

which is a contradiction with the inequality (2.6). Therefore it should be $fx = x$ for each $x \in C_{x_0, r}$. Consequently, $C_{x_0, r}$ is a fixed circle of f . \square

Remark 2.2. *If we consider the self-mapping f defined in Example 2.1, it is easy to verify that f satisfies the conditions of Theorem 2.1 and Theorem 2.3 for the ellipse $E_6(-3, 3) = \{-3, -1, 1, 3\}$ with the simulation function $\zeta(t, s) = \frac{2}{3}s - t$. This shows that the fixed ellipse is not unique for the number ρ defined in (2.2). On the other hand, the fixed point set $Fix(f) = \{-3, -1, 1, 3, 18\}$ contains also the ellipses $E_4(-3, 1) = \{-3, -1, 1\}$ and $E_4(-1, 3) = \{-1, 1, 3\}$ other than the ellipses $E_6(-3, 3)$ and $E_6(-1, 1)$. We deduce that the number ρ defined in (2.2) can not produce all fixed ellipses (resp. circles) for a self-mapping f .*

This remark shows also that a fixed ellipse may not be unique. Now, we give a general result which ensure the uniqueness of a fixed geometric figure (for example, a circle, an Apollonius circle, an ellipse, a hyperbola, etc.) for a self-mapping of a metric space (X, d) .

Theorem 2.5. (*The uniqueness theorem*) Let (X, d) be a metric space, the number $M(x, y)$ be defined as in (2.1) and $f : X \rightarrow X$ be a self-mapping. Assume that the fixed point set $\text{Fix}(f)$ contains a geometric figure \mathcal{F} . If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition

$$\zeta(d(fx, fy), M(x, y)) \geq 0 \quad (2.7)$$

is satisfied by f for all $x \in \mathcal{F}$ and $y \in X - \mathcal{F}$, then the figure \mathcal{F} is the unique fixed figure of f .

Proof. Assume that \mathcal{F}^* is another fixed figure of f . Let $x \in \mathcal{F}$, $y \in \mathcal{F}^*$ with $x \neq y$ be arbitrary points. Using the inequality (2.7) and the condition (ζ_2) , we find

$$0 \leq \zeta(d(fx, fy), M(x, y)) = \zeta(d(x, y), d(x, y)) < d(x, y) - d(x, y) = 0,$$

a contradiction. Hence, it should be $x = y$ for all $x \in \mathcal{F}$, $y \in \mathcal{F}^*$. This shows the uniqueness of the fixed figure \mathcal{F} of f . \square

Now we give a condition which excludes the identity map $I_X : X \rightarrow X$ defined by $I_X(x) = x$ for all $x \in X$ from the above results.

Theorem 2.6. Let (X, d) be a metric space, $f : X \rightarrow X$ be a self-mapping and $r \in [0, \infty)$ be a fixed number. If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition

$$d(x, fx) < \zeta(d(x, fx), \varphi_r(d(x, fx)) + r)$$

is satisfied by f for all $x \notin \text{Fix}(f)$ if and only if $f = I_X$.

Proof. Let $x \in X$ be an arbitrary point with $x \notin \text{Fix}(f)$. Using (2.5) and the condition (ζ_2) , we find

$$\begin{aligned} d(x, fx) &\leq \zeta(d(x, fx), \varphi_r(d(x, fx)) + r) = \zeta(d(x, fx), (d(x, fx) - r) + r) \\ &= \zeta(d(x, fx), d(x, fx)) < d(x, fx) - d(x, fx) = 0, \end{aligned}$$

a contradiction. Hence, it should be $x \in \text{Fix}(f)$ for all $x \in X$, that is, $\text{Fix}(f) = X$. This shows that $f = I_X$. Clearly, the identity map I_X satisfies the condition of the hypothesis for any simulation function $\zeta \in \mathcal{Z}$. \square

2.2. Geometric study of fixed points on S -metric and b -metric spaces. At first, we recall the concept of an S -metric space.

Definition 2.2. [26] Let X be nonempty set and $\mathcal{S} : X^3 \rightarrow [0, \infty)$ be a function satisfying the following conditions

- (1) $\mathcal{S}(x, y, z) = 0$ if and only if $x = y = z$,
- (2) $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$,

for all $x, y, z, a \in X$. Then \mathcal{S} is called an S -metric on X and the pair (X, \mathcal{S}) is called an S -metric space.

Let (X, d) be a metric space. It is known that the function $\mathcal{S}_d : X^3 \rightarrow [0, \infty)$ defined by

$$\mathcal{S}_d(x, y, z) = d(x, z) + d(y, z)$$

for all $x, y, z \in X$ is an S -metric on X [4]. The S -metric \mathcal{S}_d is called the S -metric generated by the metric d [14]. For example, let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \rightarrow [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = |x - z| + |y - z|, \quad (2.8)$$

for all $x, y, z \in \mathbb{R}$ [27]. Then (X, \mathcal{S}) is called the usual S -metric space. This S -metric is generated by the usual metric on \mathbb{R} . The main motivation of this subsection is the existence of some examples of S -metrics which are not generated by any metric. For example, let $X = \mathbb{R}$ and the function $\mathcal{S} : X^3 \rightarrow [0, \infty)$ be defined by

$$\mathcal{S}(x, y, z) = |x - z| + |x + z - 2y|, \quad (2.9)$$

for all x, y and $z \in \mathbb{R}$. Then, \mathcal{S} is an S -metric on \mathbb{R} , which is not generated by any metric, and the pair $(\mathbb{R}, \mathcal{S})$ is an S -metric space (see [14] for more details and examples).

Let (X, \mathcal{S}) be an S -metric space and $f : X \rightarrow X$ be a self-mapping. In this subsection, we give new solutions to the fixed-circle problem (resp. fixed-disc problem and fixed ellipse problem) for self-mappings of an S -metric space (resp. a b -metric space). For the S -metric case, we use the following numbers

$$\mu = \inf \{ \mathcal{S}(x, x, fx) : x \neq fx, x \in X \}, \quad (2.10)$$

$$M_S(x, y) = \max \left\{ \begin{array}{l} a\mathcal{S}(x, x, fx) + (1 - a)\mathcal{S}(y, y, fy), \\ (1 - a)\mathcal{S}(x, x, fx) + a\mathcal{S}(y, y, fy), \frac{\mathcal{S}(x, x, fy) + \mathcal{S}(y, y, fx)}{4} \end{array} \right\}, \quad 0 \leq a < 1 \quad (2.11)$$

and the following symmetry property given in [26]

$$\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x), \quad (2.12)$$

for all $x, y \in X$ on an S -metric space (X, \mathcal{S}) . Before stating our results, we recall the definitions of a circle, a disc and an ellipse on an S -metric space, respectively, as follows:

$$\begin{aligned} C_{x_0, r}^S &= \{x \in X : \mathcal{S}(x, x, x_0) = r\}, \\ D_{x_0, r}^S &= \{x \in X : \mathcal{S}(x, x, x_0) \leq r\} \end{aligned}$$

and

$$E_r^S(x_1, x_2) = \{x \in X : \mathcal{S}(x, x, x_1) + \mathcal{S}(x, x, x_2) = r\},$$

where $r \in [0, \infty)$ [16], [26]. For a self-mapping f of an S -metric space, the definition of a fixed figure (circle, disc, ellipse, etc.) can be given similar to the case introduced in the previous section (see [16] and [10] for the definitions of a fixed circle and a fixed disc).

Theorem 2.7. *Let (X, \mathcal{S}) be an S -metric space, $f : X \rightarrow X$ be a self-mapping, $\zeta \in \mathcal{Z}$ be a simulation function and the number μ be defined as in (2.10). If there exist some points $x_1, x_2 \in X$ such that*

(a) *For all $x \in E_\rho^S(x_1, x_2)$, there exists $\delta(\mu) > 0$ satisfying*

$$\frac{\mu}{2} \leq M_S(x, x_1) + M_S(x, x_2) < \frac{\mu}{2} + \delta(\mu) \implies \mathcal{S}(fx, fx, x_1) + \mathcal{S}(fx, fx, x_2) \leq \mu,$$

(b) *For all $x \in X$,*

$$\mathcal{S}(fx, fx, x) > 0 \implies \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_1)) \geq 0 \text{ and } \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_2)) \geq 0,$$

(c) *For all $x \in X$,*

$$\mathcal{S}(fx, fx, x) > 0 \implies \zeta\left(\mathcal{S}(fx, fx, x), \frac{\mathcal{S}(x, x, x_1) + \mathcal{S}(fx, fx, x_1) + \mathcal{S}(x, x, x_2) + \mathcal{S}(fx, fx, x_2)}{2}\right) \geq 0,$$

then $fx_1 = x_1$, $fx_2 = x_2$ and $\text{Fix}(f)$ contains the ellipse $E_\mu^S(x_1, x_2)$.

Proof. We have

$$M_S(x_1, x_1) = \mathcal{S}(x_1, x_1, fx_1) \text{ and } M_S(x_2, x_2) = \mathcal{S}(x_2, x_2, fx_2).$$

First, we show that $fx_1 = x_1$ and $fx_2 = x_2$. If $fx_1 \neq x_1$ and $fx_2 \neq x_2$ then $\mathcal{S}(x_1, x_1, fx_1) > 0$ and $\mathcal{S}(x_2, x_2, fx_2) > 0$. Using the symmetry condition (2.12) and the condition (ζ_2) , we obtain

$$\begin{aligned} \zeta(\mathcal{S}(fx_1, fx_1, x_1), M_S(x_1, x_1)) &= \zeta(\mathcal{S}(fx_1, fx_1, x_1), \mathcal{S}(x_1, x_1, fx_1)) \\ &< \mathcal{S}(fx_1, fx_1, x_1) - \mathcal{S}(x_1, x_1, fx_1) = 0 \end{aligned}$$

and

$$\begin{aligned} \zeta(\mathcal{S}(fx_2, fx_2, x_2), M_S(x_2, x_2)) &= \zeta(\mathcal{S}(fx_2, fx_2, x_2), d(fx_2, x_2)) \\ &< d(fx_2, x_2) - d(fx_2, x_2) = 0, \end{aligned}$$

which are contradictions by the condition (b). Hence it should be $fx_1 = x_1$ and $fx_2 = x_2$.

If $\mu = 0$, it is easy to check that $E_\mu^S(x_1, x_2) = C_{x_1, \mu}^S = \{x_1\}$ and $x_1 = x_2$. Hence, the proof is completed.

Assume that $\mu \neq 0$. Let $x \in E_\mu^S(x_1, x_2)$ be any point such that $fx \neq x$. Then $\mathcal{S}(x, x, fx) > 0$ and we have

$$M^S(x, x_1) = \max \left\{ a\mathcal{S}(x, x, fx), (1-a)\mathcal{S}(x, x, fx), \frac{\mathcal{S}(x, x, fx_1) + \mathcal{S}(x_1, x_1, fx)}{4} \right\}$$

and

$$M^S(x, x_2) = \max \left\{ a\mathcal{S}(x, x, fx), (1-a)\mathcal{S}(x, x, fx), \frac{\mathcal{S}(x, x, fx_2) + \mathcal{S}(x_2, x_2, fx)}{4} \right\}.$$

Using the condition (a), we get

$$\frac{\mu}{2} \leq M^S(x, x_1) + M^S(x, x_2) < \frac{\mu}{2} + \delta(\mu) \implies \mathcal{S}(fx, fx, x_1) + \mathcal{S}(fx, fx, x_2) \leq \mu. \quad (2.13)$$

Now, using the inequality (2.13), the conditions (c), (ζ_2) and the symmetry condition (2.12), we obtain

$$\begin{aligned} 0 &\leq \zeta \left(\mathcal{S}(fx, fx, x), \frac{\mathcal{S}(x, x, x_1) + \mathcal{S}(fx, fx, x_1) + \mathcal{S}(x, x, x_2) + \mathcal{S}(fx, fx, x_2)}{2} \right) \\ &< \frac{\mathcal{S}(x, x, x_1) + \mathcal{S}(fx, fx, x_1) + \mathcal{S}(x, x, x_2) + \mathcal{S}(fx, fx, x_2)}{2} - \mathcal{S}(fx, fx, x) \\ &= \frac{\mathcal{S}(x, x, x_1) + \mathcal{S}(x, x, x_2)}{2} + \frac{\mathcal{S}(fx, fx, x_1) + \mathcal{S}(fx, fx, x_2)}{2} - \mathcal{S}(fx, fx, x) \\ &\leq \frac{\mu}{2} + \frac{\mu}{2} - \mathcal{S}(fx, fx, x) = \mu - \mathcal{S}(fx, fx, x) \end{aligned}$$

and so

$$\mathcal{S}(fx, fx, x) < \mu,$$

which is a contradiction by the definition of the number μ . This contradiction leads to $fx = x$. Consequently, we have $E_\mu^S(x_1, x_2) \subset \text{Fix}(f)$. \square

If $x_1 = x_2$ then we have $E_\mu^S(x_1, x_2) = C_{x_1, \frac{\mu}{2}}$ and Theorem 2.7 is reduced to a fixed-circle theorem as follows:

Theorem 2.8. *Let (X, \mathcal{S}) be an S -metric space, $f : X \rightarrow X$ be a self-mapping and the number μ be defined as in (2.10). If there exist a simulation function $\zeta \in \mathcal{Z}$ and a point $x_0 \in X$ such that*

(i) *For all $x \in C_{x_0, \mu}^S$, there exists a $\delta(\mu) > 0$ satisfying*

$$\frac{\mu}{4} \leq M_S(x, x_0) < \frac{\mu}{4} + \delta(\mu) \implies \mathcal{S}(fx, fx, x_0) \leq \mu,$$

(ii) *For all $x \in X$,*

$$\mathcal{S}(fx, fx, x) > 0 \implies \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_0)) \geq 0,$$

then $x_0 \in \text{Fix}(f)$ and the circle $C_{x_0, \mu}^S$ is a fixed circle of f .

Proof. If $fx_0 \neq x_0$ then using the symmetry property (2.12) and the condition (ζ_2) , we get

$$M_S(x_0, x_0) = \mathcal{S}(x_0, x_0, fx_0) = \mathcal{S}(fx_0, fx_0, x_0)$$

and

$$\begin{aligned}\zeta(\mathcal{S}(fx_0, fx_0, x_0), M_S(x_0, x_0)) &= \zeta(\mathcal{S}(fx_0, fx_0, x_0), \mathcal{S}(fx_0, fx_0, x_0)) \\ &< \mathcal{S}(fx_0, fx_0, x_0) - \mathcal{S}(fx_0, fx_0, x_0) = 0.\end{aligned}$$

This last inequality is a contradiction by (ii). Therefore, it should be $fx_0 = x_0$. This shows that the circle $C_{x_0, \mu}^S = \{x_0\}$ is a fixed circle of f when $\mu = 0$. Now, let $\mu > 0$ and $x \in C_{x_0, \mu}^S$ be any element. To show that f fixes the circle $C_{x_0, \mu}^S$, we suppose that $fx \neq x$ for any $x \in C_{x_0, \mu}^S$. Since $x_0 \in \text{Fix}(f)$, we have

$$\begin{aligned}M_S(x, x_0) &= \max \left\{ a\mathcal{S}(x, x, fx), (1-a)\mathcal{S}(x, x, fx), \frac{\mathcal{S}(x, x, fx_0) + \mathcal{S}(x_0, x_0, fx)}{4} \right\} \\ &= \max \left\{ a\mathcal{S}(x, x, fx), (1-a)\mathcal{S}(x, x, fx), \frac{\mu + \mathcal{S}(x_0, x_0, fx)}{4} \right\}.\end{aligned}$$

Using the conditions (i), (ii), (ζ_2) and the symmetry property (2.12), we find

$$\begin{aligned}0 &< \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_0)) \\ &< M_S(x, x_0) - \mathcal{S}(fx, fx, x) < \mu - \mathcal{S}(fx, fx, x),\end{aligned}$$

a contradiction by the definition of the number μ . Hence, we have $fx = x$. Consequently, f fixes the circle $C_{x_0, \mu}^S$. \square

Corollary 2.1. *Let (X, \mathcal{S}) be an S -metric space, $f : X \rightarrow X$ be a self-mapping and the number μ be defined as in (2.10). If there exist a simulation function $\zeta \in \mathcal{Z}$ and a point $x_0 \in X$ such that*

(i) *For all $x \in D_{x_0, \mu}^S$, there exists a $\delta(\mu) > 0$ satisfying*

$$\frac{\mu}{4} \leq M_S(x, x_0) < \frac{r}{4} + \delta(\mu) \implies \mathcal{S}(fx, fx, x_0) \leq \mu,$$

(ii) *For all $x \in X$,*

$$\mathcal{S}(fx, fx, x) > 0 \implies \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_0)) \geq 0,$$

then the disc $D_{x_0, \mu}^S$ is a fixed disc of f .

Example 2.3. *Let $X = \{-3, -1, 1, 3, 12, 18\}$ with the S -metric defined in (2.9) and consider the self-mapping f defined in Example 2.1 on this S -metric space (X, \mathcal{S}) . It is easy to check that the self-mapping f satisfies the conditions of Theorem 2.7 with the points $x_1 = -1$ and $x_2 = 1$, the simulation function $\zeta(t, s) = \frac{7}{8}s - t$ and any $a \in [0, 1)$. We have*

$$\mu = \inf \{ \mathcal{S}(x, x, fx) : x \neq fx, x \in X \} = \mathcal{S}(12, 12, 18) = 12$$

and clearly the set $\text{Fix}(f)$ contains the ellipse $E_{12}^S(-1, 1) = \{-3, 3\}$. On the other hand, the self-mapping f does not satisfy the condition (b) of Theorem 2.7 for the

ellipse $E_{12}^S(-3, 3) = \{-3, -1, 1, 3\}$ for any simulation function ζ and any $a \in [0, 1)$. Indeed, we have

$$\begin{aligned} \zeta(\mathcal{S}(fx, fx, x), M_S(x, 3)) &= \zeta(\mathcal{S}(18, 18, 12), M_S(12, 3)) \\ &= \zeta(12, 12) < 0, \end{aligned}$$

by the condition (ζ_2) for the point $x = 12$ with $\mathcal{S}(18, 18, 12) > 0$. This shows that the converse statement of Theorem 2.7 is not true everwhen.

The self-mapping f satisfies the conditions of Theorem 2.8 with the point $x_0 = -3$ and the simulation function $\zeta(t, s) = \frac{5}{6}s - t$. The circle $C_{-3,12}^S = \{3\}$ is contained in the set $\text{Fix}(f)$. On the other hand, the self-mapping f does not satisfy the condition (ii) of Theorem 2.8 with the point $x_0 = 3$ for any simulation function ζ and any $a \in [0, 1)$. Since we have

$$\begin{aligned} \zeta(\mathcal{S}(fx, fx, x), M_S(x, x_0)) &= \zeta(\mathcal{S}(18, 18, 12), M_S(12, 3)) \\ &= \zeta(12, 12) < 0, \end{aligned}$$

by the condition (ζ_2) for the point $x = 12$. But, the circle $C_{3,12}^S = \{-3\}$ is a fixed circle of f . This shows that the converse statement of Theorem 2.8 is not true everwhen.

Remark 2.3. 1) Example 2.3 shows the importance of the studies on an S -metric space. Notice that the ellipse $E_{12}(-1, 1)$ and the circle $C_{-3,12}$ are empty sets in the metric space (X, d) in Example 2.1 but, if we consider the S -metric $\mathcal{S}(x, y, z) = |x - z| + |x + z - 2y|$ on X then the ellipse $E_{12}^S(-1, 1)$ and the circle $C_{-3,12}^S$ are not empty sets and both of them are contained in the set $\text{Fix}(f)$.

2) Note that S -metric versions of Definition 2.1 and Theorem 2.3 can also be given for self-mappings on an S -metric space.

Now we give a fixed-circle theorem using the auxiliary function $\varphi_r : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ defined in (2.5).

Theorem 2.9. (X, \mathcal{S}) be an S -metric space, $\zeta \in \mathcal{Z}$ be a simulation function and $C_{x_0, r}^S$ be any circle on X with $r > 0$. If there exists a self-mapping $f : X \rightarrow X$ satisfying

- i) $\mathcal{S}(fx, fx, x_0) = r$ for each $x \in C_{x_0, r}^S$,
 - ii) $\zeta(r, \mathcal{S}(fx, fx, fy)) \geq 0$ for each $x, y \in C_{x_0, r}^S$ with $x \neq y$,
 - iii) $\zeta(\mathcal{S}(fx, fx, fy), \mathcal{S}(x, x, y) - \varphi_r(\mathcal{S}(x, x, fx))) \geq 0$ for each $x, y \in C_{x_0, r}^S$,
 - iv) f is one to one on the circle $C_{x_0, r}^S$,
- then the circle $C_{x_0, r}^S$ is a fixed circle of f .

Proof. Let $x \in C_{x_0, r}^S$ be an arbitrary point. By the condition (i), we have $fx \in C_{x_0, r}^S$. To show that $fx = x$ for all $x \in C_{x_0, r}^S$, conversely, we assume that $x \neq fx$ for any

$x \in C_{x_0, r}^S$. Then we have $\mathcal{S}(x, x, fx) > 0$. Using the conditions (ii), (iv) and (ζ_2) , we obtain

$$0 \leq \zeta(r, \mathcal{S}(fx, fx, f^2x)) < \mathcal{S}(fx, fx, f^2x) - r$$

and hence

$$r < \mathcal{S}(fx, fx, f^2x). \quad (2.14)$$

Using the definition of the function φ_r and the conditions (iii), (iv) and (ζ_2) , we get

$$\begin{aligned} 0 &\leq \zeta(\mathcal{S}(fx, fx, f^2x), \mathcal{S}(x, x, fx) - \varphi_r(\mathcal{S}(x, x, fx))) \\ &= \zeta(\mathcal{S}(fx, fx, f^2x), \mathcal{S}(x, x, fx) - (\mathcal{S}(x, x, fx) - r)) \\ &= \zeta(\mathcal{S}(fx, fx, f^2x), r) < r - \mathcal{S}(fx, fx, f^2x) \end{aligned}$$

and hence

$$\mathcal{S}(fx, fx, f^2x) < r,$$

which is a contradiction with the inequality (2.14). Consequently, it should be $fx = x$ for each $x \in C_{x_0, r}^S$, that is, $C_{x_0, r}^S$ is a fixed circle of f . \square

Now we give a general result which ensure the uniqueness of a geometric figure contained in the set $Fix(f)$ for a self-mapping of an S -metric space (X, \mathcal{S}) .

Theorem 2.10. *(The uniqueness theorem) Let (X, \mathcal{S}) be an S -metric space, the number $M_S(x, y)$ be defined as in (2.11) and $f : X \rightarrow X$ be a self-mapping. Assume that the fixed point set $Fix(f)$ contains a geometric figure \mathcal{F} . If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition*

$$\zeta(\mathcal{S}(fx, fx, fy), M_S(x, y)) \geq 0 \quad (2.15)$$

is satisfied by f for all $x \in \mathcal{F}$ and $y \in X - \mathcal{F}$, then the figure \mathcal{F} is the unique fixed figure of f .

Proof. On the contrary, we suppose that \mathcal{F}^* is another fixed figure of the self-mapping f . Let $x \in \mathcal{F}$, $y \in \mathcal{F}^*$ with $x \neq y$ be arbitrary points. Using the inequality (2.15) and the condition (ζ_2) , we get

$$0 \leq \zeta(\mathcal{S}(fx, fx, fy), M_S(x, y)) = \zeta(\mathcal{S}(x, x, y), \mathcal{S}(x, x, y)) < \mathcal{S}(x, x, y) - \mathcal{S}(x, x, y) = 0,$$

a contradiction. Hence, it should be $x = y$ for all $x \in \mathcal{F}$, $y \in \mathcal{F}^*$. This shows the uniqueness of the fixed figure \mathcal{F} of f . \square

We give a condition which excludes the identity map $I_X : X \rightarrow X$ defined by $I_X(x) = x$ for all $x \in X$ from the above results.

Theorem 2.11. *Let (X, \mathcal{S}) be an S -metric space, $f : X \rightarrow X$ be a self-mapping and $r \in [0, \infty)$ be a fixed number. If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition*

$$\mathcal{S}(x, x, fx) < \zeta(\mathcal{S}(x, x, fx), \varphi_r(\mathcal{S}(x, x, fx)) + r)$$

is satisfied by f for all $x \notin \text{Fix}(f)$ if and only if $f = I_X$.

Proof. The proof is similar to the proof of Theorem 2.6. □

Remark 2.4. 1) *Let (X, \mathcal{S}) be an S -metric space. Suppose that the S -metric \mathcal{S} is generated by a metric d . Then, for $0 \leq a < 1$ we have*

$$\begin{aligned} M_S(x, x_0) &= \max \left\{ \begin{array}{l} a\mathcal{S}(x, x, fx) + (1-a)\mathcal{S}(x_0, x_0, fx_0), \\ (1-a)\mathcal{S}(x, x, fx) + a\mathcal{S}(x_0, x_0, fx_0), \frac{\mathcal{S}(x, x, fx) + \mathcal{S}(x_0, x_0, fx_0)}{4} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} 2ad(x, fx) + 2(1-a)d(x_0, fx_0), \\ 2(1-a)d(x, fx) + 2ad(x_0, fx_0), \frac{d(x, fx) + d(x_0, fx_0)}{2} \end{array} \right\} \end{aligned}$$

and so

$$M(x, x_0) \leq M_S(x, x_0).$$

Consequently, Theorem 2.8 (resp. Corollary 2.1) is a generalization of Theorem 3.2 (resp. Corollary 3.2) given in [21].

2) *Similar definition of the notion of a fixed figure (circle, disc, ellipse and so on) can be given for a self-mapping of a b -metric space.*

3) *From [27], we know that the function defined by*

$$d^S(x, y) = S(x, x, y) = 2d(x, y),$$

for all $x, y \in X$, defines a b -metric on an S -metric space (X, \mathcal{S}) with $b = \frac{3}{2}$. If we consider Theorem 2.8 and Theorem 2.9 together with this fact, then we get the following fixed-circle (resp. fixed-disc) results on a b -metric space using the number

$$M_{d^S}(x, y) = \max \left\{ \begin{array}{l} ad^S(x, fx) + (1-a)d^S(y, fy), \\ (1-a)d^S(x, fx) + a(1-a)d^S(y, fy), \frac{d^S(x, fx) + d^S(y, fy)}{2} \end{array} \right\}, \quad 0 \leq a < 1. \quad (2.16)$$

Theorem 2.12. *Let (X, d^S) be a b -metric space, $f : X \rightarrow X$ a self-mapping and μ be defined as*

$$\mu = \inf \{d^S(x, fx) : x \neq fx, x \in X\}. \quad (2.17)$$

If there exist a simulation function $\zeta \in \mathcal{Z}$ and a point $x_0 \in X$ such that

(i) *For all $x \in C_{x_0, \mu}^{d^S}$, there exists a $\delta(\mu) > 0$ satisfying*

$$\frac{\mu}{4} \leq M_{d^S}(x, x_0) < \frac{\mu}{4} + \delta(\mu) \implies d^S(fx, x_0) \leq \mu,$$

where

$$C_{x_0, \mu}^{d^S} = \{x \in X : d^S(x, x_0) = \mu\}$$

and

$$M_{d^S}(x, x_0) = \max \left\{ \begin{array}{l} ad^S(x, fx) + (1-a)d^S(x_0, fx_0), \\ (1-a)d^S(x, fx) + ad^S(x_0, fx_0), \frac{d^S(x, fx_0) + d^S(x_0, fx)}{4} \end{array} \right\}, \quad 0 \leq a < 1,$$

(ii) For all $x \in X$,

$$d^S(fx, x) > 0 \implies \zeta(d^S(fx, x), M_{d^S}(x, x_0)) \geq 0,$$

then $x_0 \in \text{Fix}(f)$ and the circle $C_{x_0, \mu}^{d^S}$ is a fixed circle of f .

Corollary 2.2. Let (X, d^S) be a b -metric space, $f : X \rightarrow X$ be a self-mapping and the number μ be defined as in (2.17). If there exist a simulation function $\zeta \in \mathcal{Z}$ and a point $x_0 \in X$ such that

(i) For all $x \in D_{x_0, \mu}^{d^S}$, there exists a $\delta(\mu) > 0$ satisfying

$$\frac{\mu}{4} \leq M_{d^S}(x, x_0) < \frac{\mu}{4} + \delta(\mu) \implies d^S(fx, x_0) \leq \mu,$$

where

$$D_{x_0, \mu}^{d^S} = \{x \in X : d^S(x, x_0) \leq \mu\},$$

(ii) For all $x \in X$,

$$d^S(fx, x) > 0 \implies \zeta(d^S(fx, x), M_{d^S}(x, x_0)) \geq 0,$$

then the disc $D_{x_0, \mu}^{d^S}$ is a fixed disc of f .

Theorem 2.13. Let (X, d^S) be a b -metric space, $\zeta \in \mathcal{Z}$ be a simulation function and $C_{x_0, r}^{d^S}$ be any circle on X with $r > 0$. If there exists a self-mapping $f : X \rightarrow X$ satisfying

i) $d^S(fx, x_0) = r$ for each $x \in C_{x_0, r}^{d^S}$,

ii) $\zeta(r, d^S(fx, fy)) \geq 0$ for each $x, y \in C_{x_0, r}^{d^S}$ with $x \neq y$,

iii) $\zeta(d^S(fx, fy), d^S(x, y) - \varphi_r(d^S(x, fx))) \geq 0$ for each $x, y \in C_{x_0, r}^{d^S}$,

iv) f is one to one on the circle $C_{x_0, r}^{d^S}$,

then the circle $C_{x_0, r}^{d^S}$ is a fixed circle of f .

Theorem 2.14. Let (X, d^S) be a b -metric space, the number $M_{d^S}(x, y)$ be defined as in (2.16) and $f : X \rightarrow X$ be a self-mapping. Assume that the fixed point set $\text{Fix}(f)$ contains a geometric figure \mathcal{F} . If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition

$$\zeta(d^S(fx, fy), M_{d^S}(x, y)) \geq 0$$

is satisfied by f for all $x \in \mathcal{F}$ and $y \in X - \mathcal{F}$, then the figure \mathcal{F} is the unique fixed figure of f .

Theorem 2.15. *Let (X, d^S) be a b -metric space, $f : X \rightarrow X$ be a self-mapping with the fixed point set $Fix(f)$ and $r \in [0, \infty)$ be a fixed number. If there exists a simulation function $\zeta \in \mathcal{Z}$ such that the condition*

$$d^S(x, fx) < \zeta(d^S(x, fx), \varphi_r(d^S(x, fx)) + r)$$

is satisfied by f for all $x \notin Fix(f)$ if and only if $f = I_X$.

3. Conclusions and Future Works

In this paper, we have obtained new results on the study of geometric properties of the fixed point set of a self-mapping on a metric (resp. S -metric, b -metric) space via the properties of the set of simulation functions. As a future work, the determination of new conditions which ensure a geometric figure to be fixed by a self-mapping can be considered using similar approaches. Further possible applications of our theoretic results can be done on the applied sciences using the geometric properties of fixed points. For example, in [9], the existence of a fixed point for every recurrent neural network was shown using Brouwer's Fixed Point Theorem and a geometric approach was used to locate where the fixed points are (see [9] for more details). Therefore, theoretic fixed figure results are important in the study of neural networks.

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